Fixating Groups

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Abstract

A group of bijections G acting on a set X is said *with fixed points* (abbreviated as GAF from the french "groupe à point fixe") if any element of G has at least one fixed point. The G group is called *globally fixed* (abbreviated as GAG) if there is $x \in X$ fixed by all elements of G. The group G is said *fixating* if any subgroup of G which is a GAF is automatically a GAG. The article explores which groups are fixating. The situation depends on the assumptions made on the group of bijections and on the support set X. For example the group of isometries of the Euclidean space $\mathbb{R}^n$ is fixating for $n \leqslant 3$ but no longer for $n \geqslant 4$. The case of isometries of elliptic and hyperbolic spaces is also considered, as well as that of isometries of some discrete sets.

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1 Introduction

It is easy to find a group of bijective transformations, each having a fixed point but without a common fixed point: for instance, in the symmetric group of $\{1, 2, 3, 4, 5\}$, the subgroup generated by the cycle (123) and the double transposition $(12)(45)$, or the group of rotations of the two dimensional sphere, or the group of homeomorphisms of the unit disk. Such a group will be called *eccentric*. Beside the existence of a fixed point for each bijection of the group, which additional information would allow to conclude to the existence of a common fixed point? This additional information can be the invariance of a geometric structure, the commutativity of the group or another algebraic property, the uniqueness of the fixed point for each nontrivial bijection, or a combination of the former informations.

The invariance of a geometric structure can often be stated as follows: The group of bijections is a subgroup of a larger group G. The fact that this information implies the existence of a common fixed point can be seen as a property of the larger group G. We shall say that a group G of bijections of a set X is *fixating* if it contains no eccentric subgroup.

To our knowledge, the notion of fixating group has not been considered in earlier works. Notice that this notion is not intrinsic to the group but depends on the action as a group of bijections. The aim of this paper is to explore which are fixating groups of bijections and, to a lesser extent, to find some sufficient conditions for a group of bijection to have a global fixed point.

About fixating group, we shall see that many things can happen, depending on the nature and the dimension of the set X, and on the nature of the bijections. We have paid particular attention to isometry groups of some classical spaces.

When X is a metric space, $\mathrm{Isom}\, X$ denotes the group of isometries of X. If moreover X can be oriented, $\mathrm{Isom}^+ X$ denotes the subgroup of isometries preserving the orientation. Given an integer $n \geqslant 1$, we denote $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{Z}^n$ the lattice of integer points in $\mathbb{R}^n$, $\mathbb{H}_n$ the n-dimensional hyperbolic space, $\mathbb{S}_n$ the n-dimensional sphere, and $\mathbb{R}\mathbf{P}_n$ the n-dimensional projective space (we chose to use the exponent n for cartesian products only). Our results about these spaces are the following.

▷ The groups $\operatorname{Isom}\mathbb{R}^n$ and $\operatorname{Isom}^+\mathbb{R}^n$ are fixating if and only if $n \leqslant 3$.

▷ The groups $\operatorname{Isom}\mathbb{H}_n$ and $\operatorname{Isom}^+\mathbb{H}_n$ are fixating if $n \leqslant 3$, and nonfixating if $n \geqslant 5$. When $n = 4$, the question is open.

One could think that, for each family of spaces $\mathbb{F}_n = \mathbb{R}^n$, $\mathbb{H}_n$, $\mathbb{S}_n$ or $\mathbb{Z}^n$, there exists a critical integer n_0 such that the isometry group of $\mathbb{F}_n$ is fixating if and only if $n \leqslant n_0$. This holds for $\mathbb{R}^n$ (with $n_0 = 3$) and $\mathbb{H}_n$ (with $n_0 = 3$ or 4) but neither for $\mathbb{S}_n$ nor for $\mathbb{Z}^n$:

▷ The group $\operatorname{Isom}^+\mathbb{S}_n$ is fixating if and only if $n = 1$ or 3.

▷ The group $\operatorname{Isom}\mathbb{Z}^n$ is fixating for all $n \geqslant 1$, whether $\mathbb{Z}^n$ is equipped with the Euclidean norm or the L^1 norm.

An important ingredient for the existence of global fixed points is the *median inequality*, see formula (1) at the beginning of Section 4. This inequality holds in Euclidean or hyperbolic spaces, but neither in spheres nor in projective spaces.

This inequality has been introduced by F. Bruhat and J. Tits in [5] and their result is: If a metric space (E, d) is complete and if the median inequality holds, then each isometry group with a bounded orbit admits a global fixed point, see Theorem 4.3 and Corollary 4.4.

Moreover, we show that an isometry group G in a complete metric space is fixating provided it has a normal fixating subgroup H such that G/H is cyclic, see Corollary 4.8. This is the reason why $\operatorname{Isom}\mathbb{R}^n$ and $\operatorname{Isom}^+\mathbb{R}^n$ are fixating for the same dimension n; the same holds for $\operatorname{Isom}\mathbb{H}_n$ and $\operatorname{Isom}^+\mathbb{H}_n$. The median inequality is crucial for this result: We shall see that $\operatorname{Isom}^+\mathbb{S}_3$ is fixating while $\operatorname{Isom}\mathbb{S}_3$ is not. Indeed,

▷ The group $\operatorname{Isom}\mathbb{S}_n$ is fixating if and only if $n = 1$.

About the projective space, we obtain:

▷ The group $\operatorname{Isom}\mathbb{R}\mathbf{P}_n$ is fixating if and only if $n = 1$;

▷ The group $\operatorname{Isom}^+\mathbb{R}\mathbf{P}_n$ is fixating if $n = 1$, and not if $n = 2$ or $n \geqslant 4$. We do not know whether $\operatorname{Isom}^+\mathbb{R}\mathbf{P}_3$ is fixating or not.

Another important ingredient is the existence of some free subgroups in the linear group. It is used to construct eccentric groups. A very general result by A. Borel [4] ensures this existence, although some elementary results lead to explicit examples that are enough for our need.

The article is organized as follows. Section 2 is devoted to notations and some preliminary results. Section 3 is about affine groups and Section 4 about isometry groups. Isometry groups of classical spaces are studied in details in Section 5: first in Euclidean spaces, then in hyperbolic spaces and at last in elliptic spaces. Section 6 is devoted to results in discrete spaces. We first deal with permutation groups, then with $\operatorname{Isom}\mathbb{Z}^n$, and at last with some graphs. Concerning graphs, we first present a result by J.-P. Serre about fixed points of finitely generated isometry groups of trees [20], then we extend this result to a family of colored graphs. Section 7 provides exercises; the solutions are given in Appendix 8.4. Appendix 8.1 is a short introduction to hyperbolic geometry and Appendices 8.2 and 8.3 give details of some proofs.

We leave several questions open. It may happen that some answers are already in the existing literature, or that the reader solves some of them. In that case, we would be grateful to the reader who will inform us!

2 Preliminaries

2.1 Notations

Groups. Given a group G, we denote $H \leqslant G$ to say that H is a subgroup of G and $H \trianglelefteq G$ to say that H is a normal subgroup of G.

If A is a subset of a group G, the subgroup of G generated by A is denoted by $\langle A\rangle$. When A contains a small number of elements, we will omit the braces. Thus we have

$$\langle f,g\rangle = \bigcap_{f,g\in H\leqslant G} H = \big\{ f^{i_1}g^{j_1}\dots f^{i_n}g^{j_n}\ ;\ n\in\mathbb{N},\ i_k,j_k\in\mathbb{Z}\big\}.$$

We call *cyclic* a group, finite or not, generated by a single element.

The *commutator* of f and g is $[f,g]=f^{-1}g^{-1}fg$.

For a set X, $\mathrm{Bij}\,X$ denotes the group of bijections of X.

Metric spaces. In a metric space (X,d), an *isometry* $f:X\to X$ is a *bijection* such that $d(f(x),f(y))=d(x,y)$ for all $x,y\in X$. As already said in the introduction, $\mathrm{Isom}\,X$ is the group of isometries of X, and Isom^+X is the sub-group of those preserving the orientation if X is orientable.

For an element $x\in X$ and a real number $r>0$, $B(x,r)=\{y\in X\ ;\ d(x,y)<r\}$ denotes the open ball of center x and radius r and $B'(x,r)=\{y\in X\ ;\ d(x,y)\leqslant r\}$ is the corresponding closed ball. Given two points a and b of X, $\mathrm{Med}\,(a,b)$ denotes the mediator of a and b:

$$\mathrm{Med}\,(a,b)=\{c\in X\ ;\ d(a,c)=d(b,c)\}.$$

Affine and Euclidean Spaces. Given an affine bijection $f:\mathbb{R}^n\to\mathbb{R}^n$, $\overrightarrow{f}$ is the associated linear map defined by $\overrightarrow{f}(x)=f(x)-f(0)$. The mapping $f\mapsto\overrightarrow{f}$ is a morphism of groups. In particular, we have $[\overrightarrow{f},\overrightarrow{g}]=\overrightarrow{[f,g]}$.

Similarly, if F is an affine subspace of $\mathbb{R}^n$, then $\overrightarrow{F}$ denotes the associated vector subspace.

For a subset $A\subset\mathbb{R}^n$, we denote by $\mathrm{Aff}\,A$ the affine space generated by A, i.e. the intersection of all affine subspaces of $\mathbb{R}^n$ containing A. If $A=\{a_1,\dots,a_n\}$, we shall write $\mathrm{Aff}\,(a_1,\dots,a_n)$ instead of $\mathrm{Aff}\,\big(\{a_1,\dots,a_n\}\big)$.

Hyperbolic Spaces. We will use the model of the upper half-space with the Poincaré metric, see Appendix 8.1.

2.2 GAF, GAG and fixating group

A *group of bijections* (X,G) is the data of a set X and a subgroup G of $\mathrm{Bij}\,X$. Given a bijection $g:X\to X$, its set of fixed points is

$$\mathrm{Fix}\,g=\{x\in X\ ;\ g(x)=x\}.$$

A group of bijections (X,G) is called a *fixed point group* (abbreviated as GAF) if $\mathrm{Fix}\,g$ is not empty for all $g\in G$. We say that (X,G) is a *group with global fixed points* (in short, a GAG) if

$$\mathrm{Fix}\,G:=\bigcap_{g\in G}\mathrm{Fix}\,g\neq\emptyset.$$

A GAF which is not a GAG is called *eccentric*. With the above vocabulary, we say that (X,G) is *fixating* if, for any subgroup $H\leqslant G$, we have

$$(X,H)\ \text{GAF}\ \Leftrightarrow (X,H)\ \text{GAG},$$

i.e. if it does not contain any eccentric subgroup. Note that any subgroup of a fixating group is fixating. We shall omit the set X when the context is clear. In the same way, we will say that an action ρ of a group G on a set X is fixating if $(X,\rho(G))$ is a fixating group of bijections. We will use the following result several times.

Proposition 2.1. *Let f, g be two bijections on a set X.*

a. *If f and g commute, then $g(\operatorname{Fix} f) = \operatorname{Fix} f$.*

b. *If $\operatorname{Fix} f$ is a singleton $\{x_0\}$, then $x_0 \in \operatorname{Fix} g$ for any g commuting with f.*

c. *Let G be a group of bijections on X and let $H \trianglelefteq G$. For all $g \in G$, we have $g(\operatorname{Fix} H) = \operatorname{Fix} H$.*

Proof. **a**. If $x \in \operatorname{Fix} f$ then $g(x) = g(f(x)) = f(g(x))$ hence $g(x) \in \operatorname{Fix} f$. This proves $g(\operatorname{Fix} f) \subseteq \operatorname{Fix} f$. Besides, g^{-1} also commutes with f, so $g^{-1}(\operatorname{Fix} f) \subseteq \operatorname{Fix} f$. We then obtain $\operatorname{Fix} f = g(g^{-1}(\operatorname{Fix} f)) \subseteq g(\operatorname{Fix} f)$, hence the equality.

b. Results from item **a**.

c. Let $g \in G$. It is enough to show that $g(x) \in \operatorname{Fix} h$ for all $x \in \operatorname{Fix} H$ and all $h \in H$; this will imply $g(\operatorname{Fix} H) \subseteq \operatorname{Fix} H$. Then, application to g^{-1} yields the desired equality. For such a x and such a h, we have $k = g^{-1}hg \in H$, hence $x \in \operatorname{Fix} k$. Therefore we have $g(x) = g(k(x)) = h(g(x))$, hence $g(x) \in \operatorname{Fix} h$. □

We immediately deduce from item b a first sufficient condition for a group of bijections to be a GAG.

Proposition 2.2. *Let G be a group of bijections on X. If G is Abelian and if there exists $f_0 \in G$ having a single fixed point, then G is a* GAG.

We will see in Section 3 that each of the words “Abelian” and “unique” is needed.

Let us end this section with remarks of an algebraic nature.

1. The concept “fixating” is compatible with the product: If (X_1, G_1) and (X_2, G_2) are two fixating groups of bijections, it is easily checked that the action of the product group $G_1 \times G_2$ on $X_1 \times X_2$ is fixating.

2. The notion “fixating”, however, is not compatible with the induction of Frobenius [10]. Precisely, let G be a group, H a subgroup of G, and R a system of class representatives modulo H. An action of H on a set Y induces an action of G on $X = R \times Y$ defined by $g(r, y) = (r', h(y))$ where $r' \in R$ and $h \in H$ are uniquely determined by $gr = r'h$.

Section 6.1 on permutation groups provides an example where the action (Y, H) is fixating, but the induced action is not. We consider the group G of permutations of $\{1, 2, 3, 4, 5\}$ and H the subgroup of permutations fixing 5, seen as acting on $Y = \{1, 2, 3, 4\}$. As system of representatives, we choose $r_i = (i\,5)$ (the transposition) for $1 \leqslant 4$ and $r_5 = \mathbf{id}$.

On the one hand, (Y, H) is fixating by Proposition 6.1. On the other hand, (X, G) is not: Let K be the subgroup of G generated by the permutations (123) and $(12)(45)$. One finds

$$K = \{\mathbf{id}, (123), (132), (12)(45), (13)(45), (23)(45)\},$$

$\operatorname{Fix}(12)(45) = \{(r_3, 5)\}$, $\operatorname{Fix}(13)(45) = \{(r_2, 5)\}$, $\operatorname{Fix}(23)(45) = \{(r_1, 5)\}$, and $\operatorname{Fix}(123) = \operatorname{Fix}(132) = \{(r_4, 4), (r_4, 5), (r_5, 4), (r_5, 5)\}$. hence K is an eccentric subgroup of (X, G).

3. The notion “fixating” strongly depends on the set on which the group acts. In Exercise 7.2, we introduce an intrinsic notion of globalization: A group is said to be *super-fixating* if, for any set X and any morphism $\rho : G \to \operatorname{Bij} X$, the pair $(X, \rho(G))$ is fixating. This notion finally has a rather limited interest: The result of Exercise 7.2 is that a group is super-fixating if and only if it is cyclic (finite or not). On the other hand, the additive group $\mathbb{Q}$ is *finitely super-fixating* in the following sense: If X is a finite set and $\rho : \mathbb{Q} \to \operatorname{Bij} X$ a morphism, then $(X, \rho(\mathbb{Q}))$ is fixating, cf. Exercise 7.3.

3 Groups of affine bijections

The affine structure is the most fundamental of the geometric structures. It is therefore natural to begin our study with groups of affine bijections of $\mathbb{R}^n$. The results are rather negative except in dimension one.

Proposition 3.1. *The group of affine bijections of* $\mathbb{R}$ *is fixating.*

Proof. Let H be a group of affine bijections of $\mathbb{R}$ which is a GAF. Since an affine bijection of $\mathbb{R}$ different from the identity has at most one fixed point, it is enough to prove that H is Abelian, then to apply Proposition 2.2. If f and g are in H then the commutator $[f, g]$, which also has a fixed point, is not a nontrivial translation. Since the linear group on $\mathbb{R}$ is Abelian, it can only be the identity. $\square$

The commutativity of the linear group and the uniqueness of the fixed points are the two ingredients of the previous proposition. Both ingredients are characteristic of the dimension one. In larger dimension, adding only one of the hypotheses — commutativity or uniqueness of the fixed points — is not enough to prove that a GAF is a GAG as shown in Examples 3.2 and 3.3 below.

Example 3.2. *Let* $a \in \mathbb{R} \setminus \mathbb{Q}$ *and let* f, g *be the affine transvections of* $\mathbb{R}^2$ *given by*

$$f(x, y) = (x + y + 1, y) \quad \text{and} \quad g(x, y) = (x + ay, y).$$

Then the group $G = \langle f, g\rangle$ *is Abelian and eccentric.*

Proof. It is straightforward that G is Abelian. For $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, we have $f^m g^n(x, y) = \big(x + (m + na)y + m, y\big)$, hence $\operatorname{Fix}(f^m g^n)$ is nonempty: It is the straight line $D_{m,n}$ of equation $y = \frac{-m}{m+na}$. It follows that G is a GAF. We also have $D_{m,n} \cap D_{m',n'} = \emptyset$ as soon as $mn' \neq m'n$, hence G is not a GAG. $\square$

Remarks.

1. If $\mathbb{R}^2$ is equipped with the discrete distance, given by $d(a, b) = 1$ if $a \neq b$ and $d(a, a) = 0$, then the group G of the previous example is an eccentric Abelian group of isometries. Example 4.10 of the next section is an example of an eccentric and Abelian group of isometries in a Hilbert vector space of infinite dimension. On the other hand, Theorem 4.9 shows that there does not exist any eccentric Abelian group of isometries of an Euclidean or hyperbolic space of finite dimension.

2. The above shows that the group of affine bijections of $\mathbb{R}^n$ (with $n \geqslant 1$) is fixating if and only if $n = 1$: For $n \geqslant 3$, just complete the previous maps f and g by the identity on the last $n - 2$ components, as will be done in the proof of Proposition 5.6.

Example 3.3. *Let* $b \in \mathbb{R}^2 \setminus \{0\}$ *and let* $f, g \in \operatorname{Bij}\mathbb{R}^2$ *be the affine bijections*

$$f : x \mapsto \overrightarrow{f}(x) \qquad \text{and} \qquad g : x \mapsto \overrightarrow{g}(x) + b,$$

where $\overrightarrow{f}$ *and* $\overrightarrow{g}$ *are the elements of* $\mathrm{SL}(2, \mathbb{R})$ *with matrices*

$$\operatorname{Mat}(\overrightarrow{f}) = A = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \qquad \operatorname{Mat}(\overrightarrow{g}) = B = \begin{pmatrix} -1 & -1 \\ 5 & 4 \end{pmatrix}.$$

Then the group $G_1 = \langle f, g\rangle$ *is eccentric. More precisely, every element of* $G_1 \setminus \{\mathbf{id}\}$ *has a unique fixed point but* G_1 *is not a* GAG.

We recall that $\mathrm{SL}(2, \mathbb{Z})$ is the set of 2 by 2 matrices with integer coefficients and determinant equal to 1. The proof is based on the following two results.

Lemma 3.4. ([16] Chapter VIII) *Let G_0 be the subgroup of* $\mathrm{SL}(2,\mathbb{Z})$ *generated by the matrices A and B above. Then G_0 is free and any matrix $M \in G_0 \setminus \{I\}$ has a trace different from* 2.

The proof is in Appendix 8.2. We deduce that 1 is not an eigenvalue of M for any matrix $M \in G_0 \setminus \{I\}$, since $\det M = 1$.

Lemma 3.5. *Let h be an affine bijection on $\mathbb{R}^n$ such that* 1 *is not an eigenvalue of $\overrightarrow{h}$. Then* $\operatorname{Fix} h$ *is a singleton.*

Proof. We have $h(x) = x \Leftrightarrow x - \overrightarrow{h}(x) = h(0) \Leftrightarrow x = (\mathbf{id} - \overrightarrow{h})^{-1}(h(0))$. □

Proof of Example 3.3. Let $h \in G_1 \setminus \{\mathbf{id}\}$ and let $\gamma_1, \gamma_1' \dots \gamma_r, \gamma_r' \in \mathbb{Z}$, all nonzero except possibly γ_1 and γ_r', such that $h = f^{\gamma_1} g^{\gamma_1'} \cdots g^{\gamma_r'}$. We have $\overrightarrow{h} = \overrightarrow{f}^{\gamma_1} \overrightarrow{g}^{\gamma_1'} \cdots \overrightarrow{g}^{\gamma_r'} \neq \mathbf{id}$, so $\overrightarrow{h}$ does not have 1 as eigenvalue according to Lemma 3.4, hence h has a unique fixed point by Lemma 3.5. However, there is no fixed point common to all elements of G_1 since the fixed points of f and g are distinct. □

To finish this section, let us recall two classical results on the existence of global fixed points for groups or sets of affine applications. The first result is the Markov-Kakutani theorem and the second is the Kakutani theorem, see [13, 7, 18, 12]. These two results require a compactness assumption and an additional hypothesis. Exercise 7.12 presents a finite version of these results. It is due to R. Antetomaso [1].

Theorem 3.6. (Markov-Kakutani, see for example [18] Theorem 5.23, p.140) *Let K be a convex compact nonempty subset of a separated topological vector space E, and let G be a set of* affine and continuous *maps that commute and leave K stable. Then $K \cap \operatorname{Fix} G$ is nonempty.*

Theorem 3.7. (Kakutani, cf. [18] Theorem 5.11, p.127) *Let K be a compact convex subset of a locally convex topological vector space E and G be an equicontinuous group of affine bijections leaving K stable. Then $K \cap \operatorname{Fix} G$ is nonempty.*

The literature contains a large number of recent works containing supplements and extensions of these results. Among these, below is a complement to Theorem 3.6 due to Anzai and Ishikama [2]: Under the assumptions of Theorem 3.6, with moreover E locally convex, if G is a finitely generated group, $G = \langle T_1, \dots, T_n \rangle$, then, for all $\alpha_j \in]0,1[$ satisfying $\sum_{j=1}^n \alpha_j = 1$, we have $\operatorname{Fix}\left(\sum_{j=1}^n \alpha_j T_j\right) = \operatorname{Fix} G$.

The theorem below can be deduced from Theorem 3.7 but we give an independent proof.

Theorem 3.8. *A group of affine bijections of $\mathbb{R}^n$ having a bounded orbit is a* GAG.

Proof. Let G be such a group and let x be an element of $\mathbb{R}^n$ whose orbit $\mathcal{O}_x = \{f(x)\ ;\ f \in G\}$ is bounded. Let K be the closed convex hull of $\mathcal{O}_x$. Like $\mathcal{O}_x$, K is invariant by all maps $f \in G$. Let $F = \operatorname{Aff} K$, the affine subspace generated by K. The interior of K relative to F is nonempty, hence $\lambda_F(K) > 0$, where λ_F denotes the Lebesgue measure of F (In the case where K is a singleton, λ_F is the counting measure). Since K is compact, we also have $\lambda_F(K) < +\infty$. The maps $f \in G$ are affine, so send the measure λ_F on a multiple of itself. Since they send K on K, they preserve λ_F. The centroid of K for the restriction of λ_F to K, defined by $\frac{1}{\lambda_F(K)} \int_K x d\lambda_F(x)$, is therefore fixed by all $f \in G$. □

4 Groups of isometries

4.1 The median inequality

The Bruhat-Tits fixed point theorem [5] gives a sufficient condition for a group of isometries on a metric space to be a GAG: It is enough that the space satisfies the median inequality below and that the group has a bounded orbit.

Definition 4.1. We say that a metric space (X,d) *satisfies the median inequality* if

$$\forall x,y \in X\ \exists m \in X\ \forall z \in X \quad d(z,m)^2 \leqslant \tfrac{1}{2}\big(d(z,x)^2 + d(z,y)^2\big) - \tfrac{1}{4}\, d(x,y)^2. \tag{1}$$

It is easy to prove that the point m is unique and that $d(x,m) = d(y,m) = \frac{1}{2}\, d(x,y)$, cf. Exercise 7.4. We say that m is *the midpoint of* $\{x,y\}$ and we write it $m(x,y)$.

When X is a Euclidean space, or more generally a pre-Hilbert space, (1) is actually an equality, called the *parallelogram identity*, and m is the usual midpoint of the segment $[x,y]$, cf. Exercise 7.5.a. Conversely, it is known that a normed vector space satisfying (1) is necessarily pre-Hilbert, cf. Exercise 7.5.b.

