On simultaneous inhomogeneous Diophantine approximation

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Abstract. *Let $(\alpha_1, \ldots, \alpha_k)$ be a real k-tuple with $k \geq 2$. We study the Hausdorff dimension of sets of real numbers ξ for which there are infinitely many integer k-tuples $(n_1, \ldots, n_k)$ such that $n_1\alpha_1 + \ldots + n_k\alpha_k - \xi$ is very close to an integer. We further investigate sets of real k-tuples $(\xi_1, \ldots, \xi_k)$ for which there are infinitely many integers n such that $n\alpha_1 - \xi_1, \ldots, n\alpha_k - \xi_k$ are all very close to an integer.*

1. Introduction

Let c and τ be real numbers. A pair (α, ξ) of real numbers is called (c, τ)-approximable if there exist infinitely many integers q such that $\|q\alpha - \xi\| < c\,|q|^{-\tau}$. If α and ξ are irrational real numbers such that ξ is not of the form $\xi = m\alpha + n$ for integers m, n, then a theorem of Minkowski (cf. e.g. Cas1, p. 48) asserts that there exist infinitely many integers q such that

$$\|q\alpha - \xi\| < \frac{1}{4|q|},$$

where $\| \cdot \|$ denotes the distance to the nearest integer. Thus, almost all pairs (α, ξ) are $(1/4, 1)$-approximable. Here and throughout the present paper, 'almost all' refers to the Lebesgue measure on the ambient space.

Dodson [7] proved that the set of pairs (α, ξ) in $\mathbf{R}^2$ which are $(1, \tau)$-approximable for some $\tau > 1$ has Lebesgue measure zero and Hausdorff dimension two. This is the so-called 'doubly metric' statement, and we may as well adopt two further points of view, which are 'singly metric'. The first one consists in considering that ξ is fixed and in looking at the set of real numbers α for which (α, ξ) is $(1, \tau)$-approximable. This is the most classical point of view in inhomogeneous Diophantine approximation, which has been considered by many authors. For instance, Levesley [13], extending a classical result of Jarník [10] and Besicovitch [3] dealing with the homogeneous case $\xi = 0$, proved that, for any $\tau > 1$ and any fixed ξ, the Hausdorff dimension of the set of α such that the pair (α, ξ) is $(1, \tau)$-approximable is equal to $2/(1 + \tau)$. We stress that this value does not depend on ξ.

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The second point of view has been investigated by Bernik and Dodson [2], p. 105. Their results have been improved upon by Schmeling and Troubetskoy [14] and, independently and at the same time (by means of a different approach), by Bugeaud [4] who showed that, for any $\tau > 1$ and any fixed irrational number α, the set

$$\mathcal{T}_\tau(\alpha) := \left\{ \xi \in \mathbf{R} : \|n\alpha - \xi\| < \frac{1}{n^\tau} \quad \text{holds for infinitely many } n \in \mathbf{Z}_{\geq 1} \right\}$$

has Hausdorff dimension $1/\tau$. We stress that this value does not depend on α.

These questions can as well be addressed in a multidimensional setting, by considering either inhomogeneous approximation of linear forms, or simultaneous rational inhomogeneous approximation, or, even, simultaneous inhomogeneous approximation of linear forms. In the 'doubly metric' case and in the first 'singly metric' case mentioned above, satisfactory answers have been given by Dodson [7] and Levesley [13], respectively. However, no multidimensional extension of the statements established in [14] and [4] has been studied up to now, and it is the purpose of the present work to report various results on this question.

Let $k \geq 1$ be an integer and let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a k-tuple of real numbers. For real numbers $v > 1$ and $w > 0$, we set

$$\mathcal{V}_v(\underline{\alpha}) := \left\{ \xi \in \mathbf{R} : \|n_1\alpha_1 + \ldots + n_k\alpha_k - \xi\| < \frac{c}{(\max_{1\leq i \leq k}\{|n_i|\})^v} \right.$$
$$\left. \text{holds for some } c > 0 \text{ and infinitely many } (n_1, \ldots, n_k) \in \mathbf{Z}^k \right\}$$

and

$$\mathcal{W}_w(\underline{\alpha}) := \left\{ (\xi_1, \ldots, \xi_k) \in \mathbf{R}^k : \max_{1 \leq i \leq k} \{\|n\alpha_i - \xi_i\|\} < \frac{c}{|n|^w} \right.$$
$$\left. \text{holds for some } c > 0 \text{ and infinitely many } n \in \mathbf{Z} \right\}.$$

Observe that, for $k = 1$ and $w = v > 1$, both sets do not coincide with the set $\mathcal{T}_v(\alpha)$. Actually, it is much more natural to work with $\mathcal{V}_v((\alpha))$ rather than with $\mathcal{T}_v(\alpha)$, since, for instance, the former set is clearly invariant by rational translations. In addition, there is no reason for considering only the positive integers. However, it is easily seen that both sets have the same Hausdorff dimension, namely $1/v$.

First, we recall a result of Cassels, which describes the 'almost everywhere' situation. According to [5], p. 92, a system $L_j(\underline{x})$ of n linear forms in m variables is *singular* if, for every $\varepsilon > 0$, the set of inequalities

$$\|L_j(\underline{x})\| < \varepsilon X^{-m/n}, \quad |x_i| \leq X$$

has a non-zero integer solution $\underline{x}$ for all X sufficiently large (in terms of ε). Otherwise, the system is called *regular* (see Section 6 below). It follows from the Borel–Cantelli Lemma (see e.g. [5], p. 92) that the set of singular systems has Lebesgue measure zero in the mn-dimensional space. The following result follows from [5], Theorem XIII, p. 93, by taking $n = 1$ or $m = 1$.

Theorem A. *Let $k \geq 1$ be an integer. For almost all real k-tuples $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$, we have*

$$\mathcal{V}_k(\underline{\alpha}) = \mathbf{R} \qquad \text{and} \qquad \mathcal{W}_{1/k}(\underline{\alpha}) = \mathbf{R}^k.$$

In the present work, we are mainly interested in exceptional k-tuples, that is, k-tuples $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ for which either $\mathcal{V}_k(\underline{\alpha})$ is considerably smaller than $\mathbf{R}$, or $\mathcal{W}_{1/k}(\underline{\alpha})$ is considerably smaller than $\mathbf{R}^k$. These are necessarily singular tuples, and since singular k-tuples only exist when $k \geq 2$ ([5], p. 94), k must be greater than or equal to 2. We prove that, for $k = 2$ or 3, there exist real k-tuples $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with $1, \alpha_1, \ldots, \alpha_k$ linearly independent over the rationals and such that the Hausdorff dimension of the set $\mathcal{V}_k(\underline{\alpha})$ is equal to $1/k$. In view of the results from [14] and [4], the dimension cannot be smaller. Furthermore, we prove that, for any arbitrarily small positive w, there exist real k-tuples $\underline{\alpha}$ with $1, \alpha_1, \ldots, \alpha_k$ linearly independent over the rationals and such that the set $\mathcal{W}_w(\underline{\alpha})$ is small, in the sense that its Hausdorff dimension is at most equal to 1. This considerably strengthens and generalizes a result of Khintchine, who proved [11] (see also [5], Theorem XV) that, for $k = 2$ and $w > 0$ arbitrary, there exist pairs (α_1, α_2) such that the set $\mathcal{W}_w(\alpha_1, \alpha_2)$ is not $\mathbf{R}^2$.

The present paper is organized as follows. Section 2 is devoted to the statement of the results, together with some additional remarks. Theorem 1, concerning with inhomogeneous approximation of linear forms, is proved in Section 3. Section 4 is devoted to the proof of Theorem 2, on inhomogeneous simultaneous rational approximation, and Section 5 to that of Theorem 3, which shows that, to some extend, Theorem 2 is best possible. Finally, Theorems 4 and 5, which deal with metric results, are proved in Section 6.

2. Statement of the results

We begin by stating an application of the Hausdorff–Cantelli Lemma, that provides us with upper bounds for the Hausdorff dimension by an easy covering argument.

Proposition 1. *Let $k \geq 2$ be an integer and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a real k-tuple. Let v and w be positive real numbers. We then have*

$$\dim \mathcal{V}_v(\underline{\alpha}) \leq \min\left\{1, \frac{k}{v}\right\} \quad \text{and} \quad \dim \mathcal{W}_w(\underline{\alpha}) \leq \min\left\{k, \frac{1}{w}\right\}.$$

Often, the upper bounds given by the Hausdorff–Cantelli Lemma coincide with the exact value of the Hausdorff dimension, thus Proposition 1 (whose easy proof is postponed to the beginning of Section 6) tells us what are the expected values for the Hausdorff dimensions of the sets $\mathcal{V}_v(\underline{\alpha})$ and $\mathcal{W}_w(\underline{\alpha})$.

We first turn our attention to inhomogeneous approximation of linear forms.

Theorem 1. *Let $k = 2$ or 3. Let $v > 1$ be real. There exist real k-tuples $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ such that $1, \alpha_1, \ldots, \alpha_k$ are linearly independent over the rationals and*

$$\dim \mathcal{V}_v(\underline{\alpha}) = \frac{1}{v}.$$

It follows from the result on the sets $\mathcal{T}_\tau(\alpha)$ recalled in the Introduction that, for any real number $v > 1$ and any irrational number α_1, the Hausdorff dimension of the set $\mathcal{V}_v(\alpha_1)$ is equal to $1/v$. Consequently, the Hausdorff dimension of any set $\mathcal{V}_v(\alpha_1, \ldots, \alpha_k)$ as in Theorem 1 is at least equal to $1/v$.

The assumption that $1, \alpha_1, \ldots, \alpha_k$ are linearly independent over the rationals (which occurs in the statements of Theorems 1 to 3) ensures that the result is non-trivial, since e.g. $\dim \mathcal{V}_v(\alpha, 2\alpha, \ldots, k\alpha) = 1/v$ holds for any $v > 1$, any $k \geq 2$, and any irrational real number α, by the results from [14] and [4].

Theorem 1 shows that there exist real k-tuples $\underline{\alpha}$ for which the upper bound given by Proposition 1 for the Hausdorff dimension of $\mathcal{V}_v(\underline{\alpha})$ is considerably larger than the exact value.

We are convinced that Theorem 1 holds for all integers $k \geq 2$. However, we only succeeded in establishing it for $k = 2$ and for $k = 3$. Our method of proof is quite technical and it presumably works as well for $k \geq 4$.

Remark. An interesting question remains. For any real numbers v and d with $v > 1$ and $1/v < d < k/v$, does there exist a k-tuple $\underline{\alpha}$ for which $\dim \mathcal{V}_v(\underline{\alpha})$ is equal to d?

