Best simultaneous Diophantine approximations and multidimensional continued fraction expansions

Nicolas Chevallier

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Abstract

The first goal of this paper is to review the properties of the one dimensional continued fraction expansion that can be recovered or partially recovered using only the best approximation property. The second goal is to study some multidimensional continued fraction expansions of the same kind as Lagarias' multidimensional expansion. We complete Lagarias result about convergence and explain how the LLL algorithm can be used to defined such an expansion.

1 Introduction

Given an irrational number θ there exists a unique sequence $a_0 \in \mathbb{Z}$, $a_1 > 0$, $a_2 > 0, ...$ of integers such that the sequence of irreducible fractions

$$\frac{p_0}{q_0} = a_0,\ \frac{p_1}{q_1} = a_0 + \frac{1}{a_1},\ \frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, ...$$

converges to θ. This sequence of fractions, called the *continued fraction expansion* of θ, enjoys many remarkable properties, and since Jacobi's first extension, many tries have been made to define multidimensional generalizations. Most of these generalizations start with one of the three following properties of the continued fraction expansion.

1. The sequence $(a_n)_{n\geq 0}$ can be easily computed from the iterates of the Gauss map $T:\ ]0,1[\to [0,1[,\ x \mapsto \{\frac{1}{x}\}$ where $\{y\} \in [0,1[$ is the fractional part of y.

2. For all $n \in \mathbb{N}$, $\det \begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \pm 1$ (unimodularity property).

3. The set of denominators q_n, $n \geq 0$, is the set of integers $q \geq 1$ such that, $\forall 1 \leq k < q$, $d(kx, \mathbb{Z}) > d(qx, \mathbb{Z})$ (best approximation property).

Property 1 leads to classical multidimensional continued fraction expansions such as Jacobi-Perron's expansion, Brun's expansion, Selmer's expansions....

Poincaré ([Poi]) introduced a geometric viewpoint which enlightens the unimodularity property. Many works use this geometric viewpoint, *e.g.,* Brentjes defined a multidimensional continued fraction expansion of an element θ in $\mathbb{R}^d$ as a sequence of $\mathbb{Z}^{d+1}$-bases, the positive cone of which contains the half-line $\mathbb{R}_+(\theta, 1)$. One basis is deduced from the previous one by adding to one of the basis vectors an integer multiple of another basis vector (see [Bren]).

Fewer works start with Property 3 which leads to best simultaneous Diophantine approximations. Lagarias was the first to study best Diophantine approximations for their own sake. This article is in line of the works of Lagarias and has two goals. The first one is to review positive and negative results about the properties of the one-dimensional continued fraction expansion that can be recovered in higher dimensions using only the best approximation property. The second goal is to explain the links between three objects.

1. Best Diophantine approximations,

2. The action of the diagonal flow

$$g_t = \begin{pmatrix} e^t I_d & 0 \\ 0 & e^{-dt} \end{pmatrix}$$

on the space of lattices $GL(d+1,\mathbb{R})/GL(d+1,\mathbb{Z})$,

3. Multidimensional continued fraction expansions, with a special emphasis on Lagarias' multidimensional continued fraction expansion.

The article is divided into two parts. The first part which is reduced to Section 2 is devoted to best simultaneous Diophantine approximations of an element θ in $\mathbb{R}^d$ and the second part, *i.e.* Sections 3, 4, 5 and 6, is devoted to the results using lattices in the spaces $\mathbb{R}^{d+1}$ to study the Diophantine properties of θ. All the results of the first part are well-known as well as many of the results of the second part. In the second part, we adopt a more general presentation than Lagarias, we complete his convergence result, and we give a multidimensional expansion based on Lagarias' ideas and the LLL algorithm.

In the first part, we choose to include most of the easy proofs. They depend only on the Dirichlet pigeonhole principle and on the triangle inequality but these proofs are dispersed in many papers. In the second part, the new results and the essential results leading to Lagarias' expansion are proven.

At last, we must say that nothing about best Diophantine approximations to a linear form or to a set of linear forms will be found in this paper.

2 Best Diophantine approximations

2.1 Definitions

Let N be a norm on $\mathbb{R}^d$, let $d(.,.)$ denote the distance associated with N and denote by $B(a,r)$ and $B'(a,r)$ the open and closed balls of center a and radius r.

Definition 1 Let $\theta \in \mathbb{R}^d$.
1. A positive integer q is a *best simultaneous Diophantine approximation denominator* of θ (associated with the norm N) if

$$\forall k \in \{1,...,q-1\},\ d(q\theta,\mathbb{Z}^d) < d(k\theta,\mathbb{Z}^d).$$

2. An element (P,q) in $\mathbb{Z}^d \times \mathbb{Z}$ is a *best Diophantine approximation vector* of θ if q is a best simultaneous Diophantine approximation denominator of θ and if

$$N(q\theta - P) = d(q\theta,\mathbb{Z}^d).$$

For short, we will always write best Diophantine approximation instead of best simultaneous Diophantine approximation denominator.