A combinatorial tree with its usual distance does not satisfy (1) (an edge has no midpoint) but its realization as a real metric space satisfies it. Trees are also the only graphs with this property. Complete Riemannian manifolds which are simply connected and with a negative sectional curvature, especially hyperbolic spaces with their usual distance, satisfy (1), see [5]. This is a consequence of the comparison theorem of Rauch [8], cf. Exercise 7.6.

The median inequality makes it possible to associate a single center with any bounded subset of a complete metric space. Let (X,d) a metric space and A be a nonempty bounded subset of X. For every $x \in X$, let

$$r(x,A) = \inf\{r > 0\ ;\ A \subseteq B'(x,r)\} = \sup\{d(x,a)\ ;\ a \in A\}.$$

We define the *radius of* A by

$$r_A = \inf\{r(x,A)\ ;\ x \in X\}.$$

If there exists $x_0 \in X$ such that $r_A = r(x_0,A)$, we will say that x_0 is a *center of* A. In this case, the closed ball $B'(x_0, r_A)$ is a *ball circumscribed to* A.

Lemma 4.2. [5] *If* (X,d) *is a complete metric space satisfying* (1)*, then any nonempty bounded subset of* X *has a unique center.*

Proof. Let A be a nonempty bounded subset of X, let $x,y \in X$ and let m be the midpoint of $\{x,y\}$. Writing (1) for all $a \in A$, we get

$$\begin{aligned} d(m,a)^2 &\leqslant \tfrac{1}{2}\big(d(x,a)^2 + d(y,a)^2\big) - \tfrac{1}{4} d(x,y)^2 \\ &\leqslant \tfrac{1}{2}\big(r(x,A)^2 + r(y,A)^2\big) - \tfrac{1}{4}\, d(x,y)^2, \end{aligned}$$

from which we successively deduce

$$r_A^2 \leqslant r(m,A)^2 \leqslant \tfrac{1}{2}\big(r(x,A)^2 + r(y,A)^2\big) - \tfrac{1}{4}\, d(x,y)^2,$$

and

$$\tfrac{1}{2}\, d(x,y)^2 \leqslant r(x,A)^2 + r(y,A)^2 - 2r_A^2. \tag{2}$$

We deduce the uniqueness of a possible center: If x and y are centers of A, then $r(x,A) = r(y,A) = r_A$, and (2) implies $x = y$.

To prove the existence, let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $r(x_n,A)$ tends to r_A as n tends to $+\infty$. Taking $x = x_n$ and $y = x_{n+p}$ in (2), we obtain

$$\tfrac{1}{2}\, d(x_n, x_{n+p})^2 \leqslant r(x_n,A)^2 + r(x_{n+p},A)^2 - 2r_A^2 \to 0$$

uniformly in p as n goes to infinity. Thus the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, therefore has a limit ℓ verifying $r(\ell, A) = r_A$, hence ℓ is a center of A. □

The Bruhat-Tits fixed point theorem is stated as follows.

Theorem 4.3. [5] *Let G be a group of isometries of a complete metric space (X, d) verifying the median inequality* (1). *If there is a nonempty bounded subset of X which is invariant by all the elements of G, then G is a* GAG.

Proof. Let A be a nonempty bounded subset of X, invariant by any $g \in G$, and let a be the center of A. Then, for all $g \in G$, $g(B'(a, r_A)) = B'(g(a), r_A)$ is the ball circumscribed to $g(A) = A$, therefore by uniqueness of the center, $g(a) = a$. As a consequence $a \in \operatorname{Fix} G$, hence G is a GAG. □

We immediately deduce the

Corollary 4.4. *Let G be a group of isometries of an Euclidean or hyperbolic space. If G has a bounded orbit, then G is a* GAG.

Some of the following results will be used in Section 5.

Definition 4.5. Let (E, d) be a complete metric space verifying (1). A subset C of E is called *half-convex* if, for all $x, y \in C$, the midpoint of $\{x, y\}$ is in C.

It can be easily shown that a half-convex closed subset of a normed vector space is convex in the usual sense.

The following proposition asserts the existence and uniqueness of an "orthogonal projection" on the set of fixed points of a group of isometries. We split it into three statements, each having its own interest. We recall that $d(x, A) = \inf_{a\in A} d(x, a)$.

Proposition 4.6. *Let (E, d) be a complete metric space verifying* (1).

a. *If $C \subset E$ is a closed half-convex part of E then, for all $x \in E$, there exists a unique $y \in C$ such that $D(x, C) = d(x, y)$. This point is denoted by $y = \pi_C x$.*

b. *If g is an isometry of E, then $\operatorname{Fix} g$ is closed and half-convex.*

c. *If G is a group of isometries of E, then $\operatorname{Fix} G$ is closed and half-convex.*

Proof. **a**. Let us put $\delta = d(x, C)$. By definition, for any $\varepsilon > 0$, there is $y \in C$ such that $d(x, y) < \delta + \varepsilon$. If $z \in C$ also satisfies this inequality, since $d\big(x, m(y, z)\big) \geqslant \delta$, (1) then gives

$$d(y, z)^2 \leqslant 8\delta\varepsilon + 4\varepsilon^2. \tag{3}$$

For each $n \in \mathbb{N}^*$, let $y_n \in C$ be such that $d(x, y_n) < \delta + \frac{1}{n}$. According to the above, the sequence $(y_n)_{n\in\mathbb{N}^*}$ thereby defined is Cauchy, hence converges to a point $y \in C$ satisfying $d(x, y) = \delta$; this proves the existence. Inequality (3) also proves the uniqueness.

b and **c**. Easy verification. □

Theorem 4.7. *Let (E, d) be a complete metric space verifying* (1), *let G be a group of isometries of E and let $H \trianglelefteq G$ be such that G/H is cyclic. If G is a* GAF *and H a* GAG, *then G is a* GAG.

Proof. Let $\varepsilon \in G$ be such that εH generates G/H. Denote $F = \operatorname{Fix} H$. Since H is normal in G, we have $g(F) = F$ for all $g \in G$ according to Proposition 2.1.c, in particular $\varepsilon(F) = F$.

Let $x \in \operatorname{Fix} \varepsilon$. By uniqueness of the orthogonal projection and since ε is an isometry, one has $\varepsilon(\pi_F x) = \pi_{\varepsilon(F)}\varepsilon(x) = \pi_F x$, hence $\pi_F x \in \operatorname{Fix} \varepsilon$. Since $G = \langle \varepsilon, H\rangle$, we obtain $\pi_F x \in \operatorname{Fix} G$. □

Corollary 4.8. *Let (E, d) be a complete metric space satisfying* (1) *and let $H \trianglelefteq G \leqslant \operatorname{Isom} E$ be such that G/H is solvable and finite. If H is fixating, then G is fixating. In particular a group of isometries of E is fixating as soon as it contains a fixating subgroup of index* 2.

Proof. We first assume that the quotient G/H is cyclic. Let $G_1 \leqslant G$ be a GAF; then $H_1 = G_1 \cap H$ is a GAF, therefore a GAG since H is fixating. Besides, G_1/H_1 is isomorphic to a subgroup of G/H, therefore cyclic, hence G_1 is a GAG according to Theorem 4.7.

Since finite Abelian groups are products of cyclic groups, under the hypothesis G/H solvable and finite, there is a finite sequence $H = H_0 \trianglelefteq \cdots \trianglelefteq H_n = G$ such that, for each $i = 1, \dots, n$, the quotient H_i/H_{i-1} is cyclic. The result is then successively applied to the cyclic quotients.

For the last assertion, if H is a subgroup of G of index 2, then H is normal in G and G/H is cyclic of order 2. □

Remarks.

1. Our proof is not valid if G/H is only supposed to be solvable. The right notion in our context is G/H *polycyclic* [19], that is, G/H admits a finite sequence $\{e\} = H_0 \trianglelefteq \cdots \trianglelefteq H_n = G/H$ with H_i/H_{i-1} cyclic. We do not know whether our result holds when G/H is only assumed solvable and of finite type.

2. The fact that the ambient space satisfies (1) is essential: We will see in Section 5.3 that $\operatorname{Isom}^+ \mathbb{S}_3$ is fixating, while $\operatorname{Isom} \mathbb{S}_3$ is not.

4.2 Solvable subgroups: The Euclidean and hyperbolic cases

We fix an integer $n \geqslant 0$. The notation $\mathbb{F}_n$ will indicate either the Euclidean space $\mathbb{R}^n$, or the hyperbolic space $\mathbb{H}_n$. We will use that $\mathbb{F}_n$ satisfies (1).

Theorem 4.9. *Let G be an* Abelian *group of isometries of $\mathbb{F}_n$. If G is a* GAF, *then G is a* GAG.

Proof. We proceed by induction on the dimension n. For $n = 0$ the result is trivial. Now, let $n \geqslant 1$ and assume the property true for all $k < n$.

If $G = \{\mathbf{id}\}$, we are done. Otherwise, let $f \in G \setminus \{\mathbf{id}\}$. Then $F = \operatorname{Fix} f$ is a strict subspace (affine or hyperbolic) of $\mathbb{F}_n$, of dimension $k < n$. Let $g \in G$. As f and g commute, we have $g(F) = F$ according to Proposition 2.1.a. So for all $g \in G$, the restriction of g to F, denoted by $g_{|F}$, is well defined from F to F and is an isometry of F which is itself isometric to $\mathbb{F}_k$.

By hypothesis, $\operatorname{Fix} g$ is nonempty. Let $x_g \in \operatorname{Fix} g$ and set $y_g = \pi_F x_g$. Since g is an isometry and by uniqueness of the orthogonal projection, we have $g(y_g) = \Pi_{g(F)} g(x_g) = \Pi_F x_g = y_g$. Thus, for every $g \in G$, $g_{|F}$ has at least one fixed point y_g.

Let $G_F = \{g_{|F} \ ; \ g \in G\}$. Then G_F is a GAF on a space of dimension $k < n$, hence is a GAG by the induction hypothesis. Since $\operatorname{Fix} G_F = F \cap \operatorname{Fix} G$, we deduce that $\operatorname{Fix} G$ is nonempty, hence G is a GAG. □

The following example shows that the finite dimension is necessary.

Example 4.10. Let $E = \ell^2(\mathbb{N}, \mathbb{R})$, the space of square summable real sequences; it is a Hilbert space. Let h_k be the symmetry of center 1 on the k-th component, i.e. the isometry of E defined by

$$h_k(x_0, x_1, \dots) = (x_0, \dots, x_{k-1}, 2 - x_k, x_{k+1}, \dots).$$

Let $G_n = \langle h_0, \dots, h_n \rangle$ and let $G = \bigcup_{n \in \mathbb{N}} G_n$. It is immediate that G is Abelian. Let $(e_n)_{n \in \mathbb{N}}$ denote the canonical basis of E and $s_n = \sum_{k=0}^{n} e_k$. We have $s_n \in \operatorname{Fix} f$ for all $f \in G_n$, so G is a GAF. By contradiction, if G were a GAG and if $x = (x_0, x_1, \dots) \in \operatorname{Fix} G$, then for all $n \in \mathbb{N}$ we would have $x_n = 1$, but the constant sequence equal to 1 is not in E, hence G is eccentric.

Theorem 4.9 can be generalized by changing the word "Abelian" into "solvable".

Theorem 4.11. *Let G be a* solvable *group of isometries of $\mathbb{F}_n$. If G is a* GAF, *then G is a* GAG.

Proof. Recall that a group G is said *solvable* if there is a finite and growing sequence of subgroups, $\{\mathbf{e}_G\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_p = G$ (i.e. each normal in the next one) such that all the quotients H_{k+1}/H_k are Abelian. The *solvability index of G* is the smallest integer $p \geqslant 0$ with this property. This integer is reached for example by taking the sequence of derived groups: We put $G_0 = G$ and, for $k \geqslant 0$, $G_{k+1} = G'_k = [G_k, G_k]$, the group generated by the commutators of G_k. Finally, we choose $H_k = G_{p-k}$.

The proof is by induction on the solvability index of G. The property is trivially true for $p = 0$. Assume its truth for any group of solvability index $p-1$ and let us show it for G.

Let $G_1 = [G, G]$. Since G is a GAF, G_1 is a GAF, hence a GAG by the induction hypothesis, hence $F = \operatorname{Fix} G_1$ is nonempty. Since G_1 is normal in G, we have $g(F) = F$ for all $g \in G$.

Now the group G/G_1 acts naturally on F: If $\overline{g} = \{gh \,;\, h \in G_1\}$ is an element of G/G_1 and if $x \in F$, then $\overline{g}(x) := g(x)$ does not depend on the choice of the representative $g \in \overline{g}$ since $h(x) = x$ for all $h \in G_1$.

We claim that the pair $(F, G/G_1)$ is a GAF. Indeed, since G is a GAF, if $g \in G$ and $x \in \operatorname{Fix} g$ then, as in the proof of Theorem 4.9, by uniqueness of the orthogonal projection, the projection $\pi_F x$ is also in $\operatorname{Fix} g$. As a consequence, $F \cap \operatorname{Fix} g \neq \emptyset$, which gives $\operatorname{Fix} \overline{g} \neq \emptyset$ for all $\overline{g} \in G/G_1$ seen as isometry of F.

Since G/G_1 is an Abelian group and F is a space (Euclidean or hyperbolic) of finite dimension, G/G_1 is a GAG according to Theorem 4.9. Any fixed global point $x \in F$ of G/G_1 is then fixed by any element of G, so G is itself a GAG. $\square$

5 Groups of isometries of the classical spaces

In the whole Section 5, n is a strictly positive integer.

5.1 The Euclidean case

Recall that $\operatorname{Isom} \mathbb{R}^n$ is the group of isometries of $\mathbb{R}^n$ equipped with the usual Euclidean distance, and $\operatorname{Isom}^+ \mathbb{R}^n$ is the subgroup of those preserving the orientation. It is known that the elements of $\operatorname{Isom} \mathbb{R}^n$ are affine applications, cf. Exercise 7.9. In this Section 5.1, we prove the following result.

Theorem 5.1. *The group* $\operatorname{Isom} \mathbb{R}^n$ *is fixating if and only if $n \leqslant 3$.*

We leave in Exercise 7.1 to the reader the pleasure to show that $\operatorname{Isom} \mathbb{R}^2$ is fixating. We will show successively that $\operatorname{Isom} \mathbb{R}^3$ is fixating, then $\operatorname{Isom}^+ \mathbb{R}^4$ is nonfixating, which will imply that $\operatorname{Isom} \mathbb{R}^n$ is not fixating for $n \geqslant 4$.

The case of the dimension 3

Let us recall that the elements $f \in \operatorname{Isom}^+ \mathbb{R}^3$ such that $\operatorname{Fix} f \neq \emptyset$ are either the identity or the rotations around an axis $\operatorname{Fix} f$. Those with an empty set of fixed points are the translations and the screw dispacements, or more concisely screws (a *screw* is the Abelian product of a rotation r and a translation of nonzero vector parallel to the axis of r).

The following lemma is a key step to prove that $\operatorname{Isom} \mathbb{R}^3$ is fixating. We will have this same step in the proof that the group of isometries of the hyperbolic space of dimension 3 is fixating, see Lemma 5.13.

Lemma 5.2. *If $f, g \in \operatorname{Isom}^+ \mathbb{R}^3$ are such that* $\operatorname{Fix} f \cap \operatorname{Fix} g = \emptyset$, *then there is $h \in \langle f, g \rangle$ such that* $\operatorname{Fix} h = \emptyset$.

Proof. If $\mathrm{Fix}\, f$, $\mathrm{Fix}\, g$ or $\mathrm{Fix}\,(f^{-1}g)$ is empty, we are done. Otherwise, let $a \in \mathrm{Fix}\,(f^{-1}g)$ and let $b = f(a) = g(a)$. We have $b \neq a$ since $\mathrm{Fix}\, f \cap \mathrm{Fix}\, g = \emptyset$. So for all $c \in \mathrm{Fix}\, f$ we have $d(a, c) = d(f(a), f(c)) = d(b, c)$. Therefore $\mathrm{Fix}\, f$ is in $\mathrm{Med}\,(a, b)$, the mediator plane of a and b. The same holds for $\mathrm{Fix}\, g$. Since $\mathrm{Fix}\, f \cap \mathrm{Fix}\, g = \emptyset$, $\mathrm{Fix}\, f$ and $\mathrm{Fix}\, g$ are two parallel lines. It follows that $\overrightarrow{[f, g]} = \mathbf{id}$. If f and g were commuting, we would have $f(\mathrm{Fix}\, g) = \mathrm{Fix}\, g$ by Proposition 2.1.a, in contradiction with $f \neq \mathbf{id}$. It follows that $[f, g]$ is a nontrivial translation, hence $\mathrm{Fix}\,[f, g] = \emptyset$. □

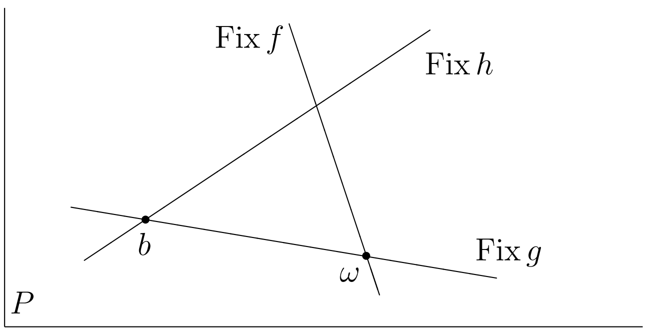
Proposition 5.3. *The group* $\mathrm{Isom}\, \mathbb{R}^3$ *is fixating.*

Proof. According to Corollary 4.8, it suffices to show that $\mathrm{Isom}^+ \mathbb{R}^3$ is fixating. Let $G \leqslant \mathrm{Isom}^+ \mathbb{R}^3$ be a GAF. We must show that $\mathrm{Fix}\, G \neq \emptyset$. Lemma 5.2 already implies $\mathrm{Fix}\, f \cap \mathrm{Fix}\, g \neq \emptyset$ for all $f, g \in G$.

We call *half-turn* a symmetry about a straight line, called *axis*. We will use the following fact: If the product of two half-turns is a half-turn, then their axes are orthogonal and secant.

If $G \setminus \{\mathbf{id}\}$ contains only half-turns then, either $G = \{\mathbf{id}, f\}$ where f is a half-turn, or $G = \{\mathbf{id}, f_1, f_2, f_3\}$ where f_1, f_2, f_3 are three half-turns of axes orthogonal and pairwise secant, so secant all three in one point, hence G is a GAG.

We assume now that there exists $f \in G \setminus \{\mathbf{id}\}$ that is not a half-turn. If $\mathrm{Fix}\, g = \mathrm{Fix}\, f$ for all $g \in G \setminus \{\mathbf{id}\}$, we have finished: $\mathrm{Fix}\, G = \mathrm{Fix}\, f \neq \emptyset$. It is now assumed that there exists $g \in G \setminus \{\mathbf{id}\}$ such that $\mathrm{Fix}\, g \neq \mathrm{Fix}\, f$. Then $\mathrm{Fix}\, f$ and $\mathrm{Fix}\, g$ are two straight lines crossing at some point denoted by ω. Set $P = \mathrm{Aff}\,(\mathrm{Fix}\, f \cup \mathrm{Fix}\, g)$, the affine plane containing $\mathrm{Fix}\, f$ and $\mathrm{Fix}\, g$.



We will be done if we show that $\omega \in \mathrm{Fix}\, G$. By contradiction, otherwise, let $h \in G$ be such that $\omega \notin \mathrm{Fix}\, h$. Since $\mathrm{Fix}\, f \cap \mathrm{Fix}\, h$ and $\mathrm{Fix}\, g \cap \mathrm{Fix}\, h$ are nonempty, we have $\mathrm{Fix}\, h \subset P$. Let $a \in \mathbb{R}^3$ be such that $f(a) = b \in \mathrm{Fix}\, g \cap \mathrm{Fix}\, h$. We have $b \in P$ and $b \notin \mathrm{Fix}\, f$, but f is not a half-turn, so $a \notin P$. Let $\widetilde{g} = f^{-1} g f$. We have $\widetilde{g}(a) = a$ and $\widetilde{g}(\omega) = \omega$, hence $\mathrm{Fix}\, \widetilde{g} = \mathrm{Aff}\,(a, \omega)$, which is a disjoint line of $\mathrm{Fix}\, h$, a contradiction. □

Remark. This result can also be proved by copying the proof of Lemma 5.14.

The case of the higher dimensions

Proposition 5.4. *The group* $\mathrm{Isom}^+ \mathbb{R}^4$ *is nonfixating. As a consequence, the group* $\mathrm{Isom}\, \mathbb{R}^4$ *is nonfixating.*

Proof. We reproduce below the construction by Wagon [23] of a free subgroup of rank 2 in SO_4 whose action on the sphere $\mathbb{S}_3$ is without fixed point. Let $\theta \in \mathbb{R}$ be such that $\cos\theta$ is transcendent, for example $\theta = 1$, and let σ and τ be the elements of SO_4, of matrices respectively

$$S_4 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad T_4 = \begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix}.$$

Lemma 5.5. ([23] Theorem 5.2, p.53) *The subgroup G_0 of SO_4 generated by σ and τ is free. Moreover, 1 is not an eigenvalue of any element of $G_0 \setminus \{\mathbf{id}\}$.*

For the convenience of the reader, we have written the proof in Appendix 8.3.

Now choose the affine rotations σ and $\widetilde{\tau} : x \mapsto \tau x + a$ with $a \neq 0$, for example $a = (1, 0, 0, 0)$. Let $G = \langle \sigma, \widetilde{\tau} \rangle$ the subgroup of $\mathrm{Isom}^+ \mathbb{R}^4$ generated by σ and $\widetilde{\tau}$. Then G is free and $\mathrm{Fix}\, g$ is a singleton for all $g \in G$. Since $\mathrm{Fix}\, \sigma \cap \mathrm{Fix}\, \tau = \emptyset$, we deduce that G is eccentric. □

Remark. The existence of free subgroups of $\mathrm{SL}(2, \mathbb{R})$ and SO_4 whose elements, apart from identity, never admit 1 as eigenvalue, is the essential ingredient of constructions of eccentric subgroups of affine applications or of affine isometries (Example 3.3 and Proposition 5.4). We have used explicit examples of such subgroups. These subgroups, although sometimes difficult to exhibit, are not exceptional. Indeed, it can be shown thanks to the Baire theorem that, if G is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$, then the set of pairs of elements of G generating a free group is either empty or contains a dense G_δ, i.e. a countable intersection of dense open subsets of $G \times G$. The same result holds with the additional constraint on the eigenvalue 1. Let us add that A. Borel has proved a very general result encompassing the constructions we used, see [4]:

If G is a semi-simple linear algebraic group defined on $\mathbb{R}$, then the set of n -tuples of $G(\mathbb{R})^n$ generating a free group contains a dense G_δ subset.

Proposition 5.6. *For any integer $n \geqslant 4$, the group $\mathrm{Isom}\, \mathbb{R}^n$ is nonfixating.*

Proof. For $n \geqslant 5$, just complete by $\mathbf{id}_{n-4}$ on the last $n - 4$ components: Let σ_n and τ_n be the elements of $\mathrm{Isom}^+ \mathbb{R}^n$ of matrices

$$S_n = \begin{pmatrix} S_4 & 0 \\ 0 & \mathbf{id}_{n-4} \end{pmatrix} \text{ and } T_n = \begin{pmatrix} T_4 & 0 \\ 0 & \mathbf{id}_{n-4} \end{pmatrix}$$

and let $G = \langle \sigma_n, \widetilde{\tau}_n \rangle$ be the subgroup of $\mathrm{Isom}\, \mathbb{R}^n$ generated by σ_n and $\widetilde{\tau}_n : x \mapsto \tau_n x + a$ with $a \in \mathbb{R}^n \setminus \{0\}$. It is easy to check that G is free and eccentric. □

The case of non Euclidean norms.