We now consider inhomogeneous simultaneous rational approximation, and we state a slightly sharper result than announced in the Introduction. For any function $\phi : \mathbf{Z}_{\geq 1} \to \mathbf{R}_{>0}$ and any k-tuple of real numbers $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$, set

$$\mathcal{W}_\phi(\underline{\alpha}) := \Big\{ (\xi_1, \ldots, \xi_k) \in \mathbf{R}^k : \max_{1 \leq i \leq k} \{ \| n\alpha_i - \xi_i \| \} < \phi(|n|) \\ \text{holds for infinitely many } n \in \mathbf{Z} \Big\}.$$

With this notation, for any positive real number w, the union of the sets $\mathcal{W}_{x \mapsto cx^{-w}}(\underline{\alpha})$ taken over the positive real numbers c is simply $\mathcal{W}_w(\underline{\alpha})$.

Theorem 2. *Let $k \geq 2$ be an integer. Let $\phi : \mathbf{Z}_{\geq 1} \to \mathbf{R}_{>0}$ be a function tending to 0 at infinity. There exist real k-tuples $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ such that $1, \alpha_1, \ldots, \alpha_k$ are linearly independent over the rationals and*

$$\dim \mathcal{W}_\phi(\underline{\alpha}) \leq 1.$$

Consequently, there exist k-tuples $\underline{\alpha}$ for which

$$\dim \mathcal{W}_w(\underline{\alpha}) \leq 1$$

holds for every $w > 0$.

We point out that, in Theorem 2, the function ϕ can tend to 0 arbitrarily slowly (in particular, w can be taken arbitrarily close to 0), and that it is not assumed to be non-increasing.

The existence of pairs (α_1, α_2) of real numbers such that $\mathcal{W}_\phi(\alpha_1, \alpha_2)$ is not $\mathbf{R}^2$ is due to Khintchine [11]. It follows from the proof of Theorem XV from [5], combined with metric results of Schmidt on badly approximable pairs [15], that there exist pairs (α_1, α_2) for which the complementary set of $\mathcal{W}_\phi(\alpha_1, \alpha_2)$ has Hausdorff dimension two (in $\mathbf{R}^2$). As far as we are aware, the existence of pairs (α_1, α_2) such that $\mathcal{W}_\phi(\alpha_1, \alpha_2)$ has Lebesgue measure zero was not established up to now. Theorem 2 is even stronger.

We emphasize that the constructions given in the proofs of Theorems 1 and 2 are effective, thus, it is possible to give explicit examples of k-tuples satisfying the conclusions of these theorems. Obviously, such k-tuples are singular. They illustrate how the behaviour of singular systems can differ from the behaviour of regular systems. In the light of Theorem A, Theorems 1 and 2 may appear somehow surprising.

It turns out that the upper bound for the dimension obtained in Theorem 2 is sharp.

Theorem 3. *Let $k \geq 2$ be an integer. For any real number $w > 0$ and any real k-tuple $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with $1, \alpha_1, \ldots, \alpha_k$ linearly independent over the rationals, we have*

$$\dim \mathcal{W}_w(\underline{\alpha}) = \frac{1}{w} \qquad \text{if} \quad w \geq 1$$

and

$$\dim \mathcal{W}_w(\underline{\alpha}) \geq 1 \qquad \text{if} \quad 0 < w \leq 1.$$

Actually, we prove a slightly sharper result than Theorem 3. The proof of Theorem 3 follows from the application of the (easy half of) Frostman's Lemma to a suitable Cantor–type set, constructed inductively and contained in $\mathcal{W}_w(\underline{\alpha})$. This can be viewed as a (difficult) extension of the proof of the main result from [4].

Unlike for the case of linear forms, the Hausdorff dimension of $\mathcal{W}_w(\underline{\alpha})$ does not depend on the k-tuple $\underline{\alpha}$, provided that w is large enough, namely greater than or equal to 1.

We complement Theorems 1 to 3 by two statements valid for almost all k-tuples.

Theorem 4. *Let $k \geq 2$ be an integer and $v \geq k$ be a real number. For almost every real k-tuples $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$, we have*

$$\dim \mathcal{V}_v(\underline{\alpha}) = \frac{k}{v}.$$

In view of Theorem 3, Theorem 5 below is interesting only in the range $1/k < w < 1$.

Theorem 5. *Let $k \geq 2$ be an integer and $w \geq 1/k$ be a real number. For almost every real k-tuples $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$, we have*

$$\dim \mathcal{W}_w(\underline{\alpha}) = \frac{1}{w}.$$

Theorems 4 and 5 are particular cases of a more general statement on systems of linear forms (see Theorem 6 in Section 6). They show that the upper bounds given by Proposition 1 are 'almost always' the exact values of the dimension.

Theorems 1 and 2 state that, for $k \geq 2$, $v > 1$ and $w < 1$, the Hausdorff dimensions of the sets $\mathcal{V}_v(\underline{\alpha})$ and $\mathcal{W}_w(\underline{\alpha})$ depend on $\underline{\alpha}$. This is not the case for $k = 1$, as proved in [14] and in [4]. This is neither the case when we consider the point of view taken by Levesley

[13], who showed that, for any real number ξ, any real k-tuple $(\xi_1, \ldots, \xi_k)$, and for any real numbers $v > k$ and $w > 1/k$, the Hausdorff dimension of the sets

$$\Big\{(\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k : \|n_1\alpha_1 + \ldots + n_k\alpha_k - \xi\| < \frac{1}{(\max_{1\le i\le k}\{|n_i|\})^v} \text{ holds for infinitely many } (n_1, \ldots, n_k) \in \mathbf{Z}^k\Big\}$$

and

$$\Big\{(\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k : \max_{1\le i\le k}\{\|n\alpha_i - \xi_i\|\} < \frac{1}{|n|^w} \text{ holds for infinitely many } n \in \mathbf{Z}\Big\}$$

are equal to $k - 1 + (k+1)/(v+1)$ and $(k+1)/(w+1)$, respectively, independently of the real number ξ and of the real k-tuple $(\xi_1, \ldots, \xi_k)$. To complement this result, we mention that, in the 'doubly metric' case, Dodson [7] established that, for real numbers $v > k$ and $w > 1/k$, the Hausdorff dimension of the sets

$$\Big\{(\alpha_1, \ldots, \alpha_k, \xi) \in \mathbf{R}^{k+1} : \|n_1\alpha_1 + \ldots + n_k\alpha_k - \xi\| < \frac{1}{(\max_{1\le i\le k}\{|n_i|\})^v} \text{ holds for infinitely many } (n_1, \ldots, n_k) \in \mathbf{Z}^k\Big\}$$

and

$$\Big\{(\alpha_1, \ldots, \alpha_k, \xi_1, \ldots, \xi_k) \in \mathbf{R}^k : \max_{1\le i\le k}\{\|n\alpha_i - \xi_i\|\} < \frac{1}{|n|^w} \text{ holds for infinitely many } n \in \mathbf{Z}\Big\}$$

are equal to $k + (k+1)/(v+1)$ and $k + (k+1)/(w+1)$.

Remark. In Theorems 1 and 2, we have given explicit constructions of real k-tuples with non-typical behaviour. A natural extension of our present work consists in studying the same questions, but for *dependent* k-tuples, that is, for instance, for k-tuples $\underline{\alpha} = (\alpha, \alpha^2, \ldots, \alpha^k)$, where α is a transcendental real number. It is known that, for almost all real numbers α, the sets $\mathcal{V}_v(\underline{\alpha})$ and $\mathcal{W}_w(\underline{\alpha})$ satisfy $\dim \mathcal{V}_v(\underline{\alpha}) = k/v$ and $\dim \mathcal{W}_w(\underline{\alpha}) = 1/w$, for real numbers $v \ge k$ and $w \ge 1/k$. Maybe, this statement even holds for *all* $\underline{\alpha}$ with α transcendental. We plan to investigate this problem in a further work.

Notation. Except in Section 5 (that is, for the proof of Theorem 3), we use the following notation. Let $k \ge 2$ be an integer. We endow $\mathbf{R}^k$ with the supremum norm $|\cdot|$, and, for any $\underline{x}$ in $\mathbf{R}^k$, we set

$$||\underline{x}|| = \inf\{|\underline{x} - \underline{n}| : \underline{n} \in \mathbf{Z}^k\}.$$

Clearly, $||\cdot||$ induced a distance on the k-dimensional torus $\mathbf{T}^k$, which we also denote by $||\cdot||$. If Y is a subset of $\mathbf{R}^k$ and $\underline{x}$ is a point in $\mathbf{R}^k$, we denote by $d(\underline{x}, Y)$ the distance from $\underline{x}$ to Y, defined by

$$d(\underline{x}, Y) = \inf\{\underline{y} \in Y : ||\underline{x} - \underline{y}||\}.$$

For the proof of Theorem 3, it appears to be more natural to endow $\mathbf{R}^k$ with the Euclidean norm, as it is specified at the beginning of Section 5.

3. Proof of Theorem 1

Let $v > 1$ be real. We treat only the case $k = 3$, since the case $k = 2$ is much more easy. Presumably, the argument works as well for any integer $k \geq 4$, but this is technically much more complicated. We first prove the existence of triples $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ for which the set

$$\mathcal{V}'_v(\underline{\alpha}) := \Big\{ \xi \in \mathbf{R} : \|n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 - \xi\| < \frac{1}{(\max_{1\leq i\leq 3}\{n_i\})^v}$$
$$\text{for infinitely many } (n_1, n_2, n_3) \in \mathbf{Z}^3_{\geq 1} \Big\}$$

has Hausdorff dimension $1/v$. At the end of this Section, we then briefly explain which adaptations should be made to get Theorem 1.

Throughout the proof of Theorem 1, we work on the circle $[0, 1[$, and we denote by $\{\cdot\}$ the fractional part. In order to simplify the exposition we need to fix some notation.

Notation. *Let a and b be real numbers. If $\{b\} > \{a\}$, then $[a, b]$ denotes the interval $[\{a\}, \{b\}]$, otherwise $[a, b]$ denotes the union $[\{a\}, 1[\cup[0, \{b\}]$.*

The proof of Theorem 1 essentially rests on the following elementary lemma.