Let us endow $\mathbb{R}^n$ with an arbitrary norm, and let G denote the group of isometries associated with this norm. According to the Mazur-Ulam theorem, the elements of G are affine maps, cf. [14] or Exercise 7.11 and its solution. The group $\overrightarrow{G}$ of the linear parts of the elements of G is closed and bounded in the vector space of endomorphisms of $\mathbb{R}^n$, so $\overrightarrow{G}$ is compact. By a classical argument, we can construct a scalar product which is invariant by the elements of $\overrightarrow{G}$. Therefore G can be seen as a subgroup of $\mathrm{Isom}\, \mathbb{R}^n$ and G is thus fixating if $n \leqslant 3$. When $n \geqslant 4$, Proposition 5.6 does not allow to conclude and indeed G may be fixating for some norms:

Proposition 5.7. *Let N be a norm on $\mathbb{R}^n$ and let G be the group of isometries associated with N. If the linear group associated with $\overrightarrow{G}$ is finite, then G is fixating. In particular, if N is one of the usual N_p norms with $p \in [1, +\infty] \setminus \{2\}$, then G is fixating.*

Proof. Let H be a subgroup GAF of G. Since the only translation in H is $\mathbf{id}$, the morphism $\varphi : H \to \overrightarrow{G}$, $f \mapsto \overrightarrow{f}$ is injective, so H is finite. We check that the point $\omega = \frac{1}{|H|} \sum_{h \in H} h(\overrightarrow{0})$ is fixed by all elements of H, hence H is a GAG.

If $N = N_p$ then $\overrightarrow{G}$ contains all the permutations of the axes, so a scalar product invariant by the elements of $\overrightarrow{G}$ is necessarily proportional to the usual Euclidean scalar product. The Euclidean unit sphere touches the unit ball for N_p exactly at the vertices of the hyper-octahedron $E = \{\pm \overrightarrow{e}_1, \ldots, \pm \overrightarrow{e}_n\}$. The set E is invariant by $\overrightarrow{G}$, so $\overrightarrow{G}$ is finite. □

5.2 The hyperbolic case

We use the Poincaré model of the half-space: $\mathbb{H}_n = \mathbb{R}_{>0} \times \mathbb{R}^{n-1}$ endowed with the Poincaré metric given by $ds^2 = \frac{1}{x_n^2}\sum_{i=1}^n dx_i^2$, cf. Appendix 8.1 for details.

Theorem 5.8. *If $n \leqslant 3$, then $\operatorname{Isom}\mathbb{H}_n$ is fixating. If $n \geqslant 5$, then $\operatorname{Isom}\mathbb{H}_n$ is nonfixating.*

We conjecture that the group $\operatorname{Isom}\mathbb{H}_4$ is nonfixating.

We will show first that $\operatorname{Isom}\mathbb{H}_n$ is nonfixating if $n \geqslant 5$, then that $\operatorname{Isom}\mathbb{H}_2$ and $\operatorname{Isom}\mathbb{H}_3$ are fixating. The case $n = 1$ is obvious.

The case of dimensions higher than 5

Proposition 5.9. *For any $n \geqslant 5$, the group $\operatorname{Isom}\mathbb{H}_n$ is nonfixating.*

Proof. With each isometry of the Euclidean space of dimension $n-1$, we associate an isometry of the hyperbolic space of dimension n in the following way: If f is an isometry of $\mathbb{R}^{n-1}$ then the map $F : \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \to \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ defined by $F(x,t) = (f(x),t)$ is an isometry of $\mathbb{H}_n$. The mapping $f \in \operatorname{Isom}\mathbb{R}^{n-1} \mapsto F \in \operatorname{Isom}\mathbb{H}_n$ is a morphism of groups and we have $\operatorname{Fix} F = \operatorname{Fix} f \times \mathbb{R}_{>0}$. Therefore the image by this morphism of an eccentric subgroup of $\operatorname{Isom}\mathbb{R}^{n-1}$ is an eccentric subgroup of $\operatorname{Isom}\mathbb{H}_n$. Since, for $n \geqslant 5$, $\operatorname{Isom}\mathbb{R}^{n-1}$ is nonfixating, $\operatorname{Isom}\mathbb{H}_n$ is nonfixating. □

The case of dimension 2

Although this case can be deduced from the 3-dimensional case (cf. Exercise 7.8) we chose to present proofs specific to the dimension 2 because they are more elementary and are a good introduction into hyperbolic geometry. We use the model of the complex half-plane $\mathbb{H}_2 = \{z \in \mathbb{C}\ ;\ \operatorname{Im} z > 0\}$. The positive isometries of $\mathbb{H}_2$ are the homographies

$$h_{a,b,c,d} : z \mapsto \frac{az+b}{cz+d} \quad \text{with} \quad a,b,c,d \in \mathbb{R},\ ad-bc = 1.$$

The other isometries of $\mathbb{H}_2$, called *negatives*, are the homographies composed with the symmetry $z \mapsto -\bar{z}$. The group $\operatorname{Isom}^+\mathbb{H}_2$ of positive isometries is called the *Möbius group*. We recall that the mapping

$$\Phi : \operatorname{Isom}^+\mathbb{H}_2 \to \mathrm{PSL}(2,\mathbb{R}),\ h_{a,b,c,d} \mapsto \{M, -M\} \quad \text{with} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an isomorphism of groups. We call *matrices associated with an isometry* h the elements of $\Phi(h)$. We define the *trace* of an isometry h by $\operatorname{tr} h = |\operatorname{tr} M|$ where $M \in \Phi(h)$. We will use the following result, the proof of which is in Appendix 8.1.4.

Lemma 5.10. *Let $h \in \operatorname{Isom}^+\mathbb{H}_2 \setminus \{\mathbf{id}\}$.*

a. *We have $\operatorname{Fix} h \neq \emptyset$ if and only if $\operatorname{tr} h < 2$. Moreover, in that case, $\operatorname{Fix} h$ is a singleton.*

b. *In the case where $\operatorname{Fix} h = \{i\}$, the associated matrices M and $-M \in \Phi(h)$ are the orthogonal matrices $R(\frac{\theta}{2}) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$ and $R(\frac{\theta}{2}+\pi)$. We then say that h is a* rotation of center i and angle $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, *and we will denote $h = r_\theta$.*

c. *In the general case, if $\operatorname{Fix} h = \{z_0\}$ and if $\varphi \in \operatorname{Isom}^+\mathbb{H}_2$ is such that $\varphi(i) = z_0$, then $\varphi^{-1}h\varphi$ fixes i, hence is a rotation r_θ, and the angle θ does not depend on the choice of φ. We will say that h is a* rotation of center a and angle θ.

d. *For every $x > 0$ and every $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, the rotation of center ix and angle θ is associated with the matrices* $\pm\begin{pmatrix} \cos\frac{\theta}{2} & -x\sin\frac{\theta}{2} \\ x^{-1}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$.

An element $h \in \operatorname{Isom}^+ \mathbb{H}_2$ is called *elliptic* if $\operatorname{tr} h < 2$, *parabolic* if $\operatorname{tr} h = 2$, and *hyperbolic* if $\operatorname{tr} h > 2$.

We divided by 2 the angle in the matrices so that the angle defined in the statement above corresponds to the usual notion of angle: The rotation $r_\theta : z \mapsto \frac{cz-s}{sz+c}$ with $c = \cos\frac{\theta}{2}$ and $s = \sin\frac{\theta}{2}$ turns approximately the points very close to its center i by an angle θ and not $\frac{\theta}{2}$.

In addition, we will use that two segments of $\mathbb{H}_2$ of the same length are positively isometric: If $x, y, x', y' \in \mathbb{H}_2$ are such that $d(x, y) = d(x', y')$, then there exists $f \in \operatorname{Isom}^+ \mathbb{H}_2$ such that $f(x) = x'$ and $f(y) = y'$; see Section 8.1.4 for a proof.

As a preliminary, we show the following result.

Lemma 5.11. *If $f, g \in \operatorname{Isom}^+ \mathbb{H}_2$ are such that $\operatorname{Fix} f \cap \operatorname{Fix} g = \emptyset$, then there exists $h \in \langle f, g\rangle$ such that $\operatorname{Fix} h = \emptyset$.*

Proof. If $\operatorname{Fix} f$ or $\operatorname{Fix} g$ is empty, we are done. Now we assume that $\operatorname{Fix} f$ and $\operatorname{Fix} g$ are nonempty. Thus f and g are two rotations of centers a and b respectively, with $a \neq b$. Below we prove that $\operatorname{tr}[f, g] > 2$, yielding $\operatorname{Fix}[f, g] = \emptyset$ by Lemma 5.10.a.

Let us start by sending a and $g(a)$ into $i\mathbb{R}$: Let $\varphi \in \operatorname{Isom}^+ \mathbb{H}_2$ and $x \in]0, +\infty[\setminus \{1\}$ be such that $\varphi(a) = i$ and $\varphi(g(a)) = ix$. Set $\widetilde{f} = \varphi f \varphi^{-1}$ and $\widetilde{g} = \varphi g \varphi^{-1}$. We have $[f, g] = \varphi^{-1}[\widetilde{f}, \widetilde{g}]\varphi$, hence $\operatorname{tr}[f, g] = \operatorname{tr}[\widetilde{f}, \widetilde{g}]$.

Now we write the commutator of $\widetilde{f}$ and $\widetilde{g}$ in the form $[\widetilde{f}, \widetilde{g}] = \widetilde{f}h$, with $h = \widetilde{g}\widetilde{f}^{-1}\widetilde{g}^{-1}$. The isometry $\widetilde{f}$ is the rotation of center i and angle $\theta \neq 0$; it is associated with the matrices $\pm\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ with $t = \frac{\theta}{2} \not\equiv 0 \mod \pi$. The isometry h is conjuguated to $\widetilde{f}^{-1}$, hence has an angle $-\theta$, and fixes the point ix. By Lemma 5.10.d, it follows that h is associated with the matrices $\pm\begin{pmatrix} \cos t & x\sin t \\ -x^{-1}\sin t & \cos t \end{pmatrix}$. As a consequence, the product $\widetilde{f}h$ is associated with the matrices

$$\pm\begin{pmatrix} \cos^2 t + x^{-1}\sin^2 t & (x-1)\cos t\sin t \\ (1-x^{-1})\cos t\sin t & x\sin^2 t + \cos^2 t \end{pmatrix}.$$

We then have

$$\operatorname{tr}[f, g] = \operatorname{tr}\widetilde{f}h = 2\cos^2 t + (x + x^{-1})\sin^2 t > 2. \qquad \square$$

The proof of Lemma 5.11 is based on the computation of a trace. A geometric proof is also available by an adaptation of the proof of Lemma 5.13 in the sequel.

Proposition 5.12. *The group $\operatorname{Isom} \mathbb{H}_2$ is fixating.*

Proof. By Corollary 4.8, it is enough to prove that $\operatorname{Isom}^+ \mathbb{H}_2$ is fixating. Let $G \leqslant \operatorname{Isom}^+ \mathbb{H}_2$ be a GAF; we have to prove that G is a GAG. If $G = \{\mathbf{id}\}$, the result is obvious. We now assume $G \neq \{\mathbf{id}\}$. By Lemma 5.11, all elements of $G \setminus \{\mathbf{id}\}$ are rotations of the same center, hence G is a GAG. $\square$

Remarks.

1. Ironically, the Möbius group acting on $\widehat{\mathbb{R}}$ instead of $\mathbb{H}_2$ is nonfixating, in spite of a lower dimension, cf. Exercise 7.7.

2. One can also deduce Proposition 5.12 from the forthcoming Corollary 5.15, cf. Exercise 7.8.

The case of the dimension 3

We start with the hyperbolic analogue of Lemma 5.2 on $\operatorname{Isom}^+ \mathbb{R}^3$.

Lemma 5.13. *Let $f, g \in \operatorname{Isom}^+ \mathbb{H}_3$. If $\operatorname{Fix} f \cap \operatorname{Fix} g = \emptyset$, then there exists $h \in \langle f, g\rangle$ such that* $\operatorname{Fix} h = \emptyset$.

Proof. The proof is done in several steps by proving that one of the isometries f, g, $f^{-1}g$, $gfg^{-1}f^{-1}$, or $gfgf^{-1}$ has no fixed point.

Let $h = f^{-1}g$. If $\operatorname{Fix} f$, $\operatorname{Fix} g$, or $\operatorname{Fix} h$ is empty, we are done; otherwise, let $x_0 \in \operatorname{Fix} h$. We have $f(x_0) = g(x_0)$ and, since $\operatorname{Fix} f \cap \operatorname{Fix} g$ is empty, one has $f(x_0) = g(x_0) \neq x_0$. Let $P_0 = \operatorname{Med}(x_0, f(x_0))$; it is a hyperbolic plane by Lemme 8.6. Since f and g are isometries, $\operatorname{Fix} f$ and $\operatorname{Fix} g$ are included in P_0.

We claim that $\operatorname{Fix} g \cap f(\operatorname{Fix} g)$ is empty. Otherwise, let $x_1 \in \operatorname{Fix} g$ be such that $y_1 = f(x_1) \in \operatorname{Fix} g$. According to Lemma 8.10, the sets of fixed points of elements of $\operatorname{Isom}^+ \mathbb{H}_3$ are hyperbolic lines when they are nonempty. Therefore, according to Lemma 8.8 on the projections, there is a single point $z_1 = \pi_{\operatorname{Fix} f} x_1$ realizing the distance from x_1 to $\operatorname{Fix} f$. The three points x_1, y_1, and z_1 are distinct since $\operatorname{Fix} f \cap \operatorname{Fix} g$ is empty.

Let γ denote the geodesic passing through x_1 and z_1. Since x_1 and $\operatorname{Fix} f$ are in the plane P_0, γ is a hyperbolic line of the plane P_0, orthogonal to $\operatorname{Fix} f$ in z_1 according to Lemma 8.8. Since $f(x_1) \in \operatorname{Fix} g \subset P_0$, the plane P_0 is stable by f. Therefore $f(\gamma)$ is a hyperbolic line of P_0. It is also orthogonal to $\operatorname{Fix} f$ in z_1 because f preserves angles. As a result, $f(\gamma) = \gamma$ and the restriction of f to γ is a symmetry of center z_1. Since $y_1 = f(x_1) \in f(\gamma) = \gamma$, y_1 is the symmetric on γ of x_1 with respect to z_1. It follows that $\gamma = \operatorname{Fix} g$, and therefore z_1 belongs to $\operatorname{Fix} g$ and $\operatorname{Fix} f$, in contradiction with $\operatorname{Fix} f \cap \operatorname{Fix} g$ empty.

Let $\widetilde{g} = fgf^{-1}$; it is a hyperbolic rotation of the same angle θ as g up to the sign (the angle is only defined up to the sign). We have $\operatorname{Fix} \widetilde{g} = f(\operatorname{Fix} g)$ so $\operatorname{Fix} \widetilde{g} \cap \operatorname{Fix} g = \emptyset$ from above. As before, if $\operatorname{Fix}(g\widetilde{g}^{-1})$ is empty we have finished; otherwise, with $x_1 \in \operatorname{Fix}(g\widetilde{g}^{-1})$, the plane $P = \operatorname{Med}(x_1, g(x_1))$ contains $\operatorname{Fix} g$ and $\operatorname{Fix} \widetilde{g}$.

We will show that $\operatorname{Fix}(g\widetilde{g})$ or $\operatorname{Fix}(g\widetilde{g}^{-1})$ is empty.

Let σ_P be the reflection about the plane P. According to Lemma 8.12, there is a hyperbolic plane S containing the line $\operatorname{Fix} g$ and making an angle $\pm\frac{\theta}{2}$ with P such that $g = \sigma_S\sigma_P$.

Since $\operatorname{Fix} \widetilde{g}$ is also included in P, we can find two hyperbolic planes $\widetilde{S}$ and $\widetilde{S}'$, containing $\operatorname{Fix} \widetilde{g}$ and making the angles $\pm\frac{\theta}{2}$ with P. According to Lemma 8.12, $\widetilde{g}$ or $\widetilde{g}^{-1} = \sigma_P\sigma_{\widetilde{\mathbb{S}}}$. It can be assumed without loss of generality that $\widetilde{g} = \sigma_P\sigma_{\widetilde{S}}$; then we have $\widetilde{g}^{-1} = \sigma_P\sigma_{\widetilde{S}'}$.

We claim that at least one of the intersections $S \cap \widetilde{S}$ or $S \cap \widetilde{S}'$ is empty. Indeed, conjugating by an element of $\operatorname{Isom} \mathbb{H}_3$ if necessary, we can assume that $\operatorname{Fix} g$ is a vertical half-line of endpoint a in the horizontal plane $\partial\mathbb{H}_3$; P and S are then vertical half-planes containing the half-line $\operatorname{Fix} g$. The boundary of P is an affine line Δ of the horizontal plane that contains a as well as the endpoints b and c of the hyperbolic line $\operatorname{Fix} \widetilde{g}$ (it may occur that the hyperbolic line $\operatorname{Fix} \widetilde{g}$ has only one endpoint; it is then vertical and the sequel becomes simpler). Since the hyperbolic lines $\operatorname{Fix} g$ and $\operatorname{Fix} \widetilde{g}$ do not cross, the points b and c are on the same side of a on the line Δ, so one of the points, say b, is between a and c. Let V be the vertical plane containing b and parallel to the plane S. Since the angles of the hyperbolic planes $\widetilde{S}$ and $\widetilde{S}'$ with the plane P are $\pm\frac{\theta}{2}$, one of the planes $\widetilde{S}$ or $\widetilde{S}'$ is tangent to V. Suppose it is $\widetilde{S}$; if this is not the case, simply replace $\widetilde{g}$ by $\widetilde{g}^{-1}$. The plane $\widetilde{S}$ does not cross S because it is located on one side of V and its closure contains c while S contains a. We have $g\widetilde{g} = \sigma_S\sigma_P\sigma_P\sigma_{\widetilde{S}} = \sigma_S\sigma_{\widetilde{S}}$.

We conclude by showing that $\operatorname{Fix}(\sigma_S\sigma_{\widetilde{S}})$ is empty. By contradiction, otherwise let $x \in \operatorname{Fix}(\sigma_S\sigma_{\widetilde{S}})$, hence $\sigma_S(x) = \sigma_{\widetilde{S}}(x)$. Since $\operatorname{Fix} \sigma_{\widetilde{S}} \cap \operatorname{Fix} \sigma_S = S \cap \widetilde{S} = \emptyset$, we would have $\sigma_S(x) \neq x$, hence the hyperbolic planes S and $\widetilde{S}$ would both be included in the plane $\operatorname{Med}(x, \sigma_S(x))$, so would coincide, in contradiction with $S \cap \widetilde{S} = \emptyset$. ☐

Lemma 5.14. *Let $G \leqslant \operatorname{Isom}^+ \mathbb{H}_3$ be such that $\operatorname{Fix} f \cap \operatorname{Fix} g \neq \emptyset$ for all $f, g \in G$. Then G is a* GAG.

Proof. If $\operatorname{Fix} f = \operatorname{Fix} g$ for all $f, g \in G \setminus \{\mathbf{id}\}$, we are done. Otherwise, let $f, g \in G \setminus \{\mathbf{id}\}$ with $\operatorname{Fix} f \neq \operatorname{Fix} g$, let Π be the hyperbolic plane containing the hyperbolic lines $\operatorname{Fix} f$ and $\operatorname{Fix} g$ and let $\omega \in \Pi$ be the point of intersection of these lines. It remains to prove that $\omega \in \operatorname{Fix} h$ for all $h \in G$. By contradiction, otherwise, let $h \in G$ be such that $\omega \notin \operatorname{Fix} h$, let α be such that $\operatorname{Fix} f \cap \operatorname{Fix} h = \{\alpha\}$, and let β such that $\operatorname{Fix} g \cap \operatorname{Fix} h = \{\beta\}$. Therefore we have $\alpha \neq \beta$ and $\alpha, \beta \in \Pi \cap \operatorname{Fix} h$, hence $\operatorname{Fix} h \subset \Pi$. By the way, we have shown

$$\forall k \in G \quad (\omega \notin \operatorname{Fix} k \Rightarrow \operatorname{Fix} k \subset \Pi). \tag{4}$$

With the same h, let $\delta \in \operatorname{Fix}(fh) \setminus \{\alpha\}$. We have $\delta \notin \operatorname{Fix} h$ (otherwise $\delta \in \operatorname{Fix} f \cap \operatorname{Fix} h$, but $\delta \neq \alpha$) hence $\operatorname{Fix} h \subset \operatorname{Med}(\delta, h(\delta))$. Similarly $\operatorname{Fix} f = \operatorname{Fix} f^{-1} \subset \operatorname{Med}(\delta, f^{-1}(\delta)) = \operatorname{Med}(\delta, h(\delta))$, which yields $\operatorname{Med}(\delta, h(\delta)) = \Pi$. It follows that $\delta \notin \Pi$, hence $\operatorname{Fix}(fh) \not\subset \Pi$. Now the contraposition of (4) implies $\omega \in \operatorname{Fix}(fh)$, but this implies $\omega \in \operatorname{Fix} h$, a contradiction. ☐

Corollary 5.15. *The group* $\operatorname{Isom} \mathbb{H}_3$ *is fixating.*

Proof. Since $\operatorname{Isom}^+ \mathbb{H}_3$ is a subgroup of index 2 of $\operatorname{Isom} \mathbb{H}_3$, by Corollary 4.8, it suffices to prove that $\operatorname{Isom}^+ \mathbb{H}_3$ is fixating. Let $G \leqslant \operatorname{Isom}^+ \mathbb{H}_3$ be a GAF. According to Lemma 5.13, we have $\operatorname{Fix} f \cap \operatorname{Fix} g \neq \emptyset$ for all $f, g \in G$, hence G is a GAG by Lemma 5.14. ☐

5.3 The spherical case

The sphere $\mathbb{S}_n$ is endowed with the spherical distance $d(x, y) = \arccos\langle x \mid y \rangle$. Since the function arccos is bijective, the isometries of $\mathbb{S}_n$ for the spherical distance coincide with the isometries of $\mathbb{S}_n$ for the distance induced by the Euclidean distance of $\mathbb{R}^{n+1}$. It is known that an isometry of $\mathbb{S}_n$ is the restriction to $\mathbb{S}_n$ of a vector isometry of $\mathbb{R}^{n+1}$, see [3], Chap.18 or Exercise 7.13. Precisely, if O_{n+1} denotes the group of vector isometries of $\mathbb{R}^{n+1}$, then we have a group isomorphism

$$\varphi : \mathrm{O}_{n+1} \to \operatorname{Isom} \mathbb{S}_n$$

which maps any element $f \in \mathrm{O}_{n+1}$ to its restriction to $\mathbb{S}_n$. We have $\operatorname{Fix} f = \{\overrightarrow{0}\}$ if and only if $\operatorname{Fix} \varphi(f) = \emptyset$. By abuse of language, we will say that a subgroup G of O_{n+1} is a GAF, resp. a GAG, resp. eccentric, if its image $\varphi(G)$ is a GAF, resp. a GAG, resp. an eccentric subgroup of $\operatorname{Isom} \mathbb{S}_n$.

Theorem 5.16. *The group* $\operatorname{Isom} \mathbb{S}_n$ *is fixating if and only if* $n = 1$.
The group $\operatorname{Isom}^+ \mathbb{S}_n$ *is fixating if and only if* $n = 1$ *or* $n = 3$.

Observe that Corollary 4.8 does not apply here because the median inequality does not hold in elliptic spaces. The proof of Theorem 5.16 is split into several parts.

For $n = 1$, the only GAF of $\operatorname{Isom} \mathbb{S}_1$ are the trivial group $\{\mathbf{id}\}$ and the groups $\{\mathbf{id}, s\}$ where s is a reflection in a line, which are obviously GAG.

Proposition 5.17. *If* $n \geqslant 2$, *then* $\operatorname{Isom} \mathbb{S}_n$ *is nonfixating.*

Proof. We use the framework of O_{n+1}. Let us start with $n = 2$. Let $(\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ be an orthonormal basis of $\mathbb{R}^3$. Let $f, g, h \in \mathrm{O}_3$ be the vector isometries of matrices respectively $\operatorname{Mat} f = \operatorname{diag}(1, -1, -1)$, $\operatorname{Mat} g = \operatorname{diag}(-1, 1, -1)$ and $\operatorname{Mat} h = \operatorname{diag}(-1, -1, 1)$. The set $G = \{\mathbf{id}, f, g, h\}$ is a group isomorphic to the Klein group $(\mathbb{Z}/2\mathbb{Z})^2$: We have $f^2 = g^2 = h^2 = \mathbf{id}$, $fg = gf = h$, $gh = hg = f$ and $hf = fh = g$. We therefore have $\operatorname{Fix} k \neq \{\overrightarrow{0}\}$ for all $k \in G$ but $\operatorname{Fix} G = \{\overrightarrow{0}\}$.

For $n \geqslant 3$, we complete the matrices of f and g by -1 and the matrix of h by 1: In an orthonormal basis, one chooses $\operatorname{Mat} f = \operatorname{diag}(1, -1, \ldots, -1)$, $\operatorname{Mat} g = \operatorname{diag}(-1, 1, -1, \ldots, -1)$, and $\operatorname{Mat} h = \operatorname{diag}(-1, -1, 1, 1, \ldots, 1)$. We check that $\{\mathbf{id}, f, g, h\}$ is still isomorphic to the Klein group, and that it is eccentric. $\square$

Let us now treat the case $\operatorname{Isom}^+ \mathbb{S}_n$. The isomorphism $\varphi : \mathrm{O}_{n+1} \to \operatorname{Isom} \mathbb{S}_n$ induces by restriction an isomorphism from SO_{n+1} into $\operatorname{Isom}^+ \mathbb{S}_n$, which will be denoted by the same letter.

Proposition 5.18. *If n is even, then $\operatorname{Isom}^+ \mathbb{S}_n$ is eccentric.*

Proof. Let $g \in \operatorname{Isom}^+ \mathbb{S}_n$ and let $f = \varphi^{-1}(g) \in \mathrm{SO}_{n+1}$. All the eigenvalues of f are of modulus 1 and their product is equal to 1. Moreover, if λ is an eigenvalue of f then $\overline{\lambda}$ too, with the same multiplicity. Since $n + 1$ is odd, we deduce that 1 is an eigenvalue of f, hence $\operatorname{Fix} f \neq \{\overrightarrow{0}\}$. It follows that $\operatorname{Isom}^+ \mathbb{S}_n$ is a GAF, but is obviously not a GAG. $\square$

It remains to treat the case n odd. For $n = 1$, $\operatorname{Isom}^+ \mathbb{S}_1$ is obviously fixating. We now treat the case $n \geqslant 5$, and we will end up with $n = 3$.

Proposition 5.19. *If n is odd and $n \geqslant 5$, then $\operatorname{Isom}^+ \mathbb{S}_n$ is nonfixating.*

Proof. We consider the space SO_{n+1} and we take again the example of the proof of Proposition 5.17, replacing 1 by I_2 and -1 by $-I_2$:

For $n = 5$, let $f, g, h \in \mathrm{SO}_6$ be the isometries of block diagonal matrices respectively $\operatorname{Mat} f = \operatorname{diag}(I_2, -I_2, -I_2)$, $\operatorname{Mat} g = \operatorname{diag}(-I_2, I_2, -I_2)$ and $\operatorname{Mat} h = \operatorname{diag}(-I_2, -I_2, I_2)$. We verify that $\{\mathbf{id}, f, g, h\}$ is isomorphic to the Klein group and eccentric.

For n odd, $n \geqslant 7$, we complete the matrices of f and g by $-I_2$ and the matrix of h by I_2, and the group $\{\mathbf{id}, f, g, h\}$ is still isomorphic to the Klein group and eccentric. $\square$

Proposition 5.20. *The group $\operatorname{Isom}^+ \mathbb{S}_3$ is fixating.*

Proof. By contradiction, let $G \leqslant \mathrm{SO}_4$ be an eccentric group. An element of $G \setminus \{\mathbf{id}\}$ has a plane (i.e. a subspace of dimension 2) of fixed points.

Step 1. One has $\operatorname{Fix} f \cap \operatorname{Fix} g \neq \{\overrightarrow{0}\}$ for all $f, g \in G$.

Let $\overrightarrow{u} \in \operatorname{Fix}(f^{-1}g) \setminus \{\overrightarrow{0}\}$ (which is nonempty, as G is a GAF). If $\overrightarrow{u} \in \operatorname{Fix} f \cap \operatorname{Fix} g$, we are done. Otherwise, we have $f(\overrightarrow{u}) = g(\overrightarrow{u}) \neq \overrightarrow{u}$. Then $\operatorname{Fix} f$ and $\operatorname{Fix} g$ are two planes included in the hyperplane (i.e. three dimensional subspace) $\operatorname{Med}(\overrightarrow{u}, f(\overrightarrow{u}))$, hence intersect each other.

Now set $f_0 \in G \setminus \{\mathbf{id}\}$. Since $\operatorname{Fix} G = \{\overrightarrow{0}\}$, there exists $g_0 \in G \setminus \{\mathbf{id}\}$ such that $\operatorname{Fix} f_0 \neq \operatorname{Fix} g_0$. From step 1, $\operatorname{Fix} f_0 \cap \operatorname{Fix} g_0$ is a straight line denoted by D, and $\operatorname{Fix} f_0 + \operatorname{Fix} g_0$ is a hyperplane denoted by H.

Step 2. For all $f \in G \setminus \{\mathbf{id}\}$, one has $\operatorname{Fix} f \subset H$.

Indeed, let $h_0 \in G$ be such that $\operatorname{Fix} h_0$ does not contain D. Such a h_0 exists since $\operatorname{Fix} G = \{\overrightarrow{0}\}$. We have $\operatorname{Fix} f_0 \neq \operatorname{Fix} g_0 \neq \operatorname{Fix} h_0$. From step 1, $\operatorname{Fix} f_0 \cap \operatorname{Fix} h_0$ is a straight line denoted by D', and $\operatorname{Fix} g_0 \cap \operatorname{Fix} h_0$ is a straight line denoted by D''. We have $D' \neq D''$ (otherwise $D' = D'' = D$, in contradiction with $D \not\subset \operatorname{Fix} h_0$) and $\operatorname{Fix} h_0$ is two dimensional, hence $\operatorname{Fix} h_0 = D' + D'' \subset H$. By the way, notice that the three straight lines D, D', and D' are not coplanar (otherwise we would have $\operatorname{Fix} f_0 = \operatorname{Fix} g_0 = \operatorname{Fix} h_0$), hence $H = D + D' + D''$, so that f_0, g_0 and h_0 play symmetrical roles. Now, let $f \in G \setminus \{\mathbf{id}\}$. Therefore $\operatorname{Fix} f$, which is a plane, cannot contain at the same time D, D' and D'' hence, from the above, $\operatorname{Fix} f$ is included in one of the three subspaces $\operatorname{Fix} f_0 + \operatorname{Fix} g_0$, $\operatorname{Fix} g_0 + \operatorname{Fix} h_0$ or $\operatorname{Fix} h_0 + \operatorname{Fix} f_0$, which in fact all three coincide with H.

Let Δ denote the line orthogonal to H: $\Delta = H^{\perp}$.

Step 3. One has $f(\Delta) = \Delta$ for all $f \in G$.

Otherwise, let $f \in G$ be such that $f(\Delta) \neq \Delta$ and let $g \in G \setminus \{\mathbf{id}\}$ be arbitrary. Let us show that $(\operatorname{Fix} g)^\perp = \Delta + f(\Delta)$. We have $\Delta \subset (\operatorname{Fix} h)^\perp$ for all $h \in G \setminus \{\mathbf{id}\}$ and, since f is an isometry, $f(\Delta) \subset (f(\operatorname{Fix} h))^\perp$. For $h = f^{-1}gf$, this gives $f(\Delta) \subset (f(\operatorname{Fix} h))^\perp = (\operatorname{Fix} g)^\perp$. Therefore $\Delta + f(\Delta) \subset (\operatorname{Fix} g)^\perp$, which have the same dimension, hence the equality. Since this holds for all $g \in G \setminus \{\mathbf{id}\}$, we get $\operatorname{Fix} G = (\Delta + f(\Delta))^\perp \neq \{\overrightarrow{0}\}$, a contradiction.

To sum up, we found a hyperplane H such that, for all $f \in G \setminus \{\mathbf{id}\}$, $\operatorname{Fix} f \subset H$ and $f(H^\perp) = H^\perp$. It follows that the restriction of f to $H^\perp$ is $-\mathbf{id}$, thus the only possible eigenvalues of f are 1 and -1, hence $f^2 = \mathbf{id}$ for all $f \in G$, so G is Abelian (we have $\mathbf{id} = f^2g^2 = (fg)^2$ hence, simplifying, $fg = gf$ for all $f, g \in G$).

Since G is Abelian, the elements of G are diagonalizable in a common basis, denoted by $\mathcal{B} = (\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}, \overrightarrow{\ell})$, with 1 and -1 as double eigenvalues (these are positive isometries of $\mathbb{R}^4$). Consider the endomorphisms f_1, f_2 and f_3 whose matrices in the basis $\mathcal{B}$ are respectively $\operatorname{diag}(1,1,-1,-1)$, $\operatorname{diag}(1,-1,1,-1)$, and $\operatorname{diag}(1,-1,-1,1)$. Thus G is a subgroup of $G_0 = \{\pm\mathbf{id}, \pm f_1, \pm f_2, \pm f_3\}$.

The list of the sixteen subgroups of G_0 splits into

▷ eleven GAG: $\{\mathbf{id}\}$, $\{\mathbf{id}, f_n\}$ with $1 \leqslant n \leqslant 3$, $\{\mathbf{id}, -f_n\}$ with $1 \leqslant n \leqslant 3$, $\{\mathbf{id}, f_1, f_2, f_3\}$ which fixes $\overrightarrow{i}$, $\{\mathbf{id}, f_1, -f_2, -f_3\}$ which fixes $\overrightarrow{j}$, $\{\mathbf{id}, -f_1, f_2, -f_3\}$ which fixes $\overrightarrow{k}$, $\{\mathbf{id}, -f_1, -f_2, f_3\}$ which fixes $\overrightarrow{\ell}$,

▷ and five containing $-\mathbf{id}$ hence not GAF: $\{\mathbf{id}, -\mathbf{id}\}$, $\{\mathbf{id}, f_n, -\mathbf{id}, -f_n\}$ with $1 \leqslant n \leqslant 3$, and G_0.

As a consequence G_0 contains no eccentric subgroup, a contradiction. □

5.4 The projective case

Usually, $\mathbb{R}\mathbf{P}_n$ is the set of vector lines of $\mathbb{R}^{n+1}$. In this article, we identify $\mathbb{R}\mathbf{P}_n$ with the quotient of $\mathbb{S}_n$ by the equivalence relation

$$x \sim y \Leftrightarrow x = y \quad \text{or} \quad x = -y.$$

For $x \in \mathbb{S}_n$, we write $\dot{x} = \{x, -x\}$ the corresponding class in $\mathbb{R}\mathbf{P}_n$. Given $\dot{x} = \{x, -x\}$ and $\dot{y} = \{y, -y\}$ in $\mathbb{R}\mathbf{P}_n$, the distance between $\dot{x}$ and $\dot{y}$ is then given by $d(\dot{x}, \dot{y}) = \arccos |\langle x \,|\, y\rangle|$, where $\langle x \,|\, y\rangle$ is the scalar product between x and y.

Given an isometry f of $\mathbb{S}_n$ and $x \in \mathbb{S}_n$, the class of $f(x)$ in $\mathbb{R}\mathbf{P}_n$ is the same as $f(-x) = -f(x)$, so we can define a function from $\mathbb{R}\mathbf{P}_n$ to $\mathbb{R}\mathbf{P}_n$, denoted by $\psi(f)$, which maps $\dot{x} = \{x, -x\}$ to the class of $f(x)$. It is known that the mapping

$$\psi : \operatorname{Isom} \mathbb{S}_n \to \operatorname{Isom} \mathbb{R}\mathbf{P}_n$$

defined in this manner is a surjective morphism, of kernel $\{\pm\mathbf{id}\}$; cf. [3], Chap.19. Thus the group $\operatorname{Isom} \mathbb{R}\mathbf{P}_n$ is the image of $\operatorname{Isom} \mathbb{S}_n$ by ψ, and similarly $\operatorname{Isom}^+ \mathbb{R}\mathbf{P}_n$ is the image of $\operatorname{Isom}^+ \mathbb{S}_n$ by ψ. When n is even, $-\mathbf{id} : \mathbb{S}_n \to \mathbb{S}_n$ reverses the orientation, so $\operatorname{Isom}^+ \mathbb{R}\mathbf{P}_n$ is equal to $\operatorname{Isom} \mathbb{R}\mathbf{P}_n$.

Theorem 5.21. *The group* $\operatorname{Isom} \mathbb{R}\mathbf{P}_n$ *is fixating if and only if* $n = 1$.
The group $\operatorname{Isom}^+ \mathbb{R}\mathbf{P}_n$ *is fixating if* $n = 1$ *and nonfixating if* n *is odd and greater than or equal to* 5.

Remarks.

1. We do not know whether $\operatorname{Isom}^+ \mathbb{R}\mathbf{P}_3$ is fixating or not.

2. Notice that the rotations of $\mathbb{R}\mathbf{P}_2$ have a single fixed point. Thus the rotation group $\mathbb{R}\mathbf{P}_2$ is another example of a nonfixating group, which is not Abelian and such that all elements have a single fixed point. This completes Example 3.3.

Proof. It is easy to check that $\operatorname{Isom}\mathbb{R}\mathbf{P}_1$ is fixating.

If n is even and $\dot{f} = \psi(f) \in \operatorname{Isom}\mathbb{R}\mathbf{P}_n$, then the matrix of f, seen as the vector isometry of $\mathbb{R}^{n+1}$, is odd-sized, so always admits 1 or -1 as eigenvalue, hence there exists $x \in \mathbb{S}_n$ such that $f(x) \in \{x, -x\}$. Thus any isometry of $\mathbb{R}\mathbf{P}_n$ has at least one fixed point $\dot{x}$, but no point of $\mathbb{R}\mathbf{P}_n$ can be fixed by all the elements of $\operatorname{Isom}\mathbb{R}\mathbf{P}_n$. The group $\operatorname{Isom}\mathbb{R}\mathbf{P}_n$ is therefore eccentric itself, hence nonfixating.

We now study the case n odd. For $n = 5$, set

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and consider the isometries f, g, h of $\mathbb{S}_5$ whose matrices are the following block diagonal matrices

$$\operatorname{Mat} f = \operatorname{diag}(I, R, R), \ \operatorname{Mat} g = \operatorname{diag}(R, I, R) \ \text{ and } \ \operatorname{Mat} h = \operatorname{diag}(R, R, I).$$

These are positive isometries. Let G_5 denote the subgroup of $\operatorname{Isom}^+\mathbb{S}_5$ generated by f, g and h. One finds for G_5 the following group of order 32:

$$G_5 = \{\operatorname{diag}(\pm I, \pm I, \pm I), \operatorname{diag}(\pm I, \pm R, \pm R), \operatorname{diag}(\pm R, \pm I, \pm R), \operatorname{diag}(\pm R, \pm R, \pm I)\}$$

and ones checks that its image $\psi(G_5)$ in $\operatorname{Isom}^+\mathbb{R}\mathbf{P}_5$ is eccentric. The verification is somewhat tedious but without difficulty. It follows that $\operatorname{Isom}^+\mathbb{R}\mathbf{P}_5$ is nonfixating, hence $\operatorname{Isom}\mathbb{R}\mathbf{P}_5$ neither.

For n odd, $n \geqslant 7$, we consider the positive isometries f, g, h of $\mathbb{S}_n$ whose matrices are the block diagonal matrices: $\operatorname{Mat} f = \operatorname{diag}(I, R, \ldots, R)$, $\operatorname{Mat} g = \operatorname{diag}(R, I, R, \ldots, R)$ and $\operatorname{Mat} h = \operatorname{diag}(R, R, I, R, \ldots, R)$. We verify similarly that the image $\psi(G_n)$ of the subgroup G_n of $\operatorname{Isom}^+\mathbb{S}_n$ generated by f, g and h is eccentric, so neither $\operatorname{Isom}^+\mathbb{R}\mathbf{P}_n$ nor $\operatorname{Isom}\mathbb{R}\mathbf{P}_n$ are fixating.

It remains to treat the case $n = 3$. Let $f, g \in \operatorname{Isom}\mathbb{S}_3$ of matrices

$$\operatorname{Mat} f = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \operatorname{Mat} g = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For simplicity, we write f and g as signed permutations: $f = (1\,2\,3\,4)$ and $g = (1\,2\ -3\ -4)$. The computation gives $fg = (1\,3)(2\ -4) = -gf$ and $f^2 = (1\,3)(2\,4) = -g^2$. Therefore the group generated by $\pm f$ and $\pm g$ is

$$G_3 = \{\pm\mathbf{id}, \pm f, \pm f^2, \pm f^3, \pm g, \pm fg, \pm f^2 g, \pm f^3 g\}.$$

By looking one by one all the elements of G_3, we easily check that $\psi(G_3)$ is eccentric in $\operatorname{Isom}\mathbb{R}\mathbf{P}_3$, hence $\operatorname{Isom}\mathbb{R}\mathbf{P}_3$ is nonfixating. We did not find any eccentric subgroup of $\operatorname{Isom}^+\mathbb{R}\mathbf{P}_3$, nor were we able to adapt to the projective case the proof that $\operatorname{Isom}^+\mathbb{S}_3$ is fixating (Proposition 5.20). □

6 Group acting on discrete sets

6.1 Symmetric groups

The aim of this section is the following result.

Proposition 6.1.

a. *The symmetric group $\mathcal{S}_n$ acting on $\{1, \dots, n\}$ is fixating if and only if $n \leqslant 4$.*

b. *The alternating group $\mathcal{A}_n$ of even permutations acting on $\{1, \dots, n\}$ is fixating if and only if $n \leqslant 4$.*

Proof. **a.** We split the proof for the symmetric group into five steps. We use the decomposition in cycles with disjoint supports. A cycle of order n is called a *n-cycle*.

Step 1: $\mathcal{S}_n$ is fixating when $n \leqslant 3$. For $n = 1$ and $n = 2$, there is nothing to do since there is no nontrivial subgroup. For $n = 3$, the nontrivial subgroups of $\mathcal{S}_3$ are $\mathcal{A}_3 = \langle(123)\rangle$, which is not a GAF, and the three subgroups of order 2 generated by each transposition, which are GAG, hence $\mathcal{S}_3$ has no eccentric subgroup.

Step 2: $\mathcal{S}_4$ is fixating. Let $G \leqslant \mathcal{S}_4$ be a GAF. So G contains neither double-transposition nor 4-cycle (since these permutations have no fixed point), so $G \setminus \{\mathbf{id}\}$ contains only transpositions and/or 3-cycles. Two cases occur.

- ▷ If G contains no 3-cycle, then $G = \{\mathbf{id}, \tau\}$ where τ is a transposition, and we are done.
- ▷ If G contains a 3-cycle γ, say $\gamma = (123)$, let us show by contradiction that $\operatorname{Fix} G = \{4\}$. Otherwise there exists $g \in G$ such that $g(4) \neq 4$, say $g(4) = 1$. If $g = (14)$, then $g\gamma = (1234)$, which has no fixed point. If g is a 3-cycle, say $g = (124)$, then $g\gamma = (14)(23)$, which has no fixed point either.

Step 3: $\mathcal{S}_5$ is nonfixating. Let $G = \langle f, g\rangle$ with $f = (123)$ and $g = (12)(45)$. We verify that $G = \{\mathbf{id}, f, f^2, g, h, k\}$ with $h = fg = (13)(45)$ and $k = gf = (23)(45)$, and that G is eccentric.

Remark. We can see G as $(\mathcal{S}_3 \times \mathcal{S}_2)^+$, the set of even permutations acting separately on $\{1, 2, 3\}$ and on $\{4, 5\}$. We also can interpret G as the group of isometries of the "double tetrahedron", i.e. the hexahedron obtained by gluing two regular isometric tetrahedra on one of their faces.

Step 4: $\mathcal{S}_6$ is nonfixating. Let $G = \langle f, g\rangle$ with $f = (12)(34)$ and $g = (12)(56)$. We verify that $G = \{\mathbf{id}, f, g, h\}$ with $h = (34)(56)$, and that G is eccentric.

Remark. Here too, G can be interpreted as a set of even permutations: those acting separately on $\{1, 2\}$, on $\{3, 4\}$, and on $\{5, 6\}$, and also as a group of isometries: the half-turns of axes the coordinate axes in $\mathbb{R}^3$.

Step 5: $\mathcal{S}_n$ is nonfixating when $n \geqslant 7$. Let $G = \langle f, g\rangle$ with $f = (123)(6\dots n)$ and $g = (12)(45)(6\dots n)$. We check that G is eccentric.

b. For $n \leqslant 4$, $\mathcal{A}_n$ is fixating, as a subgroup of a fixating group. For $n = 5$ and $n = 6$, both eccentric subgroups built in steps 3 and 4 are precisely subgroups of $\mathcal{A}_5$, resp. $\mathcal{A}_6$, which shows that $\mathcal{A}_5$ and $\mathcal{A}_6$ are nonfixating. For $n \geqslant 8$, n even, the cycle $(6\dots n)$ is even, so the eccentric subgroup of step 5 is still in $\mathcal{A}_n$. For $n \geqslant 9$, n odd, the cycle $(7\dots n)$ is even, so we complete as in step 5, but with the group of step 4 : we choose $G = \langle f, g\rangle$ with $f = (12)(34)(7\dots n)$ and $g = (12)(56)(7\dots n)$. We verify that G is an eccentric subgroup of $\mathcal{A}_n$. We leave to the reader the most interesting case $\mathcal{A}_7$, see Exercise 7.15. □

Remark. Proposition 6.1 shows that the groups $\mathcal{S}_n$ and $\mathcal{A}_n$ are not fixating when $n > 4$. Some of their subgroups might be fixating. For example, when $\mathbb{F}_q$ is a finite field with q elements and d is an integer, the group $G = \mathrm{GL}(d, \mathbb{F}_q)$ acts naturally on $X = \mathbb{F}_q^d \setminus \{0\}$ and can be identified to a subgroup of the group $\mathcal{S}_X$ of permutations of X. We do not know whether G is fixating except in trivial cases and the case $G = \mathrm{GL}(3, \mathbb{F}_2)$ which is not fixating, see exercise 7.16.

6.2 Isometries of $\mathbb{Z}^n$

Let $n \geqslant 1$ be an integer. We equip $\mathbb{Z}^n$ with the Euclidean norm, denoted $\| \ \|$. Any arbitrary $f \in \operatorname{Isom}\mathbb{Z}^n$ can be extended into an isometry of $\mathbb{R}^n$, denoted by the same letter, cf. Exercise 7.14 and its solution.

Theorem 6.2. *The group* $\operatorname{Isom}\mathbb{Z}^n$ *is fixating.*

Proof. By contradiction, let G be an eccentric subgroup of $\operatorname{Isom}\mathbb{Z}^n$ with n minimal.

Step 1: G is finite.
Let $\mathcal{C}_n = \{x = (x_1, \dots, x_n) \ ; \ x_i = -1, 0 \text{ ou } 1\}$ be the unit hypercube in $\mathbb{Z}^n$. Since the only translation of G is $\mathbf{id}$, the morphism from G to $\operatorname{Isom}\mathcal{C}_n$, which maps any isometry f to its linear part $\overrightarrow{f}$, is injective. Now $\operatorname{Isom}\mathcal{C}_n$ is finite, so G is finite, too.

Let N denote the cardinal of G and ω the centroid of the orbit of $\overrightarrow{0}$ by G, i.e. $\omega = \frac{1}{N}\sum_{f\in G} f(\overrightarrow{0})$.