Lemma 1. *Let $a_1, \ldots, a_n$ be positive integers. Set $p_n/q_n = [0; a_1, a_2, \ldots, a_n]$. Let τ and $v \geq 1$ be real numbers. Let $\alpha = [0; a_1, \ldots, a_n, a_{n+1}, \ldots]$ be real and set $p_{n+1}/q_{n+1} = [0; a_1, a_2, \ldots, a_{n+1}]$. We then have*

$$\bigcup_{j=0}^{q_{n+1}^{1/(2v)}} [j\alpha + \tau - \varepsilon, j\alpha + \tau + \varepsilon] \subset \bigcup_{j=0}^{q_n} [j\alpha + \tau - 2\varepsilon, j\alpha + \tau + 2\varepsilon]$$

and

$$\bigcup_{j=0}^{q_{n+1}^{1/(2v)}} [-j\alpha + \tau - \varepsilon, -j\alpha + \tau + \varepsilon] \subset \bigcup_{j=0}^{q_n} [-j\alpha + \tau - 2\varepsilon, -j\alpha + \tau + 2\varepsilon]$$

for any real number $\varepsilon \geq q_{n+1}^{-1/2}$.

Proof : The basic idea of the proof of Lemma 1 is the following. Let q_n denote the denominator of the n-th convergent of a real number α. Then the points $\{\alpha\}, \{2\alpha\}, \ldots, \{q_n\alpha\}$

are well distributed in the unit interval I: two consecutive points are distant by at least $1/(3q_n)$ and at most $3/q_n$. If the $(n+1)$-th partial quotient a_{n+1} of α is large, then for any integer q with e.g. $q_n \le q \le q_{n+1}^{1/4}$ the point $\{q\alpha\}$ is very close to some point $\{j\alpha\}$, with $1 \le j \le q_n$. This means that, for ℓ not too small, an interval centered at $\{q\alpha\}$ of length ℓ is contained in the interval centered at $\{j\alpha\}$ of length 2ℓ. We now turn to the proof.

Recall that, by the theory of continued fractions, we have $\|q_n\alpha\| < 1/q_{n+1}$. Let m be an integer with $q_n \le m \le q_{n+1}^{1/(2v)}$. Performing the Euclidean division of m by q_n, there are non-negative integers b and r with $b \le q_{n+1}^{1/(2v)}$, $0 \le r < q_n$, and $m = bq_n + r$. Consequently, we get

$$\|\{m\alpha\} - \{r\alpha\}\| \le q_{n+1}^{1/(2v)} \, \|q_n\alpha\| \le q_{n+1}^{-1+1/(2v)} \le q_{n+1}^{-1/2} \le \varepsilon,$$

since $\varepsilon \ge q_{n+1}^{-1/2}$. The lemma follows. □

We construct inductively the sequences of partial quotients of the real numbers α_1, α_2 and α_3, in such a way that we know an enumerable covering of the set $\mathcal{V}'_v(\alpha_1, \alpha_2, \alpha_3)$. The basic idea consists in building numbers α_1, α_2, and α_3 such that their sequences of denominators of convergents increase very rapidly and are far from each other.

For $j = 1, 2, 3$, we denote by $\alpha_j = [0; a_{1,j}, a_{2,j}, \ldots]$ the continued fraction expansion of α_j, and by $(p_{n,j}/q_{n,j})_{n\ge 1}$ the sequence of its convergents. In the course of the proof, we adopt the following convention. For any set of triples of integers

$$\tilde{X} = \{(j_1, j_2, j_3) : j_i \in J_i, i = 1, 2, 3\},$$

we define the union of close intervals X by

$$X = \bigcup_{(j_1,j_2,j_3)\in X} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - \max_{1\le i\le 3}\{j_i\}^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + \max_{1\le i\le 3}\{j_i\}^{-v}].$$

Assume that $a_{1,1}, \ldots, a_{n,1}$, $a_{1,2}, \ldots, a_{n,2}$ and $a_{1,3}, \ldots, a_{n,3}$ have already been constructed and that we have

$$q_{n,1} > q_{n-1,3}^{2v(n-1)^2}, \quad q_{n,2} > q_{n,1}^{2vn^2}, \quad q_{n,3} > q_{n,2}^{2vn^2}. \qquad (I_n)$$

It will be implicit that we take $a_{n+1,1}$, $a_{n+1,2}$, and $a_{n+1,3}$ large enough in order that (I_{n+1}) holds.

By Lemma 1 applied with j_2 and j_3 fixed, the union of intervals

$$U_n^{(1)} := \bigcup_{\substack{0\le j_1\le j_3^{1/n}, 0\le j_2\le j_3^{1/n} \\ q_{n,1}^n \le j_3 \le q_{n+1,1}^{1/(2v)}}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - 4j_3^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + 4j_3^{-v}]$$

contains

$$V_n^{(1)} := \bigcup_{\substack{0\le j_1\le q_{n+1,1}^{1/(2v)}, 0\le j_2\le j_3^{1/n} \\ q_{n,1}^n \le j_3 \le q_{n+1,1}^{1/(2v)}}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - 2j_3^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + 2j_3^{-v}]$$

and thus as well the union

$$X_n^{(1)} = \bigcup [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - \max_{1\le i\le 3}\{j_i\}^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + \max_{1\le i\le 3}\{j_i\}^{-v}],$$

taken over the set of triples

$$(\tilde{X}_n^{(1)}) \qquad 0 \le j_1 \le q_{n+1,1}^{1/(2v)}, \quad 0 \le j_2 \le j_3^{1/n}, \quad q_{n,1}^n \le j_3 \le q_{n+1,1}^{1/(2v)}.$$

Furthermore, applying Lemma 1 to $V_n^{(1)}$ with j_1 and j_3 fixed, we see that $V_n^{(1)}$ contains

$$\bigcup_{\substack{0\le j_1\le q_{n+1,1}^{1/(2v)}, 0\le j_2\le q_{n,2}^{1/(2v)} \\ q_{n,1}^n\le j_3\le q_{n,2}^{1/(2v)}}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - j_3^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + j_3^{-v}]$$

and thus as well the union

$$Y_n^{(1)} = \bigcup [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - \max_{1\le i\le 3}\{j_i\}^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + \max_{1\le i\le 3}\{j_i\}^{-v}].$$

taken over the set of triples

$$(\tilde{Y}_n^{(1)}) \qquad 0 \le j_1 \le q_{n+1,1}^{1/(2v)}, \quad 0 \le j_2 \le q_{n,2}^{1/(2v)}, \quad q_{n,1}^n \le j_3 \le q_{n,2}^{1/(2v)}.$$

In the same way, interverting the rôles played by j_2 and j_3, we have

$$W_n^{(1)} := \bigcup_{\substack{0\le j_1\le j_2^{1/n}, q_{n,1}^n\le j_2\le q_{n+1,1}^{1/(2v)} \\ 0\le j_3\le j_2^{1/n}}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - 4j_2^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + 4j_2^{-v}]$$

contains the union of intervals

$$Z_n^{(1)} = \bigcup [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - \max_{1\le i\le 3}\{j_i\}^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + \max_{1\le i\le 3}\{j_i\}^{-v}]$$

taken over the set of triples

$$(\tilde{Z}_n^{(1)}) \qquad 0 \le j_1 \le q_{n+1,1}^{1/(2v)}, \quad q_{n,1}^n \le j_2 \le q_{n+1,1}^{1/(2v)}, \quad 0 \le j_3 \le j_2^{1/n}.$$

and the union

$$T_n^{(1)} = \bigcup [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - \max_{1\le i\le 3}\{j_i\}^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + \max_{1\le i\le 3}\{j_i\}^{-v}]$$

taken over the set of triples

$$(\tilde{T}_n^{(1)}) \qquad 0 \le j_1 \le q_{n+1,1}^{1/(2v)}, \quad q_{n,1}^n \le j_2 \le q_{n,3}^{1/(2v)}, \quad 0 \le j_3 \le q_{n,3}^{1/(2v)}.$$

Proceeding as above and letting 2 and 3 play the rôle of the index 1, we further define finite union of intervals $U_n^{(2)}$, $W_n^{(2)}$, $U_n^{(3)}$ and $W_n^{(3)}$ taken, respectively, over the triples defined by

$$0 \le j_1 \le j_3^{1/n}, \quad 0 \le j_2 \le j_3^{1/n}, \quad q_{n,2}^n \le j_3 \le q_{n+1,2}^{1/(2v)},$$

$$q_{n,2}^n \le j_1 \le q_{n+1,2}^{1/(2v)}, \quad 0 \le j_2 \le j_1^{1/n}, \quad 0 \le j_3 \le j_1^{1/n},$$

$$q_{n,3}^n \le j_1 \le q_{n+1,3}^{1/(2v)}, \quad 0 \le j_2 \le j_1^{1/n}, \quad 0 \le j_3 \le j_1^{1/n},$$

and

$$0 \le j_1 \le j_2^{1/n}, \quad q_{n,3}^n \le j_2 \le q_{n+1,3}^{1/(2v)}, \quad 0 \le j_3 \le j_2^{1/n},$$

and of length, respectively, $8j_3^{-v}$, $8j_1^{-v}$, $8j_1^{-v}$, and $8j_2^{-v}$.

Using Lemma 1 as above, they contained, respectively, the unions of intervals corresponding to the sets of triples:

$$(\tilde{X}_n^{(2)}) \qquad 0 \le j_1 \le j_3^{1/n}, \quad 0 \le j_2 \le q_{n+1,2}^{1/(2v)}, \quad q_{n,2}^n \le j_3 \le q_{n+1,2}^{1/(2v)},$$

$$(\tilde{Z}_n^{(2)}) \qquad q_{n,2}^n \le j_1 \le q_{n+1,2}^{1/(2v)}, \quad 0 \le j_2 \le q_{n+1,2}^{1/(2v)}, \quad 0 \le j_3 \le j_1^{1/n},$$

$$(\tilde{X}_n^{(3)}) \qquad q_{n,3}^n \le j_1 \le q_{n+1,3}^{1/(2v)}, \quad 0 \le j_2 \le j_1^{1/n}, \quad 0 \le j_3 \le q_{n+1,3}^{1/(2v)}$$

and

$$(\tilde{Z}_n^{(3)}) \qquad 0 \le j_1 \le j_2^{1/n}, \quad q_{n,3}^n \le j_2 \le q_{n+1,3}^{1/(2v)}, \quad 0 \le j_3 \le q_{n+1,3}^{1/(2v)}.$$