Step 2: All coordinates of ω are congruent to $\frac{1}{2}$ modulo 1.
Indeed, ω is fixed by all the elements of G. Since G is eccentric, $\omega \notin \mathbb{Z}^n$. Let

$$\Omega = \{x \in \mathbb{Z}^n \ ; \ \|\omega - x\| = d(\omega, \mathbb{Z}^n)\}.$$

Let $I = \big\{i \in \{1, \dots, n\} \ ; \ \exists x, y \in \Omega, \ x_i \neq y_i\big\}$. We have $\omega_i \equiv \frac{1}{2} \bmod 1 \Leftrightarrow i \in I$. Indeed, if $x, y \in \Omega$ are such that $x_i \neq y_i$, then $|x_i - y_i| = 1$ and $|x_i - \omega_i| = |y_i - \omega_i| = \frac{1}{2}$. Conversely, if $\omega_i \equiv \frac{1}{2} \bmod 1$ and $x \in \Omega$, then $x_i = \omega_i \pm \frac{1}{2}$ and the point y with the same coordinates as x except the i-th equal to $\omega_i \mp \frac{1}{2}$ is also in Ω. In summary, we have

$$\Omega = \big\{(x_1, \dots, x_n) \in \mathbb{Z}^n \ ; \ x_i = \omega_i \pm \tfrac{1}{2} \text{ if } i \in I \text{ and } x_i = [\omega_i] \text{ if } i \notin I\big\},$$

Let $[\omega_i]$ be the integer closest to ω_i (unique since $\omega_i \not\equiv \frac{1}{2} \bmod 1$ when $i \notin I$). Let $E = \mathbb{Z}^n \cap \operatorname{Aff}\Omega$. It is a " lattice" isometric to $\mathbb{Z}^k$, where k is the cardinal of I. Precisely, let us set $E_i = \mathbb{Z}$ if $i \in I$ and $E_i = \{[\omega_i]\}$ otherwise; then we have $E = E_1 \times \cdots \times E_n$. For all $f \in G$ we have $f(\Omega) = \Omega$, so $f(\operatorname{Aff}\Omega) = \operatorname{Aff}\Omega$, and in addition $f(\mathbb{Z}^n) = \mathbb{Z}^n$, so $f(E) = E$. This allows to define G_E, the set of the restrictions to E of the elements in G. These are isometries of E. Since G is not a GAG, G_E is not a GAG either. Moreover, for all $f \in G$ and all $x \in \operatorname{Fix} f$, the orthogonal projection $\pi_E x$ is also in $\operatorname{Fix} f$, so that G_E is a GAF. By minimality of n, we deduce that $k = n$, hence $I = \{1, \dots, n\}$.

Step 3: One is reduced to $\omega = \overrightarrow{0}$ and one changes $\mathbb{Z}^n$ into $(2\mathbb{Z}+1)^n$.
Let $\varphi : \mathbb{Z}^n \to (2\mathbb{Z}+1)^n$ be the map defined by $x \mapsto 2x - 2\omega$ and let $f \in G$. Then the isometry $\widetilde{f} = \varphi f \varphi^{-1}$ fixes $\overrightarrow{0}$ and maps $(2\mathbb{Z}+1)^n$ into $(2\mathbb{Z}+1)^n$. In addition, a small calculation shows that $\widetilde{f} = \overrightarrow{f}$, the linear part associated with f, so φ also globally fixes the lattice $\mathbb{Z}^n$, and the entries of its matrix are only 0, 1 or -1.

To avoid multiple notations, we still denote by G the conjugate of G by φ. For each $f \in G$, the matrix of f is thus a matrix of a signed permutation: $\operatorname{Mat} f = (a_{i,j})_{1\leqslant i,j\leqslant n}$ with, on each row and each column, one and only one nonzero entry $a_{i,j}$, equal to 1 or -1.

Step 4: The diagonal coefficients of $\operatorname{Mat} f$ *are never* -1.
Since G is a GAF, $\operatorname{Fix} f$ is a nonempty subset of $(2\mathbb{Z}+1)^n$. Let $x = (x_1, \dots, x_n) \in \operatorname{Fix} f$. It is an element of $(2\mathbb{Z}+1)^n$, hence $x_i \neq 0$ for all $i = 1, \dots, n$. If $a_{i,i} \neq 0$, the i-th coordinate of the equality $f(x) = x$ gives $a_{i,i}x_i = x_i$, hence $a_{i,i} = 1$.

Denote by $\sigma : G \to \mathcal{S}_n, f \mapsto \sigma_f$ the function which maps f of matrix $(a_{i,j})_{1\leqslant i,j\leqslant n}$ to the permutation matrix $(|a_{i,j}|)_{1\leqslant i,j\leqslant 1}$. This is clearly a group homomorphism, which is injective by Step 4 (its kernel is reduced to $\mathbf{id}$).

Step 5: Towards the construction of a global fixed point.
Let $(e_1, \dots, e_n)$ denote the canonical basis of $\mathbb{R}^n$ and consider the two relations on $\{1, \dots, n\}$:
$i \sim j$ if there exists $f \in G$ such that $f(e_i) \in \{e_j, -e_j\}$,
$i \approx j$ if there exists $f \in G$ such that $f(e_i) = e_j$.

It is easy to check that they are equivalence relations. Let p be the number of classes for the relation $\sim$. For each $k = 1, \dots, p$, the class C_k for $\sim$ is partitioned into two classes for $\approx$ (possibly, one of the classes is empty). Let C_k^+ be one of these classes (arbitrarily chosen) and put $C_k^- = C_k \setminus C_k^+$. Denote

$$C^+ = C_1^+ \cup \dots \cup C_p^+ \quad \text{and} \quad C^- = C_1^- \cup \dots \cup C_p^-.$$

According to Step 4, given $i \sim j$, the equality $f(e_i) = e_j$ is not possible for one $f \in G$ and $g(e_i) = -e_j$ for another, so we have $i \approx \sigma_f(i) \Leftrightarrow f(e_i) = e_{\sigma_f(i)}$.

Let $f \in G$ be fixed and denote

$$\begin{aligned} C_f^{++} &= C^+ \cap \sigma_f^{-1}(C^+), \quad C_f^{+-} = C^+ \cap \sigma_f^{-1}(C^-), \\ C_f^{-+} &= C^- \cap \sigma_f^{-1}(C^+), \quad C_f^{--} = C^- \cap \sigma_f^{-1}(C^-). \end{aligned}$$

Thus we have $f(e_i) = e_{\sigma_f(i)}$ when $i \in C_f^{++} \cup C_f^{--}$ and $f(e_i) = -e_{\sigma_f(i)}$ when $i \in C_f^{+-} \cup C_f^{-+}$. We also have

$$C^+ = C_f^{++} \cup C_f^{+-}, \quad C^- = C_f^{-+} \cup C_f^{--}$$

and

$$\sigma_f(i) \in C^+ \Leftrightarrow i \in C_f^{++} \cup C_f^{-+}, \quad \sigma_f(i) \in C^- \Leftrightarrow i \in C_f^{+-} \cup C_f^{--}.$$

Step 6: The point $x = (x_1, \dots, x_n)$ defined by $x_i = 1$ if $i \in C^+$ and $x_i = -1$ if $i \in C^-$ is fixed by all elements of G.
Indeed, for any $f \in G$, we have

$$\begin{aligned} f(x) &= f\Big(\sum_{i \in C^+} e_i\Big) - f\Big(\sum_{i \in C^-} e_i\Big) \\ &= \Big(\sum_{i \in C_f^{++}} e_{\sigma_f(i)} - \sum_{i \in C_f^{+-}} e_{\sigma_f(i)}\Big) - \Big(\sum_{i \in C_f^{--}} e_{\sigma_f(i)} - \sum_{i \in C_f^{-+}} e_{\sigma_f(i)}\Big) \\ &= \sum_{\sigma_f(i) \in C^+} e_{\sigma_f(i)} - \sum_{\sigma_f(i) \in C^-} e_{\sigma_f(i)} \\ &= \sum_{j \in C^+} e_j - \sum_{j \in C^-} e_j = x. \end{aligned}$$

So that, $\operatorname{Fix} G$ would be nonempty, a contradiction. □

6.3 Isometries of trees

All the results of this section are taken from J.-P. Serre's book [20]. A *combinatorial tree* X is a simple undirected graph, connected and without cycles. The following fundamental property is easy to check: Given two vertices P and Q, there exists an unique injective path joining them. We denote by $[P, Q]$ this path and by $d(P, Q)$ its length, i.e. the number of its edges. The map $d : X \times X \to \mathbb{N}$ is the *combinatorial distance* on X and (X, d) is a discrete metric space. Two vertices are joined by an edge if and only if their distance is 1. We recall that an isometry of X is a bijection of X that preserves the distances.

Lemma 6.3. *Let s be an isometry with at least one fixed point. Then, for every vertex x of X, the distance $d(x, s(x))$ is even and the midpoint z of the path $[x, s(x)]$ is a fixed point of s. Moreover, z is the unique element of* $\operatorname{Fix} s$ *such that $d(x, z) = d(x, \operatorname{Fix} s)$.*

Proof. The uniqueness of the geodesic path between two vertices, and the fact that the image of an injective path by the isometry s is an injective path with the same length, ensure that $\operatorname{Fix} s$ is connected, hence is a subtree of X.

If $x \in \operatorname{Fix} s$ then the statement is clear. Suppose that $x \notin \operatorname{Fix} s$. Then there exists $b \in \operatorname{Fix} s$ such that $n = d(x, \operatorname{Fix} s) = d(x, b) \geqslant 1$. If b were not unique, one could construct a nontrivial cycle thanks to the connectivity of $\operatorname{Fix} s$. Let $[x_0, x_1, x_2, \ldots, x_n]$ be the geodesic path joining $b = x_0$ to $x = x_n$. By definition of b, $s(x_i) \neq x_i$ for all $i > 0$. The two paths $[x, b] = [x_n, x_{n-1}, \ldots, x_0]$ and $[b, s(x)] = [x_0, s(x_1), s(x_2), \ldots, s(x_n)]$ have no common vertex except $x_0 = b$. Indeed, if $x_j = s(x_i)$ with $i \neq j$, one could construct a path of length strictly smaller than n, joining b either to x or to $s(x)$. It follows that the geodesic path $[x, s(x)]$ is the concatenation of the geodesic paths $[x, b] = [x_n, x_{n-1}, \ldots, x_0]$ and $[b, s(x)] = [x_0, s(x_1), s(x_2), \ldots, s(x_n)]$. Therefore, the distance $d(x, s(x))$ is even and b is the midpoint of $[x, s(x)]$. □

Theorem 6.4. [20] *If G is a finitely generated group of isometries of X which is a* GAF, *then G is a* GAG. *Moreover,* $\operatorname{Fix} G$ *is a nonempty subtree of X.*

Proof. We proceed by induction on the number of generators of G. Suppose G is generated by an isometry s that fixes a vertex, and a subgroup G_0 having a global fixed point x_0. If $s(x_0) \neq x_0$ then, according to Lemma 6.3, the midpoint z of the geodesic path $[x_0, s(x_0)]$ is a fixed point of s. Similarly, for all $t \in G_0$, st has a fixed point and $[x_0, s(x_0)] = [x_0, s(t(x_0))]$. Therefore, z is fixed by all the isometries belonging to the set sG_0. Since s and sG_0 generate G, the point z is fixed by G. □

Remark. The assumption " G is finitely generated" is necessary. Indeed, let $[x_0, x_1, \ldots]$ be an infinite geodesic path in a tree X. Let k be an integer and put

$$N(k) = \{s \in \operatorname{Isom} X \ ; \ \forall \ell \geqslant k, \ s(x_\ell) = x_\ell\}.$$

The sequence $(N(k))_{k \in \mathbb{N}}$ is an nondecreasing sequence of subgroups of $\operatorname{Isom} X$ whose intersection is a subgroup of G. Any element s in G admits fixed points but, in general, there is no global fixed point (except in some special cases where the tree is filiform without symmetry). Especially, when X is a homogeneous tree of degree $\geqslant 3$, the group G has no global fixed point and so $\operatorname{Isom} X$ is nonfixating. In fact, the group G admits a kind of *fixed point at infinity*: the equivalence class of geodesic paths ending as $[x_0, x_1, \ldots]$.

We now give an explicit condition to obtain a GAF operating on a tree.

Proposition 6.5. *Let f and g be two isometries of a tree. Assume that f, g and $h = fg$ have fixed points. Then any element of the group generated by f and g has fixed points.*

Proof. It is enough to show that $\operatorname{Fix} f$ meets $\operatorname{Fix} g$. If $\operatorname{Fix} f \cap \operatorname{Fix} g = \emptyset$, let $[P, Q]$ be the geodesic joining $\operatorname{Fix} f$ to $\operatorname{Fix} g$. According to Lemma 6.3, P is the midpoint of the geodesic $[Q, f(Q)]$. Also $f(Q) = f(g(Q)) = h(Q)$ and, as $\operatorname{Fix} h$ is nonempty, Lemma 6.3 also implies P to be fixed by h. We deduce $fg(P) = P$, thus $g(P) = f^{-1}(P) = P$, contradicting $P \notin \operatorname{Fix} g$. □

Bounded orbits in a tree

The median inequality (1) almost holds for combinatorial trees, but an edge has no midpoint! This is why isometries whose fixed points should be midpoints of edges play a special role.

Lemma 6.6. *If X is a finite tree, then there exists either a vertex fixed by every element of* $\operatorname{Isom} X$, *or an edge stable by every element of* $\operatorname{Isom} X$.

Proof. We proceed by induction on the number of vertices. If there are one or two vertices, the result holds. If there are strictly more than two vertices, the set of the vertices of X having at least two neighbors is a nonempty subtree X', stable by the elements of $\operatorname{Isom} X$. Since there is at least one vertex that has a single neighbor in a finite tree, the induction assumption can be applied to X'. □

Definition 6.7. An isometry g of X is called an *inversion* if there exists an edge $\{a, b\}$ such that $g(a) = b$ and $g(b) = a$.

Proposition 6.8. *Let X be a tree whose all vertices have a finite degree. Let G be a subgroup of* $\operatorname{Isom} X$ *without inversion. If G has a bounded orbit, then G has a global fixed point.*

Proof. Suppose G has a bounded orbit Δ. Consider the set $T(\Delta)$ of vertices in X that are in a geodesic joining two vertices of Δ. Since Δ is finite, $T(\Delta)$ is finite, too. In addition, $T(\Delta)$ is the smallest subtree containing Δ, the *convex hull* of the orbit. Since an isometry sends a geodesic segment on a geodesic segment, the subtree $T(\Delta)$ is stable by the elements of $\operatorname{Isom} X$. Therefore, according to Lemma 6.6 applied to $T(\Delta)$, there is either a vertex or an edge invariant by all elements of G. If an edge $\{a, b\}$ is stable by G, then necessarily G fix a and b, because G does not contain an inversion. □

The following result is an easy consequence.

Corollary 6.9. *An isometry of finite order of a tree which is not an inversion has a fixed point.*

Application. Consider the Schwartz group G defined by two generators a and b related by the relations $a^A = b^B = (ab)^C = 1$, where A, B and C are integers greater than or equal to 2. Any action by isometries without inversion of G on a tree X has a global fixed point. Indeed, according to Proposition 6.8 about finite orbits, each isometry of X determined actions of a, b and ab has fixed points. Then, thanks to Proposition 6.5, we conclude that the group generated by a and b has a global fixed point.

By a similar argument, Serre proves that the group $\mathrm{SL}(3, \mathbb{Z})$ has the same property: Each action by isometries of G on a tree has a global fixed point.

Generalization. One can define a notion of Λ-tree where Λ is a totally ordered Abelian group. Then Theorem 6.4 still holds [15].

6.4 Questions about isometries in finite graphs

Let $X = (S, A)$ be a simple unoriented connected graph. The set S of its vertices is equipped with the distance d defined by the minimum number of edges joining vertices. Several simple and natural questions arise in this context:

1. Which are the finite graphs whose isometry group is fixating?

2. Which are the finite groups admitting a generating system defining a Cayley graph whose isometry group is fixating?

3. Find infinite families of graphs whose isometry group is fixating.

The first two questions are ambitious and probably difficult. On the other hand, the third question admits simple partial answers.

Consider the complete graph K_n on n vertices. Its isometry group is the symmetric group $\mathcal{S}_n$. Therefore, according to Proposition 6.1, it is fixating if and only if $n \leqslant 4$.

Let $\mathcal{C}_n$ be the graph associated with the n-dimensional hypercube $\{0, 1\}^n$ of $\mathbb{R}^n$. The set of vertices of $\mathcal{C}_n$ is $\{0, 1\}^n$ and the edges are the pairs of vertices of which exactly one coordinate differs.

Proposition 6.10. *The group* $\operatorname{Isom}\mathcal{C}_n$ *is fixating for all integer* $n \geqslant 1$.

Proof. The proof is analogous, in simpler form, to that of Theorem 6.2 on $\mathbb{Z}^n$. We prove that an isometry of $\mathcal{C}_n$ extends in a unique way to $\mathbb{R}^n$ in an isometry for the Euclidean norm, cf. Exercise 7.14. The center of the cube $\omega = (\frac{1}{2}, \dots, \frac{1}{2})$ is fixed by all the isometries of the cube. Conjugating by the map $x \mapsto 2x - 2\omega$, we reduce to the case where the isometries are linear maps whose matrices are matrices of signed permutations. We find that the matrices of the isometries of a subgroup GAF never have -1 on the diagonal, which implies the existence of a global fixed point as in Steps 5 and 6 of the proof of Theorem 6.2. □

6.5 A result about infinite graphs

Let $X = (S, A)$ be a simple undirected connected graph whose edges are colored. The color of an edge is given by a map with values in a color set $\mathcal{C}$, defined on the set A of edges. Recall that a cycle is called *simple* if no edge occurs more than once, and *elementary* if no vertex occurs more than once except the beginning and the end. It is easy to see that an elementary cycle of length at least 3 is simple and that a cycle, simple or not, always contains at least one elementary cycle. We make the following assumptions:

- The edges of an elementary cycle of the graph all have the same color,
- For each $c \in \mathcal{C}$, the connected components of the partial graph X_c, obtained by keeping only the edges of color c, are complete graphs.

For every $c \in \mathcal{C}$, we will call any connected component of the partial graph X_c *a cell of color* c. Each edge of X belongs to a single cell.

Note that the Cayley graph of the free product of two groups G_1 and G_2 satisfies the above assumptions if we choose $(G_1 \cup G_2) \setminus \{e\}$ for set of generators and if one colors the edges $\{w, wg\}$, $g \in G_1$, in blue and the others in red.

Here is a generalization of Theorem 6.4 on finitely generated groups of isometries of trees.

Theorem 6.11. *Suppose that each cell of the graph* X *has at most* 4 *vertices. Let* G *be a finitely generated subgroup of* $\operatorname{Isom} X$. *If* G *is a* GAF *then* G *is a* GAG.

Observe that the hypothesis about the cardinality of the cells is necessary: If X is a complete graph with at least five vertices then, according to Proposition 6.1, $\operatorname{Isom} X$ contains an eccentric subgroup. It is the same for the Cayley graph of the free product of two groups of which one at least has more than five elements. The proof of Theorem 6.11 needs some preliminary lemmas.

Lemma 6.12. *Given two vertices* x *and* y *of the graph* X, *there is a unique path joining them such that two consecutive edges on this path never have the same color. Moreover, this path is the only geodesic from* x *to* y. *Therefore, a path is a geodesic if, and only if, two consecutive edges never have the same color.*

Proof. Let x and y be in S. Since X is connected, there is a path joining x to y, and therefore at least one geodesic. Since the cells of the graph are complete graphs, this geodesic cannot have two consecutive edges of the same color, hence the existence.

For the uniqueness, let us first notice that all the vertices of a path verifying the color change property of the edges, are distinct. Indeed, if two vertices of the path coincided, then an elementary cycle could be extracted. By assumption on X, all the edges of this cycle have the same color, contradicting the property of color change.

If two distinct paths $[a_0 = x, \dots, a_m = y]$ and $[b_0 = x, \dots, b_n = y]$ join x to y and satisfy the property of color change, then there is an integer $i \geqslant 0$ such that $a_i = b_i$ and $a_{i+1} \neq$

b_{i+1}. Consider the first vertex a_j in the path $[a_{i+1}, \dots, a_m]$ which also belongs to the path $[b_{i+1}, \dots, b_n]$. So we have $a_j = b_k$ for some $k > i$. The integer k is chosen minimal. By choice of j and k, the vertices of the path $[a_i, a_{i+1}, \dots, a_j = b_k, b_{k-1}, \dots, b_i = a_i]$ are all distinct, except the two ends. This cycle has at least three edges (otherwise $a_{i+1} = b_{i+1}$), it is elementary and has at least two colors, a contradiction. □

Lemma 6.13. *Let s be an isometry of X having at least one fixed point and let x be a vertex of X that is not a fixed point of s. Let F be the set of points $y \in \operatorname{Fix} s$ such that $d(x, y) = d(x, \operatorname{Fix} s)$.*

a. *Then F is included in a cell whose color is the one from the last edge of the geodesic going from x to any element of F.*

b. *Moreover:*

- **(i)** *If $d(x, s(x))$ is even, then the midpoint z of the geodesic joining x to $s(x)$ belongs to F.*
- **(ii)** *If $d(x, s(x))$ is odd, then F is included in the cell Y containing the middle edge $[a, b]$ of the geodesic going from x to $s(x)$. In addition, $s(Y) = Y$ and $s(a) = b$.*

Proof. **a**. Let u and v be two points in F. Consider the geodesics $[u_0 = x, \dots, u_n = u]$ and $[v_0 = x, \dots, v_n = v]$ (with $n = d(x, \operatorname{Fix} s)$). Since s is an isometry that fixes u and v, it fixes each vertex of the geodesic $[w_0 = u, \dots, w_m = v]$ joining u to v. Let i be the smallest integer such that $u_i = v_i$ and $u_{i+1} \neq v_{i+1}$. We verify, as in the proof of Lemma 6.12, that the three branches of geodesics $[u_i, \dots, u_n = u]$, $[u, \dots, v]$ and $[v_n, \dots, v_i = u_i]$ form an elementary cycle. Therefore, all the edges of this cycle have the same color c and the vertices u and v are joined by an edge of color c. We conclude by noticing that c is the color of the edge $[u_{n-1}, u_n]$.

b. Let u be a point in F and let $[u_0 = x, \dots, u_n = u]$ be the geodesic joining x to u. According to item **a**, F is included in a cell Y whose color c is that of the edge $[u_{n-1}, u_n]$. Consider the image $[s(x), \dots, s(u_n) = u]$ of this geodesic by s. Let c' the color of the edge $[s(u_{n-1}), u]$.

(i) If $c \neq c'$ then $[u_0, \dots, u_n = s(u_n), s(u_{n-1}), \dots, s(u_0) = s(x)]$ is a geodesic since the colors of two consecutive edges of this path are never the same. Hence it is the geodesic joining x to $s(x)$. The length of this geodesic is $2n$ and its midpoint is $u \in F$.

(ii) If $c = c'$ then $s(u_{n-1})$ is a vertex of Y which is complete, so $[u_{n-1}, s(u_{n-1})]$ is also an edge Y. Two consecutive edges of the path $[u_0, \dots, u_{n-1}, s(u_{n-1}), \dots, s(u_0)]$ never have the same color. Hence, this path is the geodesic joining x to $s(x)$. By construction, the length of this geodesic is odd and its middle edge $[u_{n-1}, s(u_{n-1})]$ is of color c. It remains to prove that $s(Y) = Y$. Let y be a vertex of Y. The image by s of the triangle $u_{n-1}uy$ is a triangle that contains the edge $[s(u_{n-1}), u]$ which is an edge of Y. Therefore the image of this triangle is a triangle of Y, hence $s(y)$ is a vertex of Y. □

Proof of Theorem 6.11. We proceed by induction on the number of generators of G. Suppose that G is generated by an isometry s which fixes at least one vertex and a subgroup G_0 with a global fixed point x_0. If $s(x_0) = x_0$, we have finished. Otherwise, consider the geodesic going from x_0 to $s(x_0)$. Observe that it is also the geodesic going from x_0 to $st(x_0) = s(x_0)$ for any $t \in G_0$.

If this geodesic has an even length then, according to Lemma 6.13, its midpoint z is a fixed point of s but also of all st, $t \in G_0$. Since s is injective, $s(t(z)) = z = s(z)$ implies $t(z) = z$, so z is a fixed point of t. Thus z is a global fixed point of G.

If this geodesic has an odd length, then again according to Lemma 6.13, the cell Y containing the middle edge of this geodesic is stable by all st, $t \in G_0$. It is therefore stable by G. In addition, the cell Y contains fixed points of st. Therefore, the group H of restrictions to Y of elements of G is a GAF. Now $\operatorname{Isom} Y$ is the group of permutations of the vertices of Y and by hypothesis the cell Y has at most 4 vertices so, according to Proposition 6.1, $\operatorname{Isom} Y$ is fixating. Therefore H is a GAG, hence G is a GAG, too. □

7 Exercises

Exercise 7.1. Show that the group $\operatorname{Isom} \mathbb{R}^2$ is fixating.

Exercise 7.2. A group is called *superfixating* if, for any set X and any morphism $\rho : G \to \operatorname{Bij} X$, the couple $(X, \rho(G))$ is fixating. By considering the action of a group on all of its nontrivial parts, show that a group is superfixating if and only if it is cyclic (finite or not).

Exercise 7.3. Show that the additive group $\mathbb{Q}$ is *finitely superfixating* in the following sense: If X is a finite set and $\rho : \mathbb{Q} \to \operatorname{Bij} X$ a morphism, then $(X, \rho(\mathbb{Q}))$ is fixating.

Exercise 7.4. Prove that, in a metric space verifying the median inequality (1), the point m is unique and satisfies $d(x, m) = d(y, m) = \frac{1}{2}\, d(x, y)$.

Exercise 7.5. Let $(E, \| \ \|)$ be a normed vector space verifying the median inequality (1).

a. Prove that E satisfies the so-called *parallelogram identity*

$$\forall x, y \in E \qquad \|x + y\|^2 + \|x - y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big). \tag{5}$$

b. Deduce that E is a pre-Hilbert space (a result of M. Frechet, P. Jordan and J. von Neuman [9, 11]).

Exercise 7.6. According to the comparison theorem of Rauch [8], the classical cosine law becomes an inequality in hyperbolic trigonometry: In a hyperbolic triangle of side lengths a, b, c and angle γ opposite to the side of length c, we have

$$a^2 + b^2 - 2ab \cos \gamma = c^2. \tag{6}$$

From this inequality, show that $\mathbb{H}_n$ satisfies the median inequality (1).

Exercise 7.7. On the set $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, consider the *Möbius group*

$$M(\widehat{\mathbb{R}}) = \big\{\varphi : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}},\ x \mapsto \tfrac{ax+b}{cx+d}\ ;\ a, b, c, d \in \mathbb{R},\ ad - bc = \pm 1\big\}.$$

Show that this group is nonfixating. Hint: Consider the matrices A and B from Example 3.3 and use Lemma 8.16.

Exercise 7.8. Using only the fact that $\operatorname{Isom} \mathbb{H}_3$ is fixating, prove that $\operatorname{Isom} \mathbb{H}_2$ is fixating.

Exercise 7.9. Show that a map, a priori surjective or not, from $\mathbb{R}^n$ to $\mathbb{R}^n$ which preserves the Euclidean distance is an affine bijection of $\mathbb{R}^n$.

Exercise 7.10. Show that a function between two normed vector spaces which is continuous and preserves the midpoints is affine.

Exercise 7.11. *Mazur-Ulam theorem.* The statement is inspired by [22].
Let E be a real normed vector space.

a. Let $a, b \in E$ and m be the midpoint of $[a, b]$. Let $W_{a,b}$ the set of isometries of E fixing a and b and let $\lambda = \sup\{\|g(m) - m\| \ ; \ g \in W_{a,b}\}$.

(i) Prove that $\lambda \leqslant \|a - b\|$.

(ii) Let s_m be the symmetry of center m, i.e. such that $s_m(x) = 2m - x$ for all $x \in E$. For $g \in W_{a,b}$, we set $g^* = s_m\, g^{-1} s_m\, g$. Prove that $\|g^*(m) - m\| = 2\|g(m) - m\|$.

(iii) Show that any isometry that fixes a and b fixes m.

b. Let f be an isometry of E.

(i) Let $a, b \in E$. Denote by m the midpoint of $[a, b]$ and by m' the one of $[f(a), f(b)]$. Prove that $h = s_m\, f^{-1} s_{m'}\, f \in W_{a,b}$ and deduce that $f(m) = m'$.

(ii) Prove that f is affine.

Exercise 7.12. *Kakutani Theorem in finite dimension.* This exercise is inspired by R. Antetomaso [1]. Let E be a normed vector space of finite dimension, G a compact subgroup of $GL(E)$, and K a nonempty compact convex subset of E. We assume that $g(K) \subseteq K$ for all $g \in G$ and we aim to show that (K, G) is a GAG.

a. Let $\| \ \|_2$ be the Euclidean norm on E. For any $x \in E$, we set $\|x\| = \sup\{\|g(x)\|_2 \ ; \ g \in G\}$. Show that this defines a strictly convex norm on E, for which every element of G is an isometry.

b. Let f be an endomorphism of E such that $f(K) \subseteq K$. Let $x_1 \in K$ and let $(x_n)_{n>0}$ be the sequence defined by $x_{n+1} = f(x_n)$. Considering the sequence (σ_n) of Cesàro means of the sequence (x_n), show that f has a fixed point in K.

c. For $g \in G$, denote $V_g = \{x \in K \ ; \ g(x) \neq x\}$. By contradiction, assume that, for every $x \in K$, there exists $g \in G$ satisfying $g(x) \neq x$.

(i) Show that there exist $g_1, \dots, g_N \in G$ such that $K \subset V_{g_1} \cup \cdots \cup V_{g_N}$.

(ii) Show that there exists $a \in K$ such that $(g_1 + \cdots + g_N)(a) = Na$.

(iii) Show that $g_k(a) = a$ for all $k \in \{1, \dots, N\}$. Conclude.

Exercise 7.13. *The isometries of the sphere extend into isometries of the Euclidean space.* We equip the sphere $\mathbb{S}_n$ with the spherical distance $d(x, y) = \arccos\langle x \,|\, y\rangle$. We want to prove that every isometry of $\mathbb{S}_n$ is the restriction of a unique isometry of $\mathbb{R}^{n+1}$ endowed with the Euclidean distance. Let f be an isometry of $\mathbb{S}_n$.

a. Show that f preserves the scalar product.

b. We define $\widetilde{f}$ on $\mathbb{R}^{n+1}$ by $\widetilde{f}(x) = \|x\| f\big(\frac{x}{\|x\|}\big)$ if $x \neq 0$ and $\widetilde{f}(0) = 0$. Show that $\widetilde{f}$ preserves the scalar product.

c. Conclude.

Exercise 7.14. *Extension of an isometry.*
Let A be a subset of $\mathbb{R}^d$ and set $E = \operatorname{Aff} A$, the affine subspace generated by A. We endow A with the Euclidean distance induced by that of $\mathbb{R}^d$. We want to show that any isometry of A is the restriction of a unique affine isometry of E.

We know that there are $a_0, \dots, a_n \in A$ such that $E = \operatorname{Aff}(a_0, \dots, a_n)$, where $n = \dim E$. Thus, every point $x \in E$ is written in a unique way $x = \sum_{i=0}^{n} \lambda_i(x) a_i$, with $\sum_{i=0}^{n} \lambda_i(x) = 1$. The numbers $\lambda_i(x)$ are the *barycentric coordinates* of x. Let f be an isometry of A.

a. Show that, for all $i, j \in \{0, \dots, n\}$, we have $\langle f(a_i) - f(a_0) \,|\, f(a_j) - f(a_0) \rangle = \langle a_i - a_0 \,|\, a_j - a_0 \rangle$.

b. We define $\widetilde{f}$ on E by $\widetilde{f}(x) = \sum_{i=0}^{n} \lambda_i(x) f(a_i)$. Prove that $\widetilde{f}$ is an isometry.

c. Show that $\widetilde{f}$ extends f. Conclude.

Exercise 7.15. *The alternating group $\mathcal{A}_7$ is nonfixating.*
In step 3 of the proof of Proposition 6.1, we built an eccentric subgroup of $\mathcal{S}_5$ using permutations which act separately on $\{1, 2, 3\}$ and on $\{4, 5\}$, thanks to the following key point. The group $\mathcal{S}_3$ has a normal subgroup (the alternating group $\mathcal{A}_3$) which has the following two properties:

- ▷ The quotient $\mathcal{S}_3/\mathcal{A}_3$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
- ▷ Any element of $\mathcal{S}_3 \setminus \mathcal{A}_3$ has at least one fixed point.

We then obtained an eccentric subgroup $G = \langle f, g \rangle$ of $\mathcal{S}_5$ by taking for f an element of $\mathcal{A}_3$ on $\{1, 2, 3\}$ and the identity on $\{4, 5\}$, and for g an element of $\mathcal{S}_3 \setminus \mathcal{A}_3$ on $\{1, 2, 3\}$ and an element of $\mathcal{S}_2$ without fixed point on $\{4, 5\}$. Use a similar construction to build an eccentric subgroup of $\mathcal{A}_7$.

Exercise 7.16. *The action of the group $\mathrm{GL}(3, \mathbb{F}_2)$ on $X = \mathbb{F}_2^3 \setminus \{0\}$ is not fixating.*
Let denote (e_1, e_2, e_3) the canonical basis of $\mathbb{F}_2^3$ and for $a \subset \{1, 2, 3\}$, let denote $e_a = \sum_{i \in a} e_i$. So that $X = \{e_a : a \neq \emptyset\}$. Let f and g be the elements of $\mathrm{GL}(3, \mathbb{F}_2)$ defined by

$$f(e_1) = e_2,\ f(e_2) = e_3,\ f(e_3) = e_1 \text{ and } g(e_1) = e_{123},\ g(e_2) = e_2,\ g(e_3) = e_3.$$

and $G = \langle f, g \rangle$. Finally denote $Y = \{e_1, e_2, e_3, e_{123}\}$ and $Z = \{e_{23}, e_{13}, e_{12}\}$.

a. Show that f and g induce even permutations on X.

b. Show that the map which sends $h \in G$ to its restriction on Y, induces an injective morphism from G in the group of permutations $\mathcal{S}_Y$ of Y.

c. Show that, if the restriction of $h \in G$ to Y is a double transposition, then its restriction to Z is the identity.

d. Deduce that the action of $\mathrm{GL}(3, \mathbb{F}_2)$ is not fixating. Observe that we have an alternative proof of the fact that $\mathcal{A}_7$ is not fixating.

8 Appendices

8.1 A short introduction to hyperbolic geometry

Let $n \geqslant 2$ be an integer. A model of the n-dimensional hyperpolic space is the upper half-space

$$\mathbb{H}_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ ;\ x_n > 0\}$$

endowed with the Poincaré metric

$$ds^2 = \frac{1}{x_n^2}\left(dx_1^2 + \cdots + dx_n^2\right).$$

A calculation shows that the geodesic distance associated with the Riemannian metric is given by

$$d(x, y) = \operatorname{argcosh}\left(1 + \frac{\|x - y\|^2}{2 x_n y_n}\right) \tag{7}$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

Theorem 8.1.([6], Theorem 9.3) *The geodesics in $\mathbb{H}_n$ are the half-lines (affine lines) and the half-circles (Euclidean circles), the endpoints of which are in the horizontal hyperplane*

$$\partial\mathbb{H}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ ; \ x_n = 0\},$$

and orthogonal to this hyperplane at their endpoints.

8.1.1 The isometry group of $\mathbb{H}_n$.

The similarity transformations (similarities for short), the inversions with respect to spheres, and the reflections through affine hyperplanes, act on $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ and form a subgroup of the group of homeomorphisms of $\widehat{\mathbb{R}}^n$. This subgroup is call the *Möbius group of* $\widehat{\mathbb{R}}^n$ and is denoted by $M(\widehat{\mathbb{R}}^n)$. The inversions and the reflections are enough to generate the Möbius group. The restrictions to the upper half-space of some elements of the Möbius group give rise to the isometry group of $\mathbb{H}_n$:

Theorem 8.2.([17], Theorem 4.6.2) *The isometry group* $\operatorname{Isom}\mathbb{H}_n$ *is the group of the restrictions to $\mathbb{H}_n$ of the Möbius transformations φ such that $\varphi(\mathbb{H}_n) = \mathbb{H}_n$. It is generated by the reflections through spheres centered at points in $\partial\mathbb{H}_n$ and vertical hyperplanes.*

All results we need follow from this theorem, from the theorem about geodesics and from the formula giving the distance.

A first consequence of Theorem 8.2 is that a similarity $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(\mathbb{H}_n) = \mathbb{H}_n$ induces an isometry of $\mathbb{H}_n$. Conversely, since the only elements in $M(\widehat{\mathbb{R}}^n)$ that fix ∞ are the similarities ([17], Theorem 4.3.2), we have the following result.

Corollary 8.3. *If an isometry f of $\mathbb{H}_n$ is the restriction of a map $\widehat{f}$ in $M(\widehat{\mathbb{R}}^n)$ which fixes ∞, then the restriction $\widehat{f}$ to $\mathbb{R}^n$ is a similarity in $\mathbb{R}^n$.*

A second consequence of Theorem 8.2 is that isometries of $\mathbb{H}_n$ are smooth. On the one hand, it follows that the Poincaré metric is invariant. On the other hand, it follows that the sign of the Jacobian of an isometry is constant. This latter fact leads to the usual decomposition of the group $\operatorname{Isom}\mathbb{H}_n$. It is the union of the subgroup $\operatorname{Isom}^+\mathbb{H}_n$ of isometries with positive Jacobian and its complementary $\operatorname{Isom}^-\mathbb{H}_n$.

8.1.2 Subspaces of $\mathbb{H}_n$.

A *hyperbolic subspace of* $\mathbb{H}_n$ is a subset X of $\mathbb{H}_n$ of one of the following form:

- ▷ the empty set, $\dim X = -1$,
- ▷ a single point, $\dim X = 0$,
- ▷ the intersection of $\mathbb{H}_n$ with a vertical affine subspace A, $\dim X = \dim A$,
- ▷ the intersection of $\mathbb{H}_n$ with a vertical affine subspace A and a sphere S centered at a point of $\partial\mathbb{H}_n$, $\dim X = \dim A \cap S$ ($A = \mathbb{R}^n$ is permitted).

By abuse of language, subspaces of dimension 1 will be called *lines* and those of dimension 2 *planes*. Using that the image of a sphere by an inversion, the pole of which is in the sphere, is an affine subspace, we see that a p-dimensional hyperbolic subspace is isometric to

$$\{(x_1, \ldots, x_n) \in \mathbb{H}_n \ ; \ x_1 = x_2 = \cdots = x_{n-p+1} = 0\}$$

and therefore to $\mathbb{H}_p$. By definition, the *geodesics* are the lines of the hyperbolic space. Moreover, a hyperbolic subspace is *totally geodesic*, which means that if it contains two points x and y, then it contains the geodesic joigning x à y.

Lemma 8.4. *Every intersection of hyperbolics subspaces in $\mathbb{H}_n$ is a hyperbolic subspace.*

Proof. Every intersection of spheres and/or affine subspaces is a sphere S (or an affine subspace if there is no sphere) of the affine subspace A generated by the intersection. Observe that this subspace A can be empty or reduced to a single point. We need to check that, if the affine subspaces are all vertical, if the spheres are centered in $\partial\mathbb{H}_n$, and if $\dim A \geqslant 1$, then A is vertical and the center of S belongs to $\partial\mathbb{H}_n$. If the intersection is defined by at least two hyperbolic subspaces then it is included in a vertical hyperplane. Since such a hyperplane is isometric to $\mathbb{H}_{n-1}$, the result follows by induction on n. □

Lemma 8.5. *Let H be a subspace of $\mathbb{H}_n$ and x a point not in H. The dimension of the smallest subspace containing H and x is* $\dim H + 1$.

Proof. Consider f an isometry that sends H onto the intersection of $\mathbb{H}_n$ with a vertical affine subspace A of dimension $p = \dim H$. The smallest hyperbolic subspace containing $A \cap \mathbb{H}_n$ and $f(x)$ is $A' \cap \mathbb{H}_n$ where A' is the affine subspace generated by A and $f(x)$, which is of dimension $p + 1$. □

Lemma 8.6. *Let a and b be two distinct points in $\mathbb{H}_n$. Then the set* $\mathrm{Med}\,(a, b)$ *of points equidistant from a and b, is a hyperplane of $\mathbb{H}_n$.*

Proof. By (7), a point x is in $\mathrm{Med}\,(a, b)$ if and only if

$$d(x, a) = \operatorname{argcosh}\left(1 + \frac{\|x - a\|^2}{2x_n a_n}\right) = d(x, b) = \operatorname{argcosh}\left(1 + \frac{\|x - b\|^2}{2x_n b_n}\right),$$

therefore if and only if

$$\frac{\|x - a\|^2}{2x_n a_n} = \frac{\|x - b\|^2}{2x_n b_n}.$$

In the case where $a_n = b_n$, we get a vertical hyperplane. In the case where $a_n \neq b_n$, we get a half-sphere the center of which is in $\partial\mathbb{H}_n$. Indeed, the coefficient of x_n vanishes in the cartesian equation of the sphere, $b_n\|x - a\|^2 - a_n\|x - b\|^2 = 0$. □

Lemma 8.7. *If f is a isometry of $\mathbb{H}_n$ then* $\mathrm{Fix}\, f$ *is a hyperbolic subspace.*

Proof. By Lemma 8.4, there exists a smallest hyperbolic subspace H containing $\mathrm{Fix}\, f$. Let $x \in H$ be arbitrary. By contradiction, if $f(x) \neq x$ then $\mathrm{Med}\,(x, f(x))$ is a hyperplane which contains $\mathrm{Fix}\, f$, hence $\mathrm{Fix}\, f \subset F = H \cap \mathrm{Med}\,(x, f(x))$, but F does not contain x; this contradicts the minimality of H. Therefore, $x = f(x)$; this proves $\mathrm{Fix}\, f = H$. □

Lemma 8.8. *Let D be a line in $\mathbb{H}_n$ and $x \in \mathbb{H}_n$. There exists an unique point $y \in D$ such that*

$$d(x, y) = d(x, D) := \inf\{d(x, z) : z \in D\}.$$

Moreover, if $x \notin D$, then the geodesic through x and $y = \pi_D x$ is orthogonal to D.

Recall that the Poincaré metric is conformal to the Euclidean metric, hence the orthogonality relations are equivalent for these two metrics.

Proof. By compactness, the distance from x to D is realized: There exists $y \in D$ such that $d(x, D) = d(x, y)$. The point x and the line D are included in the hyperbolic plane P. On the one hand, the geodesic from x to y is included in P. On the other hand, this plane is isometric to the hyperbolic plane $\mathbb{H}_2$. Therefore we can suppose that x and D are in $\mathbb{H}_2$. Thanks to another isometry, we can suppose that $D = \{ti \; ; \; t > 0\}$. Denoting $x = a + bi$, we have

$$d(x, ti) = \operatorname{argch}\left(1 + \frac{a^2 + (b-t)^2}{2bt}\right).$$

A calculation shows that the function $t \to \frac{a^2+(b-t)^2}{2bt}$ reaches its minimum at $t = \sqrt{a^2+b^2}$, hence $\pi_D x = i\sqrt{a^2+b^2}$. The geodesic joigning x to its projection is the half-circle of center 0 and radius $\sqrt{a^2+b^2}$, which is indeed orthogonal to D. □

Corollary 8.9. *If F is a hyperplane of $\mathbb{H}_n$ and x is in $\mathbb{H}_n$, then there exists a unique point $\pi_F x \in F$ such that*

$$d(x, \pi_F x) = \inf\{d(x, z) \; ; \; z \in F\} = d(x, F).$$

Moreover, if $x \notin F$, then the geodesic going through x and $\pi_F x$ is orthogonal to F.

Proof. By compactness, the minimal distance is reached in at least one point. If two distinct points $y, z \in F$ give this minimal distance, then these two points also give the minimum distance from x to D, where D is the geodesic going through y and z, since $D \subset F$. This contradicts Lemma 8.8. This Lemma also implies that the geodesic from x to $\pi_F x$ is orthogonal to all the lines going through $\pi_F x$ and included in F, hence is orthogonal to F. □

8.1.3 Isometries of $\mathbb{H}_3$.

The next lemma asserts that an angle and an axis are associated with a positive isometry of $\mathbb{H}_3$ which has fixed points.

Lemma 8.10. *Let $f \in \operatorname{Isom}^+ \mathbb{H}_3$. Assume $\operatorname{Fix} f \neq \emptyset$ and $f \neq \mathbf{id}$.*

a. *Then $\operatorname{Fix} f$ is a hyperbolic line.*

b. *There exists an angle θ such that for all $x \in \operatorname{Fix} f$, the differential $df(x)$ is a rotation with angle $\pm\theta$ and with axis the affine line tangent to $\operatorname{Fix} f$.*

Proof. **a.** Let $x \in \operatorname{Fix} f$. The differential $df(x)$ is a positive isometry of $\mathbb{R}^3$, hence a rotation with an eigenvector $\vec{u}$. Consider the geodesic D through x of direction $\vec{u}$. It is invariant by f, and f has a point fix in D, hence $D \subset \operatorname{Fix} f$. If D were strictly included in $\operatorname{Fix} f$ then $\operatorname{Fix} f$ would contain a hyperbolic plane. The differential of f at a point of this plan would be a rotation with two independent eigenvectors; hence it would be the identity. It follows that f would be the identity, a contradiction, hence $\operatorname{Fix} f = D$.

b. Conjugating by an isometry, we can restrict ourself to the case where D is a vertical line. By Corollary 8.3, f is the restriction of a similarity. Since f has a half-line of fixed points, f is the restriction of a rotation r of axis D. The differential of f along this axis is then always equal to $\vec{r}$ or to its inverse. □

With the same method, one easily proves the following statement.

Lemma 8.11. *Let $f \in \operatorname{Isom}^- \mathbb{H}_3$. If $\operatorname{Fix} f \neq \emptyset$, then $\operatorname{Fix} f$ is a hyperbolic plane and f is a reflection through this plane.*

Our last statement concerning isometries of $\mathbb{H}_3$ is the following.

Lemma 8.12. *Let $f \in \operatorname{Isom}^+ \mathbb{H}_3$. Assume that $\operatorname{Fix} f \neq \emptyset$. Denote D the axis of f and θ its angle. Let P_1 and P_2 be two hyperbolic planes of intersection D, and let σ_{P_1} and σ_{P_2} be the reflexions through the planes P_1 and P_2. Denote ∂P_1 and ∂P_2 the circles or lines determined by these planes in $\partial\mathbb{H}_3$. If $\angle(\partial P_1, \partial P_2) = \pm\frac{\theta}{2} \mod \pi$ then $f = \sigma_{P_2}\sigma_{P_1}$ or $\sigma_{P_1}\sigma_{P_2}$.*

Proof. As in the previous proof, thanks to a conjugacy by an isometry, we can assume that D is a vertical half-line. It follows that P_1 and P_2 are vertical planes and that f, σ_{P_1} and σ_{P_2} are similarities and hence isometries of $\mathbb{R}^3$. Since the relation about angles remains true up to the sign, we have $f = \sigma_{P_2}\sigma_{P_1}$ or $\sigma_{P_1}\sigma_{P_2}$. $\square$

8.1.4 Isometries of $\mathbb{H}_2$.

The hyperbolic plane $\mathbb{H}_2$ is identified with the upper half-plane $\{z \in \mathbb{C}\ ;\ \operatorname{Im} z > 0\}$. With this identification, by Theorem 8.2, the positive isometries of $\mathbb{H}_2$ are the conformal transformations of the upper half-plane, i.e., the maps $h : \mathbb{H}_2 \to \mathbb{H}_2$ of the form

$$h(z) = h_{a,b,c,d}(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R},\ ad - bc = 1.$$

The negative isometries are obtained by composing the positive isometries with the reflection $z \mapsto -\overline{z}$.