They as well contain, respectively, the intervals corresponding to the sets of triples:

$$(\tilde{Y}_n^{(2)}) \qquad 0 \le j_1 \le q_{n+1,1}^{1/(2v)}, \quad 0 \le j_2 \le q_{n+1,2}^{1/(2v)}, \quad q_{n,2}^n \le j_3 \le q_{n+1,1}^{1/(2v)},$$

$$(\tilde{T}_n^{(2)}) \qquad q_{n,2}^n \le j_1 \le q_{n,3}^{1/(2v)}, \quad 0 \le j_2 \le q_{n+1,2}^{1/(2v)}, \quad 0 \le j_3 \le q_{n,3}^{1/(2v)},$$

$$(\tilde{Y}_n^{(3)}) \qquad q_{n,3}^n \le j_1 \le q_{n+1,2}^{1/(2v)}, \quad 0 \le j_2 \le q_{n+1,2}^{1/(2v)}, \quad 0 \le j_3 \le q_{n+1,3}^{1/(2v)}$$

and

$$(\tilde{T}_n^{(3)}) \qquad 0 \le j_1 \le q_{n+1,1}^{1/(2v)}, \quad q_{n,3}^n \le j_2 \le q_{n+1,1}^{1/(2v)}, \quad 0 \le j_3 \le q_{n+1,3}^{1/(2v)}.$$

Lemma 2. *The union*

$$\bigcup_{\substack{q_{n,2}^n \le j_1 \le q_{n+1,2}^{1/(2v)} \\ j_1 \ge j_2 \ge 0,\, j_1 \ge j_3 \ge 0}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - j_1^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + j_1^{-v}]$$

is included in $Z_n^{(2)} \cup Y_n^{(3)} \cup Y_n^{(2)} \cup T_n^{(2)}$. *The union*

$$\bigcup_{\substack{q_{n+1,2}^{1/(2v)} \leq j_1 \leq q_{n+1,2}^{n+1} \\ j_1 \geq j_2 \geq 0, j_1 \geq j_3 \geq 0}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - j_1^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + j_1^{-v}]$$

is included in $X_n^{(3)} \cup T_{n+1}^{(1)}$. *The union*

$$\bigcup_{\substack{q_{n,1}^{n} \leq j_2 \leq q_{n+1,1}^{1/(2v)} \\ j_2 \geq j_1 \geq 0, j_2 \geq j_3 \geq 0}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - j_2^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + j_2^{-v}]$$

is included in $Z_n^{(1)} \cup T_n^{(3)} \cup Y_n^{(2)} \cup T_n^{(1)}$. *The union*

$$\bigcup_{\substack{q_{n+1,1}^{1/(2v)} \leq j_2 \leq q_{n+1,1}^{n+1} \\ j_2 \geq j_1 \geq 0, j_2 \geq j_3 \geq 0}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - j_2^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + j_2^{-v}]$$

is included in $Y_n^{(3)} \cup Z_n^{(3)}$. *The union*

$$\bigcup_{\substack{q_{n,1}^{n} \leq j_3 \leq q_{n+1,1}^{1/(2v)} \\ j_3 \geq j_2 \geq 0, j_3 \geq j_1 \geq 0}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - j_3^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + j_3^{-v}]$$

is included in $X_n^{(1)} \cup Y_n^{(2)} \cup T_n^{(1)} \cup Y_n^{(1)}$. *The union*

$$\bigcup_{\substack{q_{n+1,1}^{1/(2v)} \leq j_3 \leq q_{n+1,1}^{n+1} \\ j_3 \geq j_2 \geq 0, j_3 \geq j_1 \geq 0}} [j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - j_3^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + j_3^{-v}]$$

is included in $X_n^{(2)} \cup Y_n^{(3)}$.

Proof : We content ourselves to check the first assertion, since the proofs of the five others are similar. Observe that the set of triples (j_1, j_2, j_3) such that $q_{n,2}^n \leq j_1 \leq q_{n+1,2}^{1/(2v)}$, $j_1 \geq j_2 \geq 0$, and $j_1 \geq j_3 \geq 0$ is contained in

$$Z_n^{(2)} \cup \{(j_1, j_2, j_3) : q_{n,2}^n \leq j_1 \leq q_{n+1,2}^{1/(2v)}, 0 \leq j_2 \leq j_1, j_1^{1/n} \leq j_3 \leq j_1\}.$$

The second of these two sets is contained in

$$Y_n^{(3)} \cup \{(j_1, j_2, j_3) : q_{n,2}^n \leq j_1 \leq q_{n,3}^n, 0 \leq j_2 \leq j_1, j_1^{1/n} \leq j_3 \leq j_1\},$$

hence, in the union

$$Y_n^{(3)} \cup Y_n^{(2)} \cup \{(j_1, j_2, j_3) : q_{n,2}^n \leq j_1 \leq q_{n,3}^n, 0 \leq j_2 \leq j_1, j_1^{1/n} \leq j_3 \leq q_{n,2}^n\}.$$

Since $j_1^{1/n} > q_{n,2}^n$ as soon as $j_1 > q_{n,2}^{n^2}$, the last set of triples reduces to the set

$$\{(j_1, j_2, j_3) : q_{n,2}^n \le j_1 \le q_{n,2}^{n^2}, 0 \le j_2 \le j_1, j_1^{1/n} \le j_3 \le q_{n,2}^n\}.$$

Since $q_{n,3} > q_{n,2}^{2vn^2}$, this is included in $T_n^{(2)}$. This completes the proof of the first assertion of the lemma. □

Lemma 3. *Let E be a Borelian subset of $\mathbf{R}^k$ and $\{U_j\}_{j\ge1}$ be a countable family of subsets of $\mathbf{R}^k$ such that*

$$E \subset \{\xi \in \mathbf{R} : \xi \in U_j \text{ for infinitely many } j \ge 1\}.$$

If s is a real number such that

$$\sum_{j\ge1} \big(\mathrm{diam}\ (U_j)\big)^s < +\infty,$$

then $\mathcal{H}^s(E) = 0$ and $\dim E \le s$.

Proof : This is the Hausdorff–Cantelli Lemma, see e.g. [2]. □

Lemma 4. *For $j = 1, 2, 3$, the sets $\limsup U_n^{(j)}$ and $\limsup W_n^{(j)}$ have Hausdorff dimension at most equal to $1/v$.*

Proof : We content ourselves to prove that $\dim\ \limsup U_n^{(1)} \le 1/v$. Let $s > 1/v$ be a real number. For any positive integer n_0, the s-measure of the set $\limsup U_n^{(1)}$ is at most equal to

$$\sum_{n\ge n_0} \sum_{j\ge q_{n,1}^n} (j^{2/n} + 2)\, j^{-vs}.$$

This double sum has the same behaviour as

$$\sum_{n\ge n_0} q_{n,1}^2\, q_{n,1}^{n(1-vs)},$$

which is convergent since $s > 1/v$. It then follows from Lemma 3 that $\dim\ \limsup U_n^{(1)} \le 1/v$. □

Now, we complete the proof of the theorem. Let ξ be in $\mathcal{V}'_v(\alpha_1, \alpha_2, \alpha_3)$ with α_1, α_2 and α_3 as above. Permuting α_1, α_2, and α_3 if needed, ξ belongs to infinitely many intervals

$$[j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 - j_1^{-v}, j_1\alpha_1 + j_2\alpha_2 + j_3\alpha_3 + j_1^{-v}]$$

with $j_1 \ge j_2$ and $j_1 \ge j_3$. In view of Lemma 2, this means that

$$\xi \in \bigcup_{0\le j\le3} \limsup U_n^{(j)} \cup \bigcup_{0\le j\le3} \limsup W_n^{(j)}.$$

The desired result follows from Lemma 4.

To go from the case of the sets $\mathcal{V}'_v$ to that of the sets $\mathcal{V}_v$, we first have to slightly increase the size of the intervals; basically, we replace j_i^{-v} by $j_i^{-v}(\log j_i)$. It is easily seen that Lemma 4 remains true with these slightly larger intervals. Furthermore, to go from the non-negative integers to the rational integers, we simply observe that, as a consequence of the second part of Lemma 1, the above discussion applies not only to $(\alpha_1, \alpha_2, \alpha_3)$, but to any of the eight triples $(\pm\alpha_1, \pm\alpha_2, \pm\alpha_3)$.

To conclude, it only remains for us to prove that 1, α_1, α_2 and α_3 are linearly independent over the rationals. Assume that there exist integers $A_1, \ldots, A_4$, not all zero, such that

$$A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + A_4 = 0.$$

For any positive integer n, we have

$$A_1 q_{n,2} q_{n,1} \alpha_1 + A_2 q_{n,1} q_{n,2} \alpha_2 + A_3 q_{n,1} q_{n,2} \alpha_3 + q_{n,1} q_{n,2} A_4 = 0.$$

Classical results from the theory of continued fractions (see e.g. [16]) imply that

$$\| A_1 q_{n,2} q_{n,1} \alpha_1 \| \ll q_{n,2} q_{n+1,1}^{-1}$$

and

$$\| A_2 q_{n,1} q_{n,2} \alpha_2 \| \ll q_{n,1} q_{n+2,1}^{-1},$$

where, as below, the numerical constants implied in $\ll$ depend only on $A_1, \ldots, A_4$. If $A_3 \neq 0$, we get that

$$\| A_3 q_{n,1} q_{n,2} \alpha_3 \| \ll q_{n,2} q_{n+1,1}^{-1} \ll (|A_3| q_{n,1} q_{n,2})^{-2},$$

for n large enough. Then, by Legendre's theorem (see e.g. [16]), $|A_3| q_{n,1} q_{n,2}$ is the denominator of a convergent to α_3. This is a contradiction, since

$$q_{n-1,3} < |A_3| q_{n,1} q_{n,2} < q_{n,3}$$

holds for n large enough. Consequently, we have $A_3 = 0$ and we argue in a similar way to show that $A_1 = A_2 = A_4 = 0$, in contradiction to our assumption. Thus, we have established that 1, α_1, α_2 and α_3 are linearly independent over the rationals. This completes the proof of the theorem. □

4. Proof of Theorem 2

For sake of simplicity we only do the proof for $k = 3$. This is much more illustrative than the case $k = 2$, and slightly less technical than the general case. At the end of this Section, we indicate which (slight) changes are necessary in order to treat the case $k \geq 4$.

Replacing ϕ by the function $\tilde{\phi}$ defined by $\tilde{\phi}(n) = \sup_{j \geq n} \phi(j)$ if necessary, we may assume without any loss of generality that ϕ is non-increasing.

We aim to construct $\underline{\alpha}$ such that the Hausdorff dimension of

$$\mathcal{W}'_\phi(\underline{\alpha}) = \Big\{ (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \max_{1 \le i \le 3} \{\|n\alpha_i - \xi_i\|\} < \phi(|n|) \\ \text{holds for infinitely many } n \in \mathbf{Z}_{\ge 1} \Big\}$$

is less than or equal to 1. Indeed, since $\mathcal{W}_\phi(\underline{\alpha}) = \mathcal{W}'_\phi(\underline{\alpha}) \cup (-\mathcal{W}'_\phi(\underline{\alpha}))$, this implies the first statement of the theorem.

Let $(f(n))_{n\ge 0}$ be an increasing sequence of integers such that $f(0) = 0$ and set

$$\underline{\alpha} = \left(\sum_{n=0}^{\infty} 2^{-f(3n)}, \sum_{n=0}^{\infty} 3^{-f(3n+1)}, \sum_{n=0}^{\infty} 5^{-f(3n+2)} \right).$$

We construct inductively the sequence $(f(n))_{n\ge 0}$ in order that the corresponding triple $\underline{\alpha}$ satisfies $\dim \mathcal{W}'_\phi(\underline{\alpha}) \le 1$.

In the following, θ denotes an element of $\mathbf{R}^3$ and Θ its projection on the torus $\mathbf{T}^3$. Consider the sequence $(\theta_n)_{n\ge 0}$ of triples defined by

$$\theta_0 = (2^{-f(0)}, 0, 0),\ \theta_1 = (2^{-f(0)}, 3^{-f(1)}, 0),\ \theta_2 = (2^{-f(0)}, 3^{-f(1)}, 5^{-f(2)}),$$

$$\theta_{3n} = \left(\sum_{m=0}^{n} 2^{-f(3m)}, \sum_{m=0}^{n-1} 3^{-f(3m+1)}, \sum_{m=0}^{n-1} 5^{-f(3m+2)} \right),$$

$$\theta_{3n+1} = \left(\sum_{m=0}^{n} 2^{-f(3m)}, \sum_{m=0}^{n} 3^{-f(3m+1)}, \sum_{m=0}^{n-1} 5^{-f(3m+2)} \right),$$

$$\theta_{3n+2} = \left(\sum_{m=0}^{n} 2^{-f(3m)}, \sum_{m=0}^{n} 3^{-f(3m+1)}, \sum_{m=0}^{n} 5^{-f(3m+2)} \right).$$

The sequence $(\theta_n)_{n\ge 0}$ converges to $\underline{\alpha}$.

For any integer $p \ge 2$, the projection Θ_p of θ_p is an element of $\mathbf{T}^3$ of finite order $Q_p = 2^{f(p_0)} 3^{f(p_1)} 5^{f(p_2)}$, where p_i denotes the largest integer $\le p$ of the form $p_i = 3m + i$.

Observe that all the exponents $f(m)$ occurring in $\eta_n := \alpha - \theta_n$ are strictly larger than $f(n)$. Thus, we can choose the sequence $(f(n))_{n\ge 0}$ such that $|\eta_n|$ decreases arbitrarily rapidly to 0. Further, we check that

$$\mathbf{Z}\theta_p + \mathbf{Z}^3 = \mathbf{Z}(2^{-f(p_0)}, 0, 0) + \mathbf{Z}(0, 3^{-f(p_1)}, 0) + \mathbf{Z}(0, 0, 5^{-f(p_2)}) =: \Gamma_p$$

holds for any integer $p \ge 2$. Therefore, the distance of any point x of $\mathbf{R}^3$ to Γ_p goes to zero as p tends to infinity. For every integer q in $\{0, \dots, Q_p\}$, we have

$$d(q\theta_p, q\underline{\alpha}) \le q|\eta_p| \le Q_p|\eta_p|.$$

Thus, the distance of any point x in $\mathbf{R}^3$ to $\mathbf{Z}\underline{\alpha} + \mathbf{Z}^3$ is at most $Q_p|\eta_p| + d(x, \Gamma_p)$. Consequently, $\mathbf{Z}\underline{\alpha} + \mathbf{Z}^3$ is everywhere dense in $\mathbf{R}^3$. This shows that 1 and the three coordinates of $\underline{\alpha}$ are linearly independent over the rationals.

The proof of Theorem 2 rests on the next three lemmas.

Lemma 5. *For any integer $p \geq 2$, the projection G_p of Γ_p in $\mathbf{T}^3$ is contained in a set D_p which is an union of segments of total length*

$$L_p = \begin{cases} 3^{f(p-2)}5^{f(p-1)} & \text{if} \quad p = 3n, \\ 5^{f(p-2)}2^{f(p-1)} & \text{if} \quad p = 3n+1, \\ 2^{f(p-2)}3^{f(p-1)} & \text{if} \quad p = 3n+2. \end{cases}$$

Furthermore, all these segments are of length 1*.*

Proof : We only treat the case $p = 3n$, since the two others are similar. We observe that G_p is included in the projection of the segments $[0,1] \times \{(a\,3^{-f(p-2)}, b\,5^{-f(p-1)})\}$, for $a = 1, \dots, 3^{f(p-2)}$ and $b = 1, \dots, 5^{f(p-1)}$. □

Lemma 6. *The sequence $(f(n))_{n\geq 0}$ may be chosen in such a way that there exists a sequence $(P_n)_{n\geq 0}$ of integers satisfying*

$$Q_n < P_n < Q_{n+1},$$

$$\phi(Q_n) \leq e^{-L_n} \qquad \text{and} \qquad \phi(P_n) \leq e^{-L_{n+1}}, \tag{1}$$

$$Q_n\, 2^{-f(n+1)} \leq \frac{1}{2}\phi(P_{n-1}) \qquad \text{and} \qquad P_n\, 2^{-f(n+1)} \leq \frac{1}{2}\phi(Q_n), \tag{2}$$

for any integer $n \geq 1$.

Proof : We proceed by induction. Assume that $f(0), \dots, f(n)$ and $P_0, \dots, P_{n-1}$ are constructed. Since L_{n+1} depends only on $f(0), \dots, f(n)$, we can choose an integer $P_n > Q_n$ such that $\phi(P_n) \leq e^{-L_{n+1}}$. Taking $f(n+1)$ large, we get (2), $\phi(Q_{n+1}) \leq e^{-L_{n+1}}$, and $Q_{n+1} > P_n$. □

Lemma 7. *Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive real numbers which tends to* 0*. If $(E_n)_{n\geq 1}$ is a sequence of sets such that $\sum_{n\geq 1} \mathcal{H}^s_{\varepsilon_n}(E_n) < \infty$. Then, we have*

$$\mathcal{H}^s\big(\limsup_{n\to+\infty} E_n\big) = 0,$$

and the Hausdorff dimension of $\limsup_{n\to+\infty} E_n$ is at most equal to s.

Proof : This is an easy consequence of the Hausdorff–Cantelli lemma (see e.g. [2], p. 68). Since

$$\mathcal{H}^s_\delta(\limsup_{n\to+\infty} E_n) \leq \sum_{n:\varepsilon_n\leq\delta} \mathcal{H}^s_{\varepsilon_n}(E_n)$$

holds for any $\delta > 0$, we get $\mathcal{H}^s(\limsup_{n\to+\infty} E_n) = 0$. □

For a subset E of $\mathbf{T}^3$ and a positive real number r, we put

$$V(E, r) = \{x \in \mathbf{T}^3 : d(x, E) \leq r\}.$$

Further, for a positive integer q, we set $E_q = \{0, \Theta, 2\Theta, \dots, q\Theta\}$.

Since $Q_n < P_n < Q_{n+1}$ and ϕ is non-increasing, $\mathcal{W}'_\phi(\underline{\alpha})$ is included in

$$\limsup_{n\to\infty} V\big(E_{P_n}, \phi(Q_n)\big) \cup \limsup_{n\to\infty} V\big(E_{Q_{n+1}}, \phi(P_n)\big).$$

To establish that the Hausdorff dimension of $\mathcal{W}'_\phi(\underline{\alpha})$ is at most 1, it is sufficient, by Lemma 7, to prove that, for any $s > 1$, there exist two sequences $(\varepsilon_n)_{n\geq 1}$ and $(\varepsilon'_n)_{n\geq 1}$ which decrease to 0 and are such that the series

$$\sum_n \mathcal{H}^s_{\varepsilon_n}(V(E_{P_n}, \phi(Q_n))) \qquad \text{and} \qquad \sum_n \mathcal{H}^s_{\varepsilon'_n}(V(E_{Q_{n+1}}, \phi(P_n)))$$

converge.

Let $q \leq P_n$ be a positive integer. Performing the Euclidean division of q by Q_n, there exist integers ℓ and a such that $q = \ell Q_n + a$ and $0 \leq a < Q_n$. Therefore,

$$\|q\Theta - a\Theta_n\| = \|(\ell Q_n + a)(\Theta_n + \eta_n) - a\Theta_n\| \leq |(\ell Q_n + a)\eta_n| \leq P_n\,|\eta_n|\,.$$

Furthermore, by (2), we get $P_n\,|\eta_n| \leq \phi(Q_n)$, thus

$$E_{P_n} \subset V\big(\{0, \Theta_n, \ldots, (Q_n - 1)\Theta_n\}, \phi(Q_n)\big).$$

It follows that

$$V(E_{P_n}, \phi(Q_n)) \subset V\big(\{0, \Theta_n, \ldots, (Q_n - 1)\Theta_n\}, 2\phi(Q_n)\big),$$

thus, by Lemma 5, we get

$$V(E_{P_n}, \phi(Q_n)) \subset V(D_n, 2\phi(Q_n)).$$

Let $s > 1$ be a real number and n be a positive integer. Throughout the remaining part of the proof, $\ll$ means that there is an implied absolute positive constant. Setting $\varepsilon_n = \phi(Q_n)$, we have

$$\mathcal{H}^s_{\varepsilon_n}\big(V(D_n, 2\phi(Q_n))\big) \ll \frac{\text{length of } D_n}{\varepsilon_n} \times \varepsilon_n^s \ll L_n \phi(Q_n)^{s-1}.$$

It then follows from (1) that

$$\mathcal{H}^s_{\varepsilon_n}\big(V(E_{P_n}, \phi(Q_n))\big) \ll L_n e^{-(s-1)L_n}.$$

This last inequality shows that the series $\sum_n \mathcal{H}^s_{\varepsilon_n}(V(E_{P_n}, Q_n^{-v}))$ converges. For the other series, set $\varepsilon'_n = \phi(P_n)$. For $q \leq Q_{n+1}$, we have

$$\|q\Theta - q\Theta_{n+1}\| \leq Q_{n+1}\,|\eta_{n+1}|\,,$$

and then, by (2),

$$\|q\Theta - q\Theta_{n+1}\| \leq \phi(P_n).$$

Therefore, we get

$$\begin{aligned} V(E_{Q_{n+1}}, \phi(P_n)) &\subset V(\{0, \Theta_{n+1}, \ldots, Q_{n+1}\Theta_{n+1}\}, 2\phi(P_n)) \\ &\subset V(D_{n+1}, 2\phi(P_n)) \end{aligned}$$

and

$$\mathcal{H}^s_{\varepsilon'_n}(V(E_{Q_{n+1}}, \phi(P_n))) \ll \frac{\text{length of } D_{n+1}}{\varepsilon'_n} \times \varepsilon'^s_n \ll L_{n+1}\phi(P_n)^{s-1}.$$

Finally, by (1), we obtain that

$$\mathcal{H}^s_{\varepsilon'_n}(V(E_{Q_{n+1}}, P_n^{-v})) \ll L_{n+1}e^{-(s-1)L_{n+1}}$$

which shows that the series $\sum_n \mathcal{H}^s_{\varepsilon'_n}(V(E_{P_n}, \phi(Q_n)))$ converges.