Lemma 8.13. *If $x, y, x', y' \in \mathbb{H}_2$ are such that $d(x, y) = d(x', y')$, then there exists $f \in \operatorname{Isom}^+ \mathbb{H}_2$ such that $f(x) = x'$ and $f(y) = y'$.*

Proof. It is enough to prove the statement in the case where $x' = i$ and $y' = it$, with $t > 1$. The homography $h \in \operatorname{Isom}^+ H_2$ defined by $h(z) = \frac{1}{\operatorname{Im} x}(z - \operatorname{Re} x)$ sends x to i. Let $\overrightarrow{u}$ be the unit vector, tangent at x to the geodesic going from x to y. By composing with a homography of the form $r(z) = \frac{\cos\theta\, z - \sin\theta}{\sin\theta\, z + \cos\theta}$ which fixes i, we can suppose that the differential of rh at x sends $\overrightarrow{u}$ to i. The isometry rh maps the geodesic segment joining x to y onto a geodesic segment of the same length starting at i with initial speed i, hence included in the imaginary axis. Therefore $rh(y) = it$ where t is the unique real number > 1 such that $d(i, it) = d(x, y) = d(x', y')$ hence $rh(y) = y'$. $\square$

Proof of Lemma 5.10.

a. The search of the fixed points of $h = h_{a,b,c,d}$ leads to the equation

$$cz^2 + (d - a)z - b = 0, \tag{8}$$

the discriminant of which is $\Delta = (d - a)^2 + 4bc = (a + d)^2 - 4 = (\operatorname{tr} h)^2 - 4$.

So, if $\operatorname{tr} h < 2$ then h has a unique fixed point $a \in \mathbb{H}_2$ (the other root of equation (8), the conjugate, is not in $\mathbb{H}_2$). The isometry h is *elliptic* and a is the *center of* h.

If $c = 0$ or if $\operatorname{tr} h = 2$ then equation (8) has a unique root in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, hence h has no fixed point. The isometry h is *parabolic*. This case contains the case $c = a - d = 0$.

If $c \neq 0$ and $\operatorname{tr} h > 2$ then (8) has two roots in $\overline{\mathbb{R}}$, hence h has no fixed point. The isometry h is *hyperbolic*.

b. The condition $h_{a,b,c,d}(i) = \frac{ai+b}{ci+d} = i$ implies $a = d$ and $b = -c$. The condition $ad - bc = 1$ then implies that there exists $\theta \in \mathbb{R}$ such that $a = \cos\frac{\theta}{2}$ and $b = \sin\frac{\theta}{2}$.

c. If φ and $\psi \in \operatorname{Isom}^+ \mathbb{H}_2$ send i to $z_0 \in \operatorname{Fix} h$ then, denoting $\varphi^{-1}h\varphi$ by r_θ, $\varphi^{-1}\psi$ by r_α (these are elements of $\operatorname{Isom}^+ \mathbb{H}_2$ fixing i, hence rotations), we obtain

$$\psi^{-1}h\psi = \psi^{-1}\varphi r_\theta \varphi^{-1}\psi = r_\alpha^{-1} r_\theta r_\alpha = r_\theta = \varphi^{-1} f \varphi.$$

d. The rotation of center ix and angle θ is $f = \varphi r_\theta \varphi^{-1}$, where $\varphi \in \operatorname{Isom}^+ \mathbb{H}_2$ sends i to ix. Let us choose $\varphi = h_{a,b,c,d}$ with $a = \sqrt{x}$, $b = c = 0$ and $d = \frac{1}{\sqrt{x}}$, i.e. the homography associated with the matrix $M(x) = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & 1/\sqrt{x} \end{pmatrix}$. Then the matrices associated with $\varphi r_\theta \varphi^{-1}$ are $\pm M$ with

$$M = M(x)^{-1} R\left(\tfrac{\theta}{2}\right) M(x) = \begin{pmatrix} \cos\frac{\theta}{2} & x\sin\frac{\theta}{2} \\ -x^{-1}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

$\square$

8.2 Proof of Lemma 3.4

Let $\mathcal{T} = \{M \in \mathrm{SL}(2,\mathbb{Z})\ ;\ \mathrm{tr}\, M = 3\}$. We first prove the following preliminary result.

Lemma 8.14. *Let $M \in \mathcal{T}$.*

a. *One has $M^{-1} \in \mathcal{T}$.*

b. **(i)** *For any integer $n \geqslant 1$, one has $M^n = \alpha_n M - \alpha_{n-1} I$, with $\alpha_0 = 0$, $\alpha_1 = 1$, and $\alpha_n = 3\alpha_{n-1} - \alpha_{n-2}$ for $n \geqslant 2$.*

(ii) *The sequence $(\alpha_n)_{n\in\mathbb{N}}$ takes its values in $\mathbb{N}$ and strictly increases. Moreover, for any $n \in \mathbb{N}$, it holds $\alpha_{n+1} - \alpha_n \geqslant 2^n$ and $\alpha_n \geqslant 2^n - 1$.*

c. *For any integer $n \geqslant 1$, one has $\mathrm{tr}\, M^n \geqslant 2^{n+1} - 1$.*

Proof. Item a results from the fact that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \Rightarrow\ M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Item b (i) is proved by induction. It is obvious for $n = 1$ and also for $n = 2$ (i.e. $M^2 = 3M - I$) by the Cayley-Hamilton theorem. Next, if we assume that $M^{n-1} = \alpha_{n-1} M - \alpha_{n-2} I$ for some $n \geqslant 2$, then

$$M^n = \alpha_{n-1} M^2 - \alpha_{n-2} M = (3\alpha_{n-1} - \alpha_{n-2})M - \alpha_{n-1} I = \alpha_n M - \alpha_{n-1} I.$$

For item b (ii), an easy induction shows that $\alpha_n \geqslant 0$ for all $n \in \mathbb{N}$. Therefore, we have

$$\alpha_{n+1} - \alpha_n = 2\alpha_n - \alpha_{n-1} \geqslant 2(\alpha_n - \alpha_{n-1}) \geqslant 2^n(\alpha_1 - \alpha_0) = 2^n,$$

hence

$$\alpha_n = \alpha_n - \alpha_0 = \sum_{k=0}^{n-1} (\alpha_{k+1} - \alpha_k) \geqslant \sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

For item c, considering the above, for any $n \geqslant 1$, one has

$$\mathrm{tr}\, M^n = \mathrm{tr}\,(\alpha_n M - \alpha_{n-1} I) = 3\alpha_n - 2\alpha_{n-1} = \alpha_n + 2(\alpha_n - \alpha_{n-1}) \geqslant 2^{n+1} - 1.$$

$\square$

Let G_0 be the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & -1 \\ 5 & 4 \end{pmatrix}.$$

The matrices A and B (as well as their inverses A^{-1} and B^{-1}) are elements of $\mathcal{T}$. We will show that any matrix $M \in G_0 \setminus \{I\}$ has a trace different from 2, which is equivalent to saying that 1 is not an eigenvalue of M (since $\det M = 1$), i.e. that $M - I$ is invertible.

Let $\varepsilon : \mathbb{R} \to \{-1, 1\}$ denote the *sign fonction*, i.e. $\varepsilon(x) = 1$ if $x \geqslant 0$, and $\varepsilon(x) = -1$ if $x < 0$. Let $\gg$ denote the *product order* on $\mathrm{SL}(2,\mathbb{Z})$: We write $X \gg Y$ if $x_{ij} \geqslant y_{ij}$ for all $i, j \in \{1, 2\}$. It is a partial order, compatible with the addition and the multiplication. Especially for $X, Y, X', Y' \in \mathrm{SL}(2,\mathbb{Z})$, one has

$$X \gg Y \gg 0 \ \text{ and } \ X' \gg Y' \gg 0 \ \Rightarrow\ XX' \gg YY'.$$

We also have $X \gg Y \ \Rightarrow\ \mathrm{tr}\, X \geqslant \mathrm{tr}\, Y$.

The following is inspired by [16], VIII 26, pp.158–162, and proves Lemma 3.4.

Lemma 8.15. *For $k, l \in \mathbb{Z}^*$, one has*

$$\varepsilon(kl)A^kB^l \gg \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9}$$

Proof. Note first that

$$A^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 4 & 1 \\ -5 & -1 \end{pmatrix} \qquad AB = \begin{pmatrix} 5 & 4 \\ 16 & 13 \end{pmatrix}$$

and that

$$AB^{-1} = \begin{pmatrix} -5 & -1 \\ -19 & -4 \end{pmatrix} \qquad A^{-1}B = \begin{pmatrix} -8 & -7 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad A^{-1}B^{-1} = \begin{pmatrix} 17 & 4 \\ 4 & 1 \end{pmatrix}.$$

Now let $k, l \geqslant 1$. With the notation of Proposition 8.14.b, one has

$$\begin{aligned} A^kB^l &= (\alpha_k A - \alpha_{k-1} I)(\alpha_l B - \alpha_{l-1} I) \\ &= \alpha_k\alpha_l AB - \alpha_k\alpha_{l-1} A - \alpha_{k-1}\alpha_l B + \alpha_{k-1}\alpha_{l-1} I \\ &= \begin{pmatrix} 5\alpha_k\alpha_l + \alpha_{k-1}\alpha_l + \alpha_{k-1}\alpha_{l-1} & 4\alpha_k\alpha_l - \alpha_k\alpha_{l-1} + \alpha_{k-1}\alpha_l \\ 16\alpha_k\alpha_l + \alpha_k\alpha_{l-1} - 5\alpha_{k-1}\alpha_l & 13\alpha_k\alpha_l - 3\alpha_k\alpha_{l-1} - 4\alpha_{k-1}\alpha_l + \alpha_{k-1}\alpha_{l-1} \end{pmatrix}. \end{aligned}$$

Therefore, since the sequence (α_n) strictly increases and $\alpha_0 = 0$, we obtain

$$A^kB^l \gg \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}. \tag{10}$$

In the same way, always for $k, l \geqslant 1$, we have

$$A^kB^{-l} = \begin{pmatrix} \alpha_{k-1}\alpha_{l-1} - 4\alpha_{k-1}\alpha_l - 5\alpha_k\alpha_l & -\alpha_k\alpha_{l-1} - \alpha_{k-1}\alpha_l - \alpha_k\alpha_l \\ \alpha_k\alpha_{l-1} + 5\alpha_{k-1}\alpha_l - 19\alpha_k\alpha_l & \alpha_{k-1}\alpha_{l-1} - 3\alpha_k\alpha_{l-1} + \alpha_{k-1}\alpha_l - 4\alpha_k\alpha_l \end{pmatrix}$$

$$A^{-k}B^l = \begin{pmatrix} \alpha_{k-1}\alpha_{l-1} - 3\alpha_k\alpha_{l-1} + \alpha_{k-1}\alpha_l - 8\alpha_k\alpha_l & \alpha_k\alpha_{l-1} + \alpha_{k-1}\alpha_l - 7\alpha_k\alpha_l \\ -\alpha_k\alpha_{l-1} - 5\alpha_{k-1}\alpha_l - \alpha_k\alpha_l & \alpha_{k-1}\alpha_{l-1} - 4\alpha_{k-1}\alpha_l - \alpha_k\alpha_l \end{pmatrix}$$

$$A^{-k}B^{-l} = \begin{pmatrix} \alpha_{k-1}\alpha_{l-1} - 3\alpha_k\alpha_{l-1} - 4\alpha_{k-1}\alpha_l + 17\alpha_k\alpha_l & \alpha_k\alpha_{l-1} - \alpha_{k-1}\alpha_l + 4\alpha_k\alpha_l \\ -\alpha_k\alpha_{l-1} + 5\alpha_{k-1}\alpha_l + 4\alpha_k\alpha_l & \alpha_{k-1}\alpha_{l-1} + \alpha_{k-1}\alpha_l + \alpha_k\alpha_l \end{pmatrix}$$

from which one draws the inequalities

$$-A^kB^{-l} \gg \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \qquad -A^{-k}B^l \gg \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A^{-k}B^{-l} \gg \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}.$$

This and (10) give the desired result. $\square$

Lemma 8.16. *Let $m \geqslant 1$ be an integer and $k_1, \dots k_m$, $l_1, \dots l_m$, be nonzero integers, except possibly k_1 and l_m. Then the matrix $M = A^{k_1}B^{l_1} \dots A^{k_m}B^{l_m}$ satisfies $|\mathrm{tr}\, M| \geqslant 3$.*

Proof. If $m = 1$, then $(k_1, l_1) \neq (0, 0)$, and the result follows from Proposition 8.14.c if $k_1l_1 = 0$, and from (9) if k_1 and l_1 are both nonzero, since then

$$|\mathrm{tr}\,(A^{k_1}B^{l_1})| = \varepsilon(k_1l_1)\mathrm{tr}\,(A^{k_1}B^{l_1}) \geqslant 6.$$

Now suppose $m \geqslant 2$.

▷ If k_1 and s_m are nonzero, then

$$|\operatorname{tr} M| = \operatorname{tr}\left(\varepsilon(k_1 l_1) A^{k_1} B^{l_1} \times \cdots \times \varepsilon(k_m l_m) A^{k_m} B^{l_m}\right) \geqslant \operatorname{tr}\begin{pmatrix} 5^m & 0 \\ 0 & 1 \end{pmatrix} = 5^m + 1 \geqslant 3.$$

▷ If $k_1 = s_m = 0$, then

$$\operatorname{tr} M = \operatorname{tr}(B^{l_1} A^{k_2} B^{l_2} \ldots A^{k_m}) = \operatorname{tr}(A^{k_2} B^{l_2} \ldots A^{k_m} B^{l_1})$$

and we fall in the previous case.

▷ If $k_1 = 0$ and $s_m \neq 0$, then

$$\operatorname{tr} M = \operatorname{tr}(B^{l_1} A^{k_2} B^{l_2} \ldots A^{k_m} B^{l_m}) = \operatorname{tr}(A^{k_2} B^{l_2} \ldots A^{k_m} B^{l_m + l_1}).$$

If $l_m + l_1 \neq 0$, we fall into one of the previous cases (depending on whether $m = 2$ or $m > 2$). If $l_m + l_1 = 0$, then

$$\operatorname{tr} M = \operatorname{tr}(A^{k_2} B^{l_2} \ldots A^{k_m}) = \operatorname{tr}(A^{k_2 + k_m} B^{l_2} \ldots A^{k_{m-1}} B^{k_{m-1}})$$

and we iterate until we are reduced to an already considered case.

▷ If $k_1 \neq 0$ and $s_m = 0$, then

$$\operatorname{tr} M = \operatorname{tr}(A^{k_1} B^{l_1} \ldots A^{k_m}) = \operatorname{tr}(A^{k_1 + k_m} B^{l_1} \ldots A^{k_{m-1}} B^{k_{m-1}})$$

and we proceed as above. □

8.3 Proof of Lemma 5.5

Let w be a nontrivial word of $G_0 = \langle \sigma, \tau \rangle$ of length $\ell \geqslant 1$, i.e. an element $w = a_1 \ldots a_\ell$, with $a_i \in \{\sigma, \sigma^{-1}, \tau, \tau^{-1}\}$ and $a_i a_{i+1} \neq \mathbf{id}$ for all $i \in \{1, \ldots, \ell - 1\}$. The word w has one of the four forms $\sigma^\pm \cdots \tau^\pm$, $\tau^\pm \cdots \sigma^\pm$, $\sigma^\pm \cdots \sigma^\pm$ or $\tau^\pm \cdots \tau^\pm$. The second form comes down to the first by considering w^{-1} instead of w, and the last two forms can be reduced to one of the first two by conjugation, unless w is simply a power of σ or τ, or a conjugate of such a power, in which case the result comes from $\theta \notin \pi\mathbb{Q}$. So we just have to show that 1 is not an eigenvalue of w when $w = \sigma^\pm \cdots \tau^\pm$. A simple induction shows that w has a matrix of the form

$$\begin{pmatrix} P & -Q & -R & -S \\ Q & P & -S & R \\ R & S & P & -Q \\ S & -R & Q & P \end{pmatrix}$$

where P, R are polynomials in $\cos\theta$ with integer coefficients, and Q, S are products of such polynomials with $\sin\theta$.

Since w is orthogonal, we have $P^2 + Q^2 + R^2 + S^2 = 1$. A simple computation then shows that the characteristic polynomial of w is $\lambda^4 - 4P\lambda^3 + (4P^2 + 2)\lambda^2 - 4P\lambda + 1$.

We show below that the degree of P is equal to ℓ, the length of w. If 1 were an eigenvalue of w, we would have $4P^2 - 8P + 4 = 0$, a contradiction since $\cos\theta$ is transcendent.

In the sequel, the notation $\asymp$ indicates that only the term of highest degree in $\cos\theta$ has been retained. Using $\cos m\theta \asymp 2^{m-1}\cos^m\theta$ and $\sin m\theta \asymp 2^{m-1}\cos^{m-1}\theta \sin\theta$, and denoting $\cos\theta$ by c and $\sin\theta$ by s, we get, with $\varepsilon = \pm 1$ and $\delta = \pm 1$

$$\sigma^{\varepsilon m} \asymp 2^{m-1} c^{m-1} \begin{pmatrix} c & -\varepsilon s & 0 & 0 \\ \varepsilon s & c & 0 & 0 \\ 0 & 0 & c & -\varepsilon s \\ 0 & 0 & \varepsilon s & c \end{pmatrix} \quad \tau^{\delta k} \asymp 2^{k-1} c^{k-1} \begin{pmatrix} c & 0 & 0 & -\delta s \\ 0 & c & -\delta s & 0 \\ 0 & \delta s & c & 0 \\ \delta s & 0 & 0 & c \end{pmatrix}.$$

Multiplying both and using $s^2 = 1 - c^2 \asymp -c^2$, we get

$$\sigma^{\varepsilon m}\tau^{\delta k} \asymp A(m,k,\varepsilon,\delta) = 2^{m+k-2}c^{m+k-1}\begin{pmatrix} c & -\varepsilon s & -\varepsilon\delta c & -\delta s \\ \varepsilon s & c & -\delta s & \varepsilon\delta c \\ \varepsilon\delta c & \delta s & c & -\varepsilon s \\ \delta s & -\varepsilon\delta c & \varepsilon s & c \end{pmatrix}.$$

Let us denote $w = \sigma^{\varepsilon_1 m_1}\tau^{\delta_1 k_1}\cdots\sigma^{\varepsilon_n m_n}\tau^{\delta_n k_n}$ with $\varepsilon_i, \delta_i = \pm 1$ and $m_i, k_i \geqslant 1$; thus the length of w is $\ell = m_1 + k_1 + \cdots + m_n + k_n$.

We assert that, for every integer $n \geqslant 1$, there exist ξ_n, μ_n, ζ_n and ν_n equal to ± 1 such that $\xi_n\zeta_n = \mu_n\nu_n$ and

$$w \asymp A_n = 2^{\ell-n-1}c^{\ell-1}\begin{pmatrix} \xi_n c & -\mu_n s & -\zeta_n c & -\nu_n s \\ \mu_n s & \xi_n c & -\nu_n s & \zeta_n c \\ \zeta_n c & \nu_n s & \xi_n c & -\mu_n s \\ \nu_n s & -\zeta_n c & \mu_n s & \xi_n c \end{pmatrix}.$$

By induction on n, let us show that this is indeed the case. For $n = 1$, this is because $A_1 = A(m_1, k_1, \varepsilon_1, \delta_1)$ and because we have $\xi_1 = 1$, $\mu_1 = \varepsilon_1$, $\zeta_1 = \varepsilon_1\delta_1$, and $\nu_1 = \delta_1$.

Now assume that the property holds for an integer $n \geqslant 1$ and let us check it for $n+1$. One sees that $A_n A(m_{n+1}, k_{n+1}, \varepsilon_{n+1}, \delta_{n+1}) \asymp A_{n+1}$ with

$$\begin{aligned}
\xi_{n+1} &= \tfrac{1}{2}(\xi_n + \varepsilon_{n+1}\mu_n - \varepsilon_{n+1}\delta_{n+1}\zeta_n + \delta_{n+1}\nu_n) \\
\mu_{n+1} &= \tfrac{1}{2}(\varepsilon_{n+1}\xi_n + \mu_n + \delta_{n+1}\zeta_n - \varepsilon_{n+1}\delta_{n+1}\nu_n) \\
\zeta_{n+1} &= \tfrac{1}{2}(\varepsilon_{n+1}\delta_{n+1}\xi_n + \delta_{n+1}\mu_n + \zeta_n - \varepsilon_{n+1}\nu_n) \\
\nu_{n+1} &= \tfrac{1}{2}(\delta_{n+1}\xi_n + \varepsilon_{n+1}\delta_{n+1}\mu_n - \varepsilon_{n+1}\zeta_n + \nu_n).
\end{aligned}$$

From these equalities, it is easily shown that $\xi_{n+1}\zeta_{n+1} = \mu_{n+1}\nu_{n+1}$. It remains to check that $\xi_{n+1}, \mu_{n+1}, \zeta_{n+1}, \nu_{n+1}$ are equal to ± 1. Noticing that $\zeta_n = \xi_n\mu_n\nu_n$, one obtains

$$\xi_{n+1} = \tfrac{1}{2}(\xi_n + \varepsilon_{n+1}\mu_n + \delta_{n+1}\nu_n - \varepsilon_{n+1}\delta_{n+1}\xi_n\mu_n\nu_n) = \pm 1$$

since, in general, if a, b, c are equal to ± 1, then $a + b + c - abc = \pm 2$. In the same way, we show that μ_{n+1}, ζ_{n+1} and ν_{n+1} are equal to ± 1, which completes the proof.

8.4 Solutions of the exercices

Exercise 7.1. From Corollary 4.8, it is enough to show that $\mathrm{Isom}^+\mathbb{R}^2$ is fixating. Let $G \leqslant \mathrm{Isom}^+\mathbb{R}^2$ be a GAF. If $G = \{\mathbf{id}\}$, then G is obviously a GAG. Otherwise, let $f \in G \setminus \{\mathbf{id}\}$. Then f is a rotation of center $a \in \mathbb{R}^2$ and nonzero angle. Let $g \in G \setminus \{\mathbf{id}\}$ be arbitrary; g is a rotation of center $b \in \mathbb{R}^2$ and nonzero angle. In addition we have $\overrightarrow{[f,g]} = \mathbf{id}$, thus $[f,g]$ is the translation of vector $\overrightarrow{cf(c)}$, with $c = g(a)$. Since G is a GAF, $\mathrm{Fix}\,[f,g]$ is not the emptyset, hence $f(c) = c$, then $c = a$, $g(a) = a$, and $a = b$. Therefore, all the elements of $G \setminus \{\mathbf{id}\}$ are rotations of center a. I follows that $\mathrm{Fix}\,G = \{a\} \neq \emptyset$ and G is a GAG.

Exercise 7.2. Let $G = \langle g \rangle$, let X be an arbitrary set and $\rho : G \to \mathrm{Bij}\,X$ be a morphism. We have $\rho(G) = \langle \rho(g) \rangle$. Let $H \leqslant \rho(G)$. Then H is cyclic, generated by some h, hence $\mathrm{Fix}\,H = \mathrm{Fix}\,h$. If H is a GAF, then $\mathrm{Fix}\,h \neq \emptyset$, hence $\mathrm{Fix}\,H \neq \emptyset$, too, hence H is a GAG.
Conversely, let G be a noncyclic group. Let us act G by multiplication on the left on the set $X = \mathcal{P}(G) \setminus \{\emptyset, G\}$ of nontrivial subsets of G: Let $\rho : G \to \mathrm{Bij}\,X$ be defined by $\rho(g)(A) = gA = \{ga\ ;\ a \in A\}$. For every $g \in G$ the subset $\langle g \rangle$ is nontrivial since G is noncyclic, and $\langle g \rangle$ is fixed by $\rho(g)$, hence $\rho(G)$ itself is a GAF, but $\rho(G)$ has no global fixed point: Given $A \in X$, let $a \in A$ and $b \notin A$. Since $b = (ba^{-1})a$, we have $b \in \rho(ba^{-1})(A)$ hence $\rho(ba^{-1})(A) \neq A$. Thus (X, G) is eccentric, therefore nonfixating.