Consequently, the Hausdorff dimension of $\mathcal{W}'_\phi(\underline{\alpha})$ is at most 1.

Taking for ϕ a function tending to 0 more slowly than any function $x \mapsto x^{-w}$ with $w > 0$, we get the second statement of the theorem.

To conclude, we briefly explain how to proceed to deal with the general case. For an arbitrary integer $k \geq 2$, we choose the first k prime numbers $p_1 = 2$, $p_2 = 3, \ldots, p_k$ and we set

$$\underline{\alpha} = (\sum_{n=0}^{\infty} p_1^{f(kn)}, \sum_{n=0}^{\infty} p_2^{f(kn+1)}, \ldots, \sum_{n=0}^{\infty} p_k^{f(kn+k-1)}).$$

We define a sequence $(\theta_n)_{n\geq 0}$ in the same way as above, that is, such that for any i and j in $\{0, \ldots, k-1\}$, the j-th coordinate of θ_{kn+i} is

$$\sum_{m=0}^{n} p_j^{f(kn+j-1)} \quad \text{if} \quad j \leq i+1,$$

and

$$\sum_{m=0}^{n-1} p_j^{f(kn+j-1)} \quad \text{if} \quad j > i+1.$$

The proof goes exactly as in the case $k = 3$. The main point is that in Lemma 6 the length L_p depends only on $f(0), \ldots, f(p-1)$.

$\square$

5. Proof of Theorem 3

In this Section, we use the following notation. Let $k \geq 2$ be an integer. We endow $\mathbf{R}^k$ with the Euclidean norm $|\cdot|_2$, and, for any $\underline{x}$ in $\mathbf{R}^k$, we set

$$||\underline{x}||_2 = \inf\{|\underline{x} - \underline{n}|_2 : \underline{n} \in \mathbf{Z}^k\}.$$

Clearly, $||\cdot||_2$ induces a distance on the k-dimensional torus $\mathbf{T}^k$, which we also denote by $||\cdot||_2$. If Y is a subset of $\mathbf{R}^k$ and $\underline{x}$ is a point in $\mathbf{R}^k$, we denote by $d_2(\underline{x}, Y)$ the distance from $\underline{x}$ to Y, defined by

$$d_2(\underline{x}, Y) = \inf\{\underline{y} \in Y : ||\underline{x} - \underline{y}||_2\}.$$

We could as well have worked with the supremum norm, as in the rest of the paper; however, since geometric arguments are applied in the present case, it seems to us more natural to use the Euclidean norm.

Furthermore, throughout this Section, the constants implied by $\asymp$, $\ll$ and $\gg$ depend only on the dimension k.

First, we introduce the notions of best approximation in $\mathbf{R}^k$ and in the torus $\mathbf{T}^k$ (see e.g. [12]). These are needed to establish Lemma 9. As in Section 4, if θ is an element of $\mathbf{R}^k$, we denote by Θ its projection on the torus $\mathbf{T}^k$.

Definition 1. *Let Θ be in $\mathbf{T}^k$. A positive integer q is a best approximation of Θ if we have $\|p\Theta\|_2 > \|q\Theta\|_2$, for every integer p with $0 < p < q$. Let θ be in $\mathbf{R}^k$. A positive integer q is a best approximation of θ if it is a best approximation of Θ, that is, if we have $\|p\Theta\|_2 > \|q\Theta\|_2$, for every integer p with $0 < p < q$.*

Let θ be in $\mathbf{R}^k$. Arranging the set of best approximations of θ in increasing order, we get an increasing sequence $(q_n)_{n\geq 0}$ of positive integers starting with $q_0 = 1$. For any positive integer n, let ε_n be the vector of $\mathbf{R}^k$ and P_n be the integer k-tuple such that

$$q_n\theta = P_n + \varepsilon_n \text{ and } |\varepsilon_n|_2 = \|q_n\Theta\|_2 .$$

Put

$$\theta_n = \theta - \frac{1}{q_n}\,\varepsilon_n = \frac{1}{q_n}P_n \qquad \text{and} \qquad r_n = |\varepsilon_n|_2 = \|q_n\theta\|_2.$$

Then, θ_n is the rational approximation of θ corresponding to the best approximation q_n, and we have obviously $\|q_n\theta_n\|_2 = 0$. We consider the lattice

$$\Lambda_n := \mathbf{Z}^k + \mathbf{Z}\theta_n,$$

which is included in $\mathbf{Q}^k$, since θ_n has rational coordinates. We denote by $\lambda_{1,n}, \dots, \lambda_{k,n}$ the successive minima of Λ_n.

Lemma 8. *The subgroup $\langle\Theta_n\rangle$ of $\mathbf{T}^k$, generated by Θ_n, has exactly q_n elements, that is, $k\Theta_n$ is non-zero for any $k = 1, \dots, q_n - 1$. Furthermore, for any $p = 0, 1, \dots, q_n - 1$, we have*

$$\|p\theta - p\theta_n\| \leq r_n.$$

Moreover, the lattice Λ_n has determinant $1/q_n$ and its first minimum $\lambda_{1,n}$ satisfies

$$2r_{n-1} \geq \lambda_{1,n} \geq r_{n-1}/2.$$

Proof : This follows from Lemme 2 from [6], since, with the notation of [6], the first minimum of Λ_n is equal to $d(0, \langle\Theta_n\rangle \setminus \{0\})$ and therefore to $r(\langle\Theta_n\rangle)$. □

Lemma 9. *The last minimum of Λ_n tends to 0 as n tends to infinity. The product $q_n r_{n-1}$ tends to infinity with n.*

Proof : For any positive integer q, let F_q denote the set $(\{0, \theta, \dots, q\theta\} + \mathbf{Z}^k)$. By Lemma 8, the distance of each point of $F_{q_n - 1}$ to Λ_n is less than r_n, and

$$\max_{x \in \mathbf{R}^k} d_2(x, \Lambda_n) \asymp \lambda_{k,n}.$$

Consequently, we have

$$\lambda_{k,n} \ll \max_{x \in \mathbf{R}^k} d_2(x, F_{q_n - 1}) + r_n,$$

and, since $\mathbf{Z}\theta + \mathbf{Z}^k$ is dense in $\mathbf{R}^k$, we get

$$\lim_{n \to \infty} \max_{x \in \mathbf{R}^k} d_2(x, F_{q_n - 1}) = 0 \text{ and } \lim_{n \to \infty} \lambda_{k,n} = 0, \tag{3}$$

thus the last minimum of Λ_n tends to 0 as n tends to infinity.

By Minkowski's second theorem on successive minima (see e.g. [5], p.156), we have

$$\det \Lambda_n = \frac{1}{q_n} \asymp \lambda_{1,n} \lambda_{2,n} \dots \lambda_{k,n} \le \lambda_{1,n} (\lambda_{k,n})^{k-1}.$$

Combined with Lemma 8, this gives

$$\frac{1}{q_n r_{n-1}} \asymp \frac{1}{q_n \lambda_{1,n}} \le (\lambda_{k,n})^{k-1}. \tag{4}$$

Since $k \ge 2$, it follows from (3) and (4) that $q_n r_{n-1}$ tends to infinity with n, as asserted. □

After these preliminaries, we turn to the proof of Theorem 3. Let $w \ge 1$ be a real number and s be any real number in $]0, 1/w[$. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$ with $1, \alpha_1, \dots, \alpha_k$ linearly independent over the rationals. We shall prove that $\mathcal{W}_w(\underline{\alpha})$ contains a Cantor–type set $\mathcal{K}$ whose Hausdorff dimension is greater than s. By the mass distribution principle (see e.g. [9], p. 24; this is also called the Frostman lemma), it is sufficient to construct a probability measure μ on $\mathcal{K}$ such that

$$\lim_{r \to 0} \frac{\mu(B(x, r))}{r^s} = 0$$

for all x in $\mathcal{K}$. We divide our inductive construction into 6 steps.

Step 1. For any positive integer q, let F_q denote the set $(\{0, \underline{\alpha}, \dots, q\underline{\alpha}\} + \mathbf{Z}^k)$ and put $E_q = F_q \cap [0,1]^k$. Let $(q_n)_{n \ge 0}$ denote the sequence of best approximations of $\underline{\alpha}$ and, for any positive integer n, put

$$\Lambda_n = \mathbf{Z}^k + \frac{\mathbf{Z}}{q_n} P_n,$$

where P_n is the point of the lattice $\mathbf{Z}^k$ for which $q_n \underline{\alpha} - P_n$ is minimal.

For any positive integer n, put

$$A_n = \left\{ x \in [0,1]^k : d_2(x, E_{q_n - 1}) \leq \frac{1}{q_n^w} \right\}.$$

Let $(n_j)_{j \geq 1}$ be an increasing sequence of positive integers, which will be chosen in Step 6, and put

$$\mathcal{K} = \bigcap_{j \geq 1} A_{n_j}.$$

First, we observe that $\mathcal{K}$ is a Cantor–type set. Indeed, the sets A_n are made of closed balls and, by the definition of best approximation, the distance between the centers of two balls composing A_n is at least r_{n-1}. Furthermore, it follows from Lemma 9 that, for n sufficiently large, A_n is a disjoint union of balls of the same radius.

To simplify the notation, for any integer $j \geq 1$, we put

$$Q_j = q_{n_j}, \ \Lambda^j = \Lambda_{n_j}, \ \rho_j = r_{n_j - 1},$$

and

$$K_j = \bigcap_{1 \leq p \leq j} A_{n_p}.$$

Step 2. Since $1, \alpha_1, \ldots, \alpha_k$ are linearly independent over the rationals, the sequence $(m\underline{\alpha})_{m \geq 1}$ is uniformly distributed in the torus $\mathbf{T}^k$. Thus, we may select n_{j+1} sufficiently large in order that each ball of K_j contains a number

$$N \in [c_k Q_{j+1} Q_j^{-wk}/2, \ 2c_k Q_{j+1} Q_j^{-wk}]$$

of balls of K_{j+1}, where c_k denotes the volume of the unit ball of $\mathbf{R}^k$. Dropping some of these balls if necessary, we can suppose that each of the balls of K_j contains exactly $N_j = [c_k Q_{j+1} Q_j^{-wk}/2]$ balls of K_{j+1}. Let $\mathcal{E}_j$ denote the set of centers of the balls composing K_j.

We define inductively a sequence of discrete probability measures $(\mu_j)_{j \geq 1}$ such that, for any $j \geq 1$, we have:

(i) The support of μ_j is equal to $\mathcal{E}_j$;

(ii) All the points of $\mathcal{E}_j$ have the same mass m_j.

Since the μ_j are probability measures, we get

$$m_{j+1} = \frac{m_j}{N_j} \asymp m_j \frac{2}{c_k} Q_j^{wk} Q_{j+1}^{-1}.$$

The sequence $(\mu_j)_{j \geq 1}$ weakly converges to a probability measure μ whose support is contained in $\mathcal{K}$.

Step 3. Let x be in $\mathcal{K}$. Let $j \geq 1$ be an integer. We wish to estimate $\mu(B(x,r))\, r^{-s}$ for r in $[\frac{1}{2} Q_{j+1}^{-w}, \frac{1}{2} Q_j^{-w}]$. There is a point P_j in $E_{Q_j - 1}$ such that the ball $B_j = B(P_j, Q_j^{-w})$ of

K_j contains x. Call $\mathcal{C}_j$ the set of centers of balls of K_{j+1} included in B_j. Since all balls of K_{j+1} have the same radius Q_{j+1}^{-w}, we have

$$\mu(B(x,r)) \le \mu_{j+1}(B(x, r + Q_{j+1}^{-w})).$$

Therefore, in order to estimate $\mu(B(x,r))$, it is sufficient to count the number of points of $\mathcal{C}_j$ lying in $B(x, r + Q_{j+1}^{-w})$.

We begin with an obvious upper bound. By the definition of ρ_{j+1}, the distance between any two points of $E_{Q_{j+1}-1}$ is at least ρ_{j+1} and since $\mathcal{C}_j$ is contained in $E_{Q_{j+1}-1}$, we get

$$\text{card } B(x,r) \cap \mathcal{C}_j \ll \max\left\{1, \left(\frac{r}{\rho_{j+1}}\right)^k\right\}. \tag{5}$$

When r/ρ_{j+1} is large, the previous estimate is useless and a sharper upper bound is required. Since the set $E_{Q_{j+1}-1}$ is close to the lattice Λ^{j+1}, we begin by counting the points of $\Lambda^{j+1} \cap B(x,r)$.

Step 4. Let $(e_1, \ldots, e_k)$ be a reduced basis of Λ^{j+1}. By 'reduced', we mean (see [1]) that the following two properties hold true:

$$|e_1|_2 \le \ldots \le |e_k|_2\,,$$

and, for $i = 1, \ldots, k$,

$$\sin(\angle(e_i, V_i)) \ge (\sqrt{3}/2)^k,$$

where V_i denotes the vector subspace spanned by the $k-1$ vectors $e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_k$. This last inequality implies that, for any real numbers $t_1, \ldots, t_k$, we have

$$\left|\sum_{i=1}^{k} t_i e_i\right|_2 \ge (\sqrt{3}/2)^k \max\{|t_i| \cdot |e_i|_2 : i = 1, \ldots, k\}. \tag{6}$$

Let $y = \sum_{i=1}^{k} a_i e_i$ be in $\Lambda^{j+1} \cap B(0,r)$. Assume first that there exists i with $1 \le i \le k-1$ such that r is in $[(\sqrt{3}/2)^k\,|e_i|_2\,, (\sqrt{3}/2)^k\,|e_{i+1}|_2\,[$. Then, by (6), we have

$$|a_j| \le \frac{r}{(\sqrt{3}/2)^k\,|e_j|_2}, \quad j = 1, \ldots, i$$

and, since the a_j's are integers,

$$a_j = 0, \quad j = i+1, \ldots, k.$$

Therefore, we get

$$\text{card } \Lambda^{j+1} \cap B(0,r) \ll \frac{r^i}{|e_1|_2 \ldots |e_i|_2}. \tag{7}$$

Assume now that $r \ge (\sqrt{3}/2)^k \, |e_k|_2$. Then, we get the upper bound

$$\text{card } \Lambda^{j+1} \cap B(0,r) \ll \frac{r^k}{|e_1|_2 \ldots |e_k|_2} \ll \frac{r^k}{\det \Lambda^{j+1}} = r^k Q_{j+1}, \tag{8}$$

by Lemma 8. Furthermore, we observe that, for any x in $\mathbf{R}^k$, we have

$$\text{card } \Lambda^{j+1} \cap B(z,r) \le \text{card } \Lambda^{j+1} \cap B(0,2r).$$

Step 5. Thanks to (5), (7) and (8), we are now able to bound $\mu(B(x,r))\, r^{-s}$ from above. We distinguish three cases.

Assume first that r is in $[1/2Q_{j+1}^{-w},\ |e_1|_2]$. With the 'obvious' estimate (5), we get

$$\begin{aligned}\frac{\mu(B(x,r))}{r^s} &\le \frac{\mu_{j+1}(B(x, r + Q_{j+1}^{-w}))}{r^s} \\ &\ll m_{j+1} \max\left\{1, \left(\frac{2r}{|e_1|_2}\right)^k\right\} r^{-s} \\ &\ll m_{j+1} r^{-s} \ll m_{j+1} Q_{j+1}^{sw} \ll m_j Q_j^{wk} Q_{j+1}^{ws-1}.\end{aligned}$$

Assume now that $r \ge |e_1|_2$ and that r is in $[(\sqrt{3}/2)^k |e_i|_2/6,\ (\sqrt{3}/2)^k |e_{i+1}|_2/6]$, for some $i = 1, \ldots, k-1$. Then, we infer from (7) that

$$\begin{aligned}\frac{\mu(B(x,r))}{r^s} &\le \frac{\mu_{j+1}(B(x, r + Q_{j+1}^{-w}))}{r^s} \\ &\ll \frac{m_{j+1} \text{card } \Lambda^{j+1} \cap (B(x, r + Q_{j+1}^{-w} + |e_1|_2)}{r^s} \\ &\ll \frac{m_{j+1}}{r^s} \frac{r^i}{|e_1|_2 \ldots |e_i|_2} = \frac{m_{j+1}}{r^s} \frac{r^i |e_{i+1}|_2 \ldots |e_k|_2}{|e_1|_2 \ldots |e_k|_2}.\end{aligned} \tag{9}$$

Since we have $|e_1|_2 \ldots |e_k|_2 \ge \det \Lambda^{j+1} = 1/Q_{j+1}$, we get from (9) that

$$\begin{aligned}\frac{\mu(B(x,r))}{r^s} &\ll m_{j+1} r^{i-s} Q_{j+1} |e_{i+1}|_2 \ldots |e_k|_2 \\ &\ll m_j \frac{Q_j^{wk}}{Q_{j+1}} |e_{i+1}|_2^{i-s} Q_{j+1} |e_{i+1}|_2 \ldots |e_k|_2 \\ &\ll m_j Q_j^{wk} |e_k|_2^{k-s} \\ &\ll m_j Q_j^{wk} \times (\text{last minimum of } \Lambda^{j+1})^{k-s}.\end{aligned}$$

Indeed, for each i, $|e_i|_2$ is greater than or equal to the i-th minimum and, since the basis is reduced, the product of the norms of the vectors of the basis is of the same order of size as the determinant of Λ^{j+1} and, thus, as the product of the minima. It follows that, for each i, $|e_i|_2$ is of the same order of size as the i-th minimum.

Finally, if r is in $[(\sqrt{3}/2)^k)|e_k|_2/6,\ Q_j^{-w}]$, we get from (8) that

$$\begin{aligned}
\frac{\mu(B(x,r))}{r^s} &\le \frac{\mu_{j+1}(B(x, r + Q_{j+1}^{-w})}{r^s} \\
&\le \frac{m_{j+1}\text{card } \Lambda^{j+1} \cap (B(x, r + Q_{j+1}^{-w} + |e_1|_2)}{r^s} \\
&\ll \frac{m_{j+1}}{r^s} r^k Q_{j+1} \ll m_j \frac{Q_j^{wk}}{Q_{j+1}} r^{k-s} Q_{j+1} \\
&\ll m_j Q_j^{wk} (Q_j^{-w})^{k-s} \ll m_{j-1} Q_{j-1}^{wk} Q_j^{sw-1}.
\end{aligned}$$

Step 6. To conclude, it is sufficient to define inductively the sequence $(n_j)_{j\ge1}$ such that:
- the uniform distribution condition stated in Step 2 holds;
- the sequence $(m_j Q_j^{wk} Q_{j+1}^{ws-1})_j$ tends to zero as j goes to infinity;
- the sequence $(m_j Q_j^{wk}(\text{last minimum of } \Lambda^{j+1})^{k-s})_j$ tends to zero as j goes to infinity.

This is possible since we have $ws - 1 < 0$ and thanks to Lemma 9. Then, the results obtained in Step 5 show that, for any x in $\mathcal{K}$, we have

$$\lim_{r\to0} \frac{\mu(B(x,r))}{r^s} = 0.$$

Taking s arbitrarily close to $1/w$, this proves that the Hausdorff dimension of $\mathcal{W}_w(\underline{\alpha})$ is at least equal to $1/w$. Theorem 3 now follows from Proposition 1. □

6. Proof of Proposition 1 and Theorems 4 and 5

Theorems 4 and 5 are particular cases of a more general result on systems of linear forms. For $u > 0$ and A in $M_{n,m}(\mathbf{R})$, set

$$\mathcal{U}_u(A) = \left\{ \underline{\xi} \in \mathbf{R}^n : \|A\underline{x} - \underline{\xi}\| \le \frac{1}{|\underline{x}|^u} \text{ holds for infinitely many } \underline{x} \text{ in } \mathbf{Z}^m \right\},$$

where $|\underline{x}|$ denotes the maximum of the absolute values of the entries of the integer m-tuple $\underline{x}$, and $A\underline{x}$ is the (usual) product of the matrix A by the column vector $\underline{x}$.

This Section is devoted to the proof of the following assertion.

Theorem 6. *Let n, m be positive integers and $u \ge m/n$ be a real number. Then for almost all A in $M_{n,m}(\mathbf{R})$, we have*

$$\dim \mathcal{U}_u(A) = \frac{m}{u}.$$

Theorems 4 and 5 follow straightforwardly from Theorem 6. Actually, we prove a slightly stronger result, since there is no positive constant c involved in the definition of $\mathcal{U}_u(A)$ (in comparison with $\mathcal{V}_v(\underline{\alpha})$ and $\mathcal{W}_w(\underline{\alpha})$).