Exercise 7.3. Denote $\rho(\mathbb{Q}) = \{g_1, \dots, g_n\}$. Let $r_i = \frac{p_i}{q_i} \in \mathbb{Q}$ such that $\rho(r_i) = g_i$, with p_i and q_i relatively prime, and denote by M the least common multiple of the q_i. Then $\rho(\mathbb{Q})$ is cyclic, generated by $\rho\big(\frac{1}{M}\big)$, hence fixating by Exercise 7.2.

Exercise 7.4. Taking $z = x$, then $z = y$ in (1), we obtain $d(x, m)$ and $d(y, m) \leqslant \frac{1}{2}\, d(x, y)$. The triangular inequality $d(x, y) \leqslant d(x, m) + d(m, y)$ yields the equalities. If m' is another point satisfying (1) then, applying (1) with $z = m'$, and using $d(x, m') = d(y, m') = \frac{1}{2}\, d(x, y)$, we get $d(m, m') \leqslant 0$, hence $m = m'$.

Exercise 7.5. *Theorem of M. Frechet, P. Jordan and J. von Neuman* [9, 11].

a. In a normed vector space, due to the fact that $m = \frac{1}{2}(x + y)$, the median inequality (1) for $z = 0$ reads as $\|x + y\|^2 \leqslant 2\big(\|x\|^2 + \|y\|^2\big) - \||x - y\|^2$. Rewritten with $x = \frac{1}{2}(a + b)$ and $y = \frac{1}{2}(a - b)$, this yields the opposite inequality.

b. Set $\langle x \,|\, y\rangle = \frac{1}{4}\big(\|x + y\|^2 - \|x - y\|^2\big)$. It is enough to verify that $\langle \ \,|\, \ \rangle$ is a positive definite symmetric bilinear form. Easily, we have $\langle x \,|\, y\rangle = \langle y \,|\, x\rangle$, $\langle x \,|\, x\rangle \geqslant 0$ and $\big(\langle x \,|\, x\rangle = 0 \Rightarrow x = 0\big)$. It remains to prove that $\langle x + x' \,|\, y\rangle = \langle x \,|\, y\rangle + \langle x' \,|\, y\rangle$ and $\langle \lambda x \,|\, y\rangle = \lambda\langle x \,|\, y\rangle$.

First, we have $\langle 0 \,|\, y\rangle = 0$. By using (5), we get

$$\begin{aligned} &\langle x + x' \,|\, y\rangle + \langle x - x' \,|\, y\rangle \\ &\qquad = \tfrac{1}{4}\big(\|x + x' + y\|^2 + \|x - x' + y\|^2 - \|x + x' - y\|^2 - \|x - x' - y\|^2\big) \\ &\qquad = \tfrac{1}{2}\big(\|x + y\|^2 + \|x'\|^2 - \|x - y\|^2 - \|x'\|^2\big) \\ &\qquad = 2\langle x \,|\, y\rangle. \end{aligned}$$

For $x' = x$, we obtain $\langle 2x \,|\, y\rangle = 2\langle x \,|\, y\rangle$.
For $x = \frac{1}{2}(a + b)$ and $x' = \frac{1}{2}(a - b)$, we deduce $\langle a \,|\, y\rangle + \langle b \,|\, y\rangle = \langle a + b \,|\, y\rangle$. Therefore $\langle kx \,|\, y\rangle = k\langle x \,|\, y\rangle$, first for integer k, then for rational k, finally for real k by continuity of the norm.

Exercise 7.6. Let xyz be a hyperbolic triangle and m the midpoint of the geodesic segment from x to y. Let $\alpha = \angle zmx$ and $\beta = \angle zmy \in \big]0, \frac{\pi}{2}\big[$; one has $\alpha + \beta = \frac{\pi}{2}$ hence $\cos\alpha + \cos\beta = 0$. For short, denote by ab the geodesic distance between two points a and b. The cosine inequality for the triangle myz is: $mz^2 + my^2 - 2mz.my\cos\alpha \leqslant yz^2$. By using $my = \frac{1}{2}xy$, we obtain $mz^2 + \frac{1}{4}xy^2 - mz.xy\cos\alpha \leqslant yz^2$. In the same way, one has $mz^2 + \frac{1}{4}xy^2 - mz.xy\cos\beta \leqslant xz^2$. By adding both inequalities, we get $2mz^2 + \frac{1}{2}xy^2 \leqslant xz^2 + yz^2$, which indeed corresponds to the median inequality (1).

Exercise 7.7. Let $f : x \mapsto \frac{1}{-x+3}$ and $g : x \mapsto \frac{-x-1}{5x+4}$; f and g have distinct fixed points. Let $k_1, l_1, \dots, k_m, l_m$ be relative integers all nonzero, except possibly k_1 and l_m. The product $h = f^{k_1} g^{l_1} \cdots f^{k_m} g^{l_m}$ is of the form $h : x \mapsto \frac{ax+b}{cx+d}$ with $ad - bc = 1$. Looking for a fixed point of h leads to the equation $cx^2 + (d - a)x - b = 0$, whose discriminant is $\Delta = (a + d)^2 - 4$. From Lemma 8.16, one has $|a + d| \geqslant 3$, hence h still has a fixed point. Therefore, the group $\langle f, g\rangle$ is eccentric.

Exercise 7.8. Let G be a GAF of $\operatorname{Isom}\mathbb{H}_2$. Let us identify $\mathbb{H}_3$ with $\mathbb{R} \times \mathbb{H}_2$. For any $g \in G$, let $\varphi(g) : \mathbb{H}_3 \to \mathbb{H}_3$, $(x, y) \mapsto (x, g(y))$. We easily check that $\varphi(g) \in \operatorname{Isom}\mathbb{H}_3$ and $\operatorname{Fix}\varphi(g) = \mathbb{R} \times \operatorname{Fix} g \neq \emptyset$. Thus $H := \varphi(G)$ is a GAF of $\operatorname{Isom}\mathbb{H}_3$, hence a GAG since $\operatorname{Isom}\mathbb{H}_3$ is globalizing, and $\operatorname{Fix} H$ is of the form $\mathbb{R} \times A$. We obtain $\operatorname{Fix} G = A \neq \emptyset$, hence G is a GAG of $\operatorname{Isom}\mathbb{H}_2$.

Exercise 7.9. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ preserving the Euclidean distance. The equality

$$\langle x \,|\, y\rangle = \tfrac{1}{2}\big(\|x\|^2 + \|y\|^2 - \|x - y\|^2\big)$$

shows that the function $g : x \mapsto f(x) - f(0)$ preserves the scalar product: One has $\langle g(x) \,|\, g(y)\rangle = \langle x \,|\, y\rangle$ for every $x, y \in \mathbb{R}^n$. Thus, for every $x, y \in \mathbb{R}^n$ and every $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \|g(x+\lambda y) - g(x) - \lambda g(y)\|^2 &= \|g(x+\lambda y)\|^2 + \|g(x)\|^2 + \lambda^2\|g(y)\|^2 - \\ &\quad 2\langle g(x+\lambda y) \,|\, g(x)\rangle - 2\lambda\langle g(x+\lambda y) \,|\, g(y)\rangle + 2\lambda\langle g(x) \,|\, g(y)\rangle \\ &= \|x+\lambda y\|^2 + \|x\|^2 + \lambda^2\|y\|^2 - \\ &\quad 2\langle x+\lambda y \,|\, x\rangle - 2\lambda\langle x+\lambda y \,|\, y\rangle + 2\lambda\langle x \,|\, y\rangle \\ &= \|x+\lambda y - x - \lambda y\|^2 = 0. \end{aligned}$$

Therefore, g is linear, hence f is affine. Since g preserves the Euclidean distance, it is injective; since we are in finite dimension, g is surjective. It is the same with f.

Exercise 7.10. Let E be a normed vector space and $f : E \to E$ a continuous function such that

$$\forall a, b \in E, \quad f\big(\tfrac{1}{2}(a+b)\big) = \tfrac{1}{2}\big(f(a) + f(b)\big). \tag{11}$$

Let $g : E \to E$, $x \mapsto f(x) - f(0)$. Then g is continuous and satisfies (11) and $g(0) = 0$. Since $g(x) = g\big(\frac{1}{2}(0+2x)\big) = \frac{1}{2}\, g(2x)$, we can rewrite (11) as

$$\forall a, b \in E, \quad g(a+b) = g(a) + g(b).$$

It follows that $g(ax) = ag(x)$ for all $a \in \mathbb{N}$, then for all $a \in \mathbb{Q}$, finally for all $a \in \mathbb{R}$ by continuity of g and by density of $\mathbb{Q}$ in $\mathbb{R}$, hence g is linear, hence f is affine.

Exercice 7.11. *Mazur-Ulam Theorem* [22].

a. **(i)** Let $g \in W_{a,b}$. One has

$$\|g(m) - m\| \leqslant \|g(m) - g(a)\| + \|a - m\| = 2\|a - m\| = \|a - b\|,$$

hence $\lambda \leqslant \|a - b\|$.

(ii) Let $g \in W_{a,b}$. Since s_m is an isometry which fixes m, one has

$$\begin{aligned} \|g^*(m) - m\| &= \|s_m\, g^{-1} s_m\, g(m) - m\| = \|g^{-1} s_m\, g(m) - m\| \\ &= \|s_m(g(m)) - g(m)\| = 2\|g(m) - m\|. \end{aligned}$$

(iii) From the above, on the one hand λ is finite, and on the other hand, for all $g \in W_{a,b}$, since s_m permutes a and b, we have $g^* \in W_{a,b}$, hence $2\|g(m) - m\| \leqslant \lambda$, hence $2\lambda \leqslant \lambda$, hence $\lambda = 0$. As a consequence, we have $g(m) = m$ for all $g \in W_{a,b}$.

b. **(i)** One has $h(a) = s_m\, f^{-1} s_{m'}(f(a)) = s_m\, f^{-1}(f(b)) = s_m(b) = a$. Similarly one proves $h(b) = b$, hence $h \in W_{a,b}$, therefore from **a** one has $h(m) = m$.
It follows that $f^{-1} s_{m'} f(m) = s_m(m) = m$, then $s_{m'}(f(m)) = f(m)$. Since $s_{m'}$ admits m' as unique fixed point, one has $f(m) = m'$.

(ii) From item (i), for all a, b in E, we have $f(\frac{1}{2}(a+b)) = \frac{1}{2}(f(a) + f(b))$. Since f is continuous, we deduce that f is affine by Exercise 7.10.

Exercise 7.12. *Kakutani Theorem in finite dimension* [1].

a. One easily checks that $\|\ \ \|$ is a norm. In order to prove that $\|\ \ \|$ is strictly convex on E, we consider $x, y \in E$ such that $x \neq \overrightarrow{0}$ and $y \notin \mathbb{R}^+ x$. Then we have, for all $g \in G$, $g(x) \neq \overrightarrow{0}$ and $g(y) \notin \mathbb{R}^+ g(x)$. Put $\varphi(g) = \|g(x)\|_2 + \|g(y)\|_2 - \|g(x) + g(y)\|_2$. The function φ is continuous and takes positive values on the compact set G, hence bounded below by some $\delta > 0$. Thus, for all $g \in G$, $\|g(x) + g(y)\|_2 \leqslant \|g(x)\|_2 + \|g(y)\|_2 - \delta \leqslant \|x\| + \|y\| - \delta$ hence, taking the supremum: $\|x + y\| \leqslant \|x\| + \|y\| - \delta$.

Finally $\|g(x) - g(y)\| = \sup_{h \in G} \|hg(x) - hg(y)\|_2 = \sup_{k \in G} \|k(x) - k(y)\|_2 = \|x - y\|$ since G is a group, showing that every element of G is an isometry.

b. Since K is convex, the Cesàro mean $\sigma_n = \frac{1}{n}(x_1 + \cdots + x_n)$ is in K. By compactness, there exists a subsequence $(\sigma_{n_k})_{k\in\mathbb{N}}$ tending to some $a \in K$. We have

$$f(\sigma_{n_k}) = \tfrac{1}{n_k}(x_2 + \cdots + x_{n_k+1}) = \sigma_{n_k} - \tfrac{1}{n_k}(x_{n_k+1} - x_1),$$

hence the sequence $(f(\sigma_{n_k}))_{k\in\mathbb{N}}$ tends to a, too. Since f is an endomorphism on a finite dimensional space, f is continuous, hence $(f(\sigma_{n_k}))_{k\in\mathbb{N}}$ tends to $f(a)$, from which we deduce that $f(a) = a$.

c. Given $g \in G$, set $V_g = \{x \in K \ ; \ g(x) \neq x\}$ and assume by contradiction that, for all $x \in K$, there exists $g \in G$ such that $g(x) \neq x$.

(i) Since an isometry is continuous, each V_g is an open subset of K. By assumption, K is the union of the V_g, $g \in G$, hence by compactness ther exist $g_1, \dots, g_N \in G$ such that $K = V_{g_1} \cup \cdots \cup V_{g_N}$.

(ii) Let $f = \frac{1}{N}(g_1 + \cdots + g_N)$. By convexity, one has $f(K) \subseteq K$, hence there exists $a \in K$ such that $f(a) = a$ by item b.

(iii) One has $\|Na\| = \|g_1(a) + \cdots + g_N(a)\| \leqslant \|g_1(a)\| + \cdots + \|g_n(a)\| = N\|a\|$ because the g_k are isometries, threrefore the inequality is an equality, hence the $g_k(a)$ are positively collinear by strict convexity of the norm, hence all equal, hence all equal to a. As a consequence, the point a would be in none of the V_{g_k}, a contradiction.

Exercise 7.13. *Isometries of the sphere.*

a. For all $x, y \in \mathbb{S}_n$, one has $\langle f(x) \,|\, f(y)\rangle = \cos\, d\big(f(x), f(y)\big) = \cos\, d(x,y) = \langle x \,|\, y\rangle$.

b. For x or $y = 0$, one has $\big\langle \widetilde{f}(x) \,|\, \widetilde{f}(y)\big\rangle = 0 = \langle x \,|\, y\rangle$. For x and $y \neq 0$, one has

$$\big\langle \widetilde{f}(x) \,|\, \widetilde{f}(y)\big\rangle = \|x\|\,\|y\|\,\big\langle f\big(\tfrac{x}{\|x\|}\big) \,\big|\, f\big(\tfrac{y}{\|y\|}\big)\big\rangle = \|x\|\,\|y\|\,\big\langle \tfrac{x}{\|x\|} \,\big|\, \tfrac{y}{\|y\|}\big\rangle = \langle x \,|\, y\rangle.$$

c. Firstly, we have

$$\|\widetilde{f}(x)\|^2 = \big\langle \widetilde{f}(x) \,\big|\, \widetilde{f}(x)\big\rangle = \langle x \,|\, x\rangle = \|x\|^2$$

for all $x \in \mathbb{R}^{n+1}$. Then for all $x, y \in \mathbb{R}^{n+1}$

$$\|\widetilde{f}(x) - \widetilde{f}(y)\|^2 = \|\widetilde{f}(x)\|^2 + \|\widetilde{f}(y)\|^2 - 2\big\langle \widetilde{f}(x) \,|\, \widetilde{f}(y)\big\rangle = \|x\|^2 + \|y\|^2 - 2\langle x \,|\, y\rangle = \|x - y\|^2,$$

By Exercise 7.9, $\widetilde{f}$ est also surjective — hence an isometry — and affine. Since $\widetilde{f}(0) = 0$, $\widetilde{f}$ is linear, hence it is determined by its values on a subset spanning $\mathbb{R}^{n+1}$, for instance $\mathbb{S}_n$, hence the uniqueness.

Exercise 7.14. *Extension of an isometry.*

a. One has $\langle f(a_i) - f(a_0) \,|\, f(a_j) - f(a_0)\rangle = \|f(a_i) - f(a_0)\|^2 + \|f(a_j) - f(a_0)\|^2 - \frac{1}{2}\|f(a_i) - f(a_j)\|^2 = \|a_i - a_0\|^2 + \|a_j - a_0\|^2 - \frac{1}{2}\|a_i - a_j\|^2 = \langle a_i - a_0 \,|\, a_j - a_0\rangle$.

b. One has

$$
\begin{aligned}
\|\widetilde{f}(x)-\widetilde{f}(y)\|^2 &= \Big\| \sum_{i=0}^{n} \big(\lambda_i(x)-\lambda_i(y)\big) f(a_i) \Big\|^2 \\
&= \Big\| \sum_{i=0}^{n} \big(\lambda_i(x)-\lambda_i(y)\big)\big(f(a_i)-f(a_0)\big) \Big\|^2 \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} \big(\lambda_i(x)-\lambda_i(y)\big)\big(\lambda_j(x)-\lambda_j(y)\big)\langle f(a_i)-f(a_0) \,|\, f(a_j)-f(a_0)\rangle \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} \big(\lambda_i(x)-\lambda_i(y)\big)\big(\lambda_j(x)-\lambda_j(y)\big)\langle a_i-a_0 \,|\, a_j-a_0\rangle \\
&= \Big\| \sum_{i=0}^{n} \big(\lambda_i(x)-\lambda_i(y)\big)(a_i-a_0) \Big\|^2 = \|x-y\|^2.
\end{aligned}
$$

By Exercise 7.9, $\widetilde{f}$ is also surjective, hence $\widetilde{f}$ is an isometry.

c. Of course we have $\widetilde{f}(a_i) = f(a_i)$ for all $i \in \{0,\dots,n\}$, by uniqueness of the barycentric coordinates. Let $a \in A$; from above, one has $\|\widetilde{f}(a) - f(a_i)\| = \|a - a_i\|$ for all $i \in \{0,\dots,n\}$, hence $\|\widetilde{f}(a) - f(a_i)\| = \|f(a) - f(a_i)\|$ since f is an isometry of A. Thus $f(a_i) \in \mathrm{Med}\big(f(a), \widetilde{f}(a)\big)$ for all $i \in \{0,\dots,n\}$, but $\mathrm{Med}\big(f(a), \widetilde{f}(a)\big)$ is an affine subspace, hence $\widetilde{f}(a) = \sum_{i=0}^{n} \lambda_i(a) f(a_i) \in \mathrm{Med}\big(f(a), \widetilde{f}(a)\big)$, hence $\widetilde{f}(a) = f(a)$, hence $\widetilde{f}$ indeed extends f. Since isometries of an affine space are affine maps by Exercise 7.9, $\widetilde{f}$ is affine, and it is the only affine map of $\mathrm{Aff}\,(a_0,\dots,a_n)$ taking the values $f(a_i)$ at points a_i, hence the uniqueness.

Exercice 7.15. *The alternating group $\mathcal{A}_7$ is nonfixating.*
The group $\mathcal{A}_4$ has a subgroup isomorphic to the Klein group,

$$K = \{\mathbf{id}, (12)(34), (13)(24), (14)(23)\},$$

such that:

- ▹ The quotient $\mathcal{A}_4/K$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.
- ▹ Any element of $\mathcal{A}_4 \setminus K$ has at least one fixed point (these are all cycles of order 3).

An eccentric subgroup $G = \langle f, g\rangle$ of $\mathcal{A}_7$ is then obtained by taking two even permutations which act separately on $\{1,2,3,4\}$ and on $\{5,6,7\}$: for f an element of K on $\{1,2,3,4\}$ and the identity on $\{5,6,7\}$, say $f = (12)(34)$, and for g an element of $\mathcal{A}_4 \setminus K$ on $\{1,2,3,4\}$ and an element of $\mathcal{A}_3$ without fixed point on $\{5,6,7\}$, say $g = (123)(567)$. We obtain for G a 12-element semi-direct product of K and $\mathcal{A}_3$.

Any elements of G whose restriction to $\{1,2,3,4\}$ has no fixed point, i.e. is in K, is the identity on $\{5,6,7\}$, so all the elements of G have a fixed point, but f and g have no common fixed point.

Exercise 7.16. *The action of the group* $\mathrm{GL}(3,\mathbb{F}_2)$ *on* $X = \mathbb{F}_2^3 \setminus \{0\}$ *is not fixating.*
Identify X to $\{1,\dots,7\}$ with

$$e_i \to i,\ e_{123} \to 4,\ e_{23} \to 5,\ e_{13} \to 6,\ e_{12} \to 7.$$

With this identification, we obtain $f = (1,2,3)(5,6,7)$ and $g = (1,4)(6,7)$ which are two even permutations.

Since the elements of G are linear and Y generates the vector space $\mathbb{F}_2^3$, if the restriction to Y of $h \in G$ is the identity then h is the identity. In addition f and g send Y in Y so the map which sends h to its restriction to Y, is a one to one morphism from G to $\mathcal{S}_Y$.

As $\operatorname{Fix} f \cap \operatorname{Fix} g = \emptyset$, to prove that the action of G is eccentric, it is enough to check that $\operatorname{Fix} h \neq \emptyset$ for all h in G . Since the elements $h \in G$ induce even permutations acting separately on Y and Z, it suffices to show that if the restriction of h to Y is a double transposition, then its restriction to Z is the identity.

We observe that

$$
\begin{aligned}
(fg)^2 &= (1,2)(3,4)\\
(gf)^2 &= (1.3)(2.4)\\
(fgf)^2 &= (1.4)(2.3)
\end{aligned}
$$

which shows that the action of G is eccentric.

References

[1] R. Antetomaso, Un théorème de point fixe en dimension finie : application aux sous-groupes compacts de $GL_n(\mathbb{R})$, *R.M.S.* **2** (1993-1994) 145–146.

[2] K. Anzai, S. Ishikawa, On common fixed points for several continuous affine mappings, *Pacific J. Math.* **72** (1977) 1–4.

[3] M. Berger, *Géométrie*, Cedic-Nathan, Paris, 1977.

[4] A. Borel, Free subgroups of semi-simple groups, *L'Enseignement Mathématique* **29** (1983) 151–164.

[5] F. Bruhat, J. Tits, Groupes réductifs sur un corps local, *Publ. Math. IHES* **41** (1972) 5–252.

[6] J. W. Cannon, W. J. Floyd, R. Kenyon, W. R. Parry, *Flavors of Geometry*, MSRI Publications, Volume **31**, 1997.

[7] N. Dunford, J. T. Schwartz, *Linear operators, 1: General Theory*, Pure and Applied Mathematics, Academic press, London, New York, Sydney, 1957.

[8] J. Cheeger, D. G. Ebin, *Comparison Theorems in Riemannian Geometry*, North–Holland Publishing Company, Amsterdam, Oxford–American Elsevier Publishing Company, Inc. New York, 1975.

[9] M. Frechet, Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert, *Ann. Math.* **36** (1935) 705–718.

[10] F.G. Frobenius, Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen, *Sitzungsberichte Preuss. Akad. Wiss.*, Berlin (1898), 5001–5015.

[11] P. Jordan, J. von Neuman, On inner products in linear, metric spaces, *Ann. Math.* **36** (1935) 719–723.

[12] S. Kakutani, Two fixed-point theorems concerning bicompact convex sets, *Proc. Imp. Acad. Tokyo* **14** (1938) 242–245.

[13] A. Markov, Quelques théorèmes sur les ensembles abéliens, *Doklady Akad. Nauk. SSSR.* **10** (1936) 311–314.

[14] S. Mazur, S. Ulam, Sur les transformations isométriques d'espaces vectoriels normés, *C. R. Acad. Sci.* **194** (1932) 946–948.

[15] J. W. Morgan, P. B. Shalem, Valuations, Trees and Degenerations of Hyperbolic Structures I, *Ann. Math.* **120** (1984) 401-476.

[16] M. Newman, *Integral matrices*, Academic Press, New-york and London, 1972.

[17] J. G. Ratcliff, *Fundations of Hyperbolic Manifolds*, Graduate Texts in Mathematics **149**, Springer-Verlag, 1994.

[18] W. Rudin, *Functional Analysis*, Mc Graw-Hill Inc., 2ème edition, 1991.

[19] D. Segal, *Polycyclic groups*, Cambridge Tracts Math. 82, Cambridge Univ. Press, 1983.

[20] J.P. Serre, *Arbres, Amalgames,* SL_2, Astérisque, S.M.F., 1977.

[21] J.-P. Serre, *Représentations linéaires des groupes finis*, Hermann, 1988.

[22] J. Väisälä, A proof of the Mazur-Ulam theorem, *Amer. Math. Monthly* **110-7** (2003) 633–635.

[23] S. Wagon, *Banach-Tarski Paradox*, Cambridge University Press, 1985.