Let A be in $M_{n,m}(\mathbf{R})$. The Hausdorff–Cantelli Lemma easily yields the upper bound $\dim \mathcal{U}_u(A) \le m/u$. Indeed, for any positive integer n, we set

$$E_n = \bigcup_{\substack{|\underline{x}|=n \\ \underline{x}\in\mathbf{Z}^m}} \left\{ \underline{\xi} \in \mathbf{R}^n : \|A\underline{x} - \underline{\xi}\| \le \frac{1}{n^u} \right\} \cap [0,1]^n$$

and $\varepsilon_n = n^{-u}$. Let $s > m/u$ be real. Observe that

$$\mathcal{H}^s_{\varepsilon_n}(E_n) \ll n^{m-1}\, \varepsilon_n^s \ll n^{m-1-us},$$

where the constant implied in $\ll$ only depends on n and m. Since $\mathcal{U}_u(A) \cap [0,1]^n = \limsup_{n\to+\infty} E_n$ and $m-1-us < -1$, we infer from Lemma 7 that

$$\mathcal{H}^s(\mathcal{U}_u(A) \cap [0,1]^n) = 0.$$

Consequently, the Hausdorff dimension of $\mathcal{U}_u(A)$ is at most equal to m/u. The same argument shows that, for any positive real number c, the Hausdorff dimension of the set

$$\left\{ \underline{\xi} \in \mathbf{R}^n : \|A\underline{x} - \underline{\xi}\| \le \frac{c}{|\underline{x}|^u} \text{ holds for infinitely many } \underline{x} \text{ in } \mathbf{Z}^m \right\}$$

is at most equal to m/u. This proves Proposition 1.

However, the reverse inequality is slightly more difficult to obtain. Our proof uses on the one hand a classical result of Cassels [5], asserting that almost all matrices A in $M_{n,m}(\mathbf{R})$ share a same approximation property. On the other hand, we use the notion of *ubiquitous systems*, introduced by Dodson, Rynne and Vickers [8] to get the expected lower bound for the dimension of $\mathcal{U}_u(A)$.

First, we recall some results about Diophantine approximation.

Definition 2. *A matrix A in $M_{n,m}(\mathbf{R})$ is regular if there exists $\delta > 0$ and infinitely many positive integers X such that*

$$\inf\{\|A\underline{x}\| : \underline{x} \in \mathbf{Z}^m,\ \underline{x} \ne 0,\ |\underline{x}| \le X\} \ge \delta\, X^{-m/n}.$$

It follows from the Borel–Cantelli Lemma that almost all matrices are regular (see e.g. [5], p. 92]).

We further need a transference theorem between homogeneous approximation and inhomogeneous approximation.

Theorem B. *Let A be in $M_{n,m}(\mathbf{R})$. Suppose that for every non-zero $\underline{x}$ in $\mathbf{Z}^m$ with $|\underline{x}| \le X_0$ we have $\|A\underline{x}\| > c$. Then, for all $\underline{\xi}$ in $\mathbf{R}^n$, there exists $\underline{x}$ in $\mathbf{Z}^m$ such that*

$$\|A\underline{x} - \underline{\xi}\| \le \frac{1}{2}(X_0^{-m}c^{-n}+1)c \qquad \text{and} \qquad |\underline{x}| \le \frac{1}{2}(X_0^{-m}c^{-n}+1)X_0.$$

Proof : This is [5], Theorem VI, p. 82. □

We shall get from Theorem B that the lower bound $\dim \mathcal{U}_u(A) \geq m/u$ holds for every regular matrix A.

We first recall some facts about ubiquitous systems. Let Ω be a bounded open subset of $\mathbf{R}^n$, $(S_\alpha)_{\alpha\in J}$, be a family of subsets of Ω, $\mu : J \to \mathbf{R}^+$ be a positive function, and $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ be a non-increasing function tending to 0 as x tends to infinity. Finally, set

$$S_\alpha(\rho) = \{\underline{\xi} \in \Omega : d(\underline{\xi}, S_\alpha) < \rho\}$$

and

$$\mathcal{L}(S_\alpha, \mu, \psi) = \{\underline{\xi} \in \Omega : d(\underline{\xi}, S_\alpha) \leq \psi(\mu(\alpha)) \text{ holds for infinitely many } \alpha \text{ in } J\}.$$

In the sequel, we denote by $\mathrm{diam}\, \mathcal{C}$ the diameter of an hypercube $\mathcal{C}$, that is, the supremum of the distance between any two points of $\mathcal{C}$. Assume that the following hypothesis is satisfied. For each j, there exists a Lebesgue measurable subset $E(j)$ and a positive number $\lambda(j)$ such that

$$\lim_{j\to\infty} |\Omega \setminus E(j)| = 0, \quad \lim_{j\to\infty} \lambda(j) = 0,$$

and for any hypercube $\mathcal{C} \subset \Omega$ with $\mathrm{diam}\, \mathcal{C} = \lambda(j)$ and $\frac{1}{2}\mathcal{C} \cap E(j) \neq \emptyset$, there exist a real number d and α in J, with $\mu(\alpha) \leq j$ and $0 \leq d \leq j$, such that for all ρ satisfying $0 < \rho \leq \lambda(j)$, we have

$$|\mathcal{C} \cap S_\alpha(\rho)| \gg \rho^{n-d} (\mathrm{diam}\, \mathcal{C})^d \tag{10}$$

and

$$|\mathcal{C}' \cap \mathcal{C} \cap S_\alpha(\rho)| \ll \rho^{n-d} (\mathrm{diam}\, \mathcal{C}')^d, \tag{11}$$

where $\mathcal{C}'$ is any hypercube in Ω with $\mathrm{diam}\, \mathcal{C}' \leq \lambda(j)$. The system (S_α, μ) is called an ubiquitous system relative to λ. The real number d is called the dimension of (S_α). The following result was proved Dodson, Rynne and Vickers [8].

Theorem C. *Suppose that (S_α, μ) is an ubiquitous system with respect to λ and that $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ is a decreasing function. Let $t = d + \gamma(n-d)$, where*

$$\gamma = \min\left\{1, \limsup_{j\to\infty} \frac{\log \lambda(j)}{\log \psi(j)}\right\}.$$

We then have

$$\dim \mathcal{L}(S_\alpha, \mu, \psi) \geq t.$$

Proof : This is Theorem 1 from [8]. □

We are now ready to prove Theorem 6.

Proof of Theorem 6. Let A be a regular matrix in $M_{n,m}(\mathbf{R})$. There exist $\delta > 0$ and an increasing sequence of integers $(x_p)_{p\geq 0}$ with $x_p \to \infty$, such that for every p we have

$$\inf\{\|A\underline{x}\| : \underline{x} \in \mathbf{Z}^m,\ \underline{x} \neq 0,\ |\underline{x}| \leq x_p\} \geq \delta\, x_p^{-m/n}.$$

By Theorem B, for any $\underline{\xi} \in \mathbf{R}^n$ there exists $\underline{x} \in \mathbf{Z}^m$ such that

$$\begin{aligned} \|A\underline{x} - \underline{\xi}\| \leq& \frac{1}{2}\big(x_p^{-m}\,(\delta\, x_p^{-m/n})^{-n} + 1\big)\delta\, x_p^{-m/n} \\ =& \frac{1}{2}(\delta^{-n} + 1)\delta\, x_p^{-m/n} = \Delta\delta x_p^{-m/n} \end{aligned} \tag{12}$$

and

$$|\underline{x}| \leq \frac{1}{2}(\delta^{-n} + 1)x_p = \Delta x_p \tag{13}$$

where $\Delta = \frac{1}{2}(\delta^{-n} + 1)$. For a positive integer p, set $S_p = \{A\underline{x} : |\underline{x}| \leq \Delta x_p\} + \mathbf{Z}^n$, and $\mu(p) = p$. We claim that (S_p, μ) is an ubiquitous system with respect to $\lambda(p) = \Delta\delta x_p^{-m/n}$ (and $\Omega =]0,1[^n$). Indeed, by (12) and (13), any hypercube $\mathcal{C}$ of diameter at least $\lambda(p) = \Delta\delta x_p^{-m/n}$ contains at least one point of S_p; therefore, we have

$$\big|\mathcal{C} \cap \{\underline{\xi} \in \Omega : d(\underline{\xi}, S_p) < \rho\}\big| \gg \rho^n,$$

for any ρ satisfying $0 < \rho \leq \lambda_p$, and (10) holds with $d = 0$. As for (11), consider the sets

$$T_p(\underline{x}) = A\underline{x} + \left\{A\underline{y} : \underline{y} \in \mathbf{Z}^m,\ \big|\underline{y}\big| \leq \frac{1}{2}x_p\right\} + \mathbf{Z}^n,\ \underline{x} \in \mathbf{Z}^m.$$

There exists an integer a depending only on δ and n such that for all p there exist $\underline{x}_1, \ldots, \underline{x}_a \in \mathbf{Z}^m$ with

$$S_p \subset \cup_{i=1}^{a} T_p(\underline{x}_i).$$

Let $\mathcal{C}'$ be an hypercube with diameter less than $\lambda(p) = \Delta\delta x_p^{-m/n}$. Since the distance between two points of $T_p(\underline{x})$ is at least $\delta x_p^{-m/n}$, the hypercube $\mathcal{C}'$ contains at most Δ^{-n} points of $T_p(\underline{x})$. We get

$$\big|\mathcal{C}' \cap \{\underline{\xi} \in \Omega : d(\underline{\xi}, T_p(\underline{x})) \leq \rho\}\big| \ll \rho^n$$

and therefore

$$\big|\mathcal{C}' \cap \{\underline{\xi} \in \Omega : d(\underline{\xi}, S_p) < \rho\}\big| \ll \rho^n.$$

Consequently, (11) holds for $d = 0$. Let $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ be a non-increasing function such that $\psi(p) = x_p^{-u}$. Theorem C yields the lower bound

$$\begin{aligned} \dim\, \mathcal{L}(S_\alpha, \mu, \psi) &\geq n \min\left\{1, \limsup_{p\to\infty} \frac{\log \lambda(p)}{\log \psi(p)}\right\} \\ &= n \min\left\{1, \limsup_{p\to\infty} \frac{\log \Delta\delta x_p^{-m/n}}{\log x_p^{-u}}\right\} = \min\left\{n, \frac{m}{u}\right\}. \end{aligned}$$

Since we have

$$\mathcal{L}(S_p, \mu, \psi) \subset \mathcal{U}_u(A) \cup (A\mathbf{Z}^m + \mathbf{Z}^n),$$

the proof is finished. □

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