If $\theta \notin \mathbb{Q}^d$, the set of best Diophantine approximations of θ is infinite. Ordering this set, we obtain a sequence $q_0 = q_0(\theta) = 1 < q_1 = q_1(\theta) < ... < q_n = q_n(\theta) < ....$ When $d=1$, by the best approximation property, the integers $q_0, q_1, ..., q_n, ...$ are the denominators of the ordinary continued fraction expansion of θ. The only slight difference is that in the ordinary continued fraction expansion it can happen that $q_0 = q_1 = 1$. In this case, the indices are shifted by one.

Notation. We shall always denote by $(q_n = q_n(\theta))_{n\geq 0}$ the sequence of best approximations associated with θ in $\mathbb{R}^d$ by Definition 1. We also denote by $r_n = r_n(\theta)$ the distance from $q_n\theta$ to $\mathbb{Z}^d$, and by $P_n = P_n(\theta)$ the point in $\mathbb{Z}^d$ such $d(q_n\theta, P_n) = r_n$. With these notations, (P_n, q_n) is a best Diophantine approximation vector. The *remainder vector* $q_n\theta - P_n$ is denoted by ε_n.

The most obvious drawback of Definition 1 is that the sequence $(q_n)_{n\geq 0}$ depends on the norm as soon as the dimension is not 1 (see Section 2.4 for an inequality between best Diophantine approximations associated with two different norms). Despite this drawback, a weak form of many properties of the one dimensional continued fraction can be recovered using only Definition 1.

2.1.1 Alternative definitions

Some authors use other definitions of best approximation vectors. These definitions, however, are essentially the same as Definition 1.

For instance, there is a definition using an auxiliary norm $\|.\|_{\mathbb{R}^{d+1}}$ on $\mathbb{R}^{d+1}$. The norms N and $\|.\|_{\mathbb{R}^{d+1}}$ are usually chosen to be the Euclidean norms or the sup norms. A nonzero vector (P,q) in $\mathbb{Z}^d \times \mathbb{N}^*$ is a best approximation vector of θ in $\mathbb{R}^d$ if, for each nonzero vector (A,b) in $\mathbb{Z}^{d+1}$, we have

$$\|(A,b)\|_{\mathbb{R}^{d+1}} < \|(P,q)\|_{\mathbb{R}^{d+1}} \Rightarrow N(b\theta - A) > N(q\theta - P)$$

and

$$\|(A,b)\|_{\mathbb{R}^{d+1}} = \|(P,q)\|_{\mathbb{R}^{d+1}} \Rightarrow N(b\theta - A) \geq N(q\theta - P).$$

Lemma 2 *For all θ in $\mathbb{R}^d$, there exists a positive real number α such that for all $(P,q) \in \mathbb{Z}^d \times \mathbb{N}^*$ with $N(q\theta - P) \leq \alpha$, the two definitions of best approximation vector are equivalent.*

Proof. To compare the two definitions, observe that the sequence of best approximation vectors with respect to the first definition is given by the successive minima of the function

$$\varphi(T) = \min\{N(A - b\theta) : (A,b) \in \mathbb{Z}^d \times \mathbb{N}^*,\ q \leq T\}$$

as the real number T increases to infinity and that the sequence of best approximation vectors with respect to the second definition is given by the successive minima of the function

$$\varphi(T) = \min\{N(A - b\theta) : (A,b) \in \mathbb{Z}^d \times \mathbb{N}^*,\ \|(A,b)\|_{\mathbb{R}^{d+1}} \leq T\}.$$

Therefore, the lemma will be proved if we can find a positive real number α such that

$$\|(A,b)\|_{\mathbb{R}^{d+1}} \leq \|(X,y)\|_{\mathbb{R}^{d+1}} \Leftrightarrow b \leq y$$

for all integer vectors (A,b) and (X,y) with b,y nonnegative and $N(A - b\theta)$, $N(X - y\theta) \leq \alpha$. To prove this property, first, normalize the norm $\|.\|_{\mathbb{R}^{d+1}}$ by $\|(\theta,1)\|_{\mathbb{R}^d} = 1$. Next the equivalence of norms shows that there exists a positive real number α such that $N(X) \leq \alpha$ implies $\|(X,0)\|_{\mathbb{R}^{d+1}} \leq \frac{1}{3}$. Now, since $(X,y) = y(\theta,1) + (X - y\theta, 0)$, the norm $\|(X,y)\|_{\mathbb{R}^{d+1}}$ lies between $|y| - \frac{1}{3}$ and $|y| + \frac{1}{3}$ for each vector (X,y) with $N(X - y\theta) \leq \alpha$. Thus, $\|(A,b)\|_{\mathbb{R}^{d+1}} \leq \|(X,y)\|_{\mathbb{R}^{d+1}} \Leftrightarrow b \leq y$ for all integer vectors (A,b) and (X,y) with b,y nonnegative and $N(A - b\theta)$, $N(X - y\theta) \leq \alpha$. □

It follows that the two definitions give rise to the same sequence of best approximation vectors up to a finite number of terms. The first definition has the advantage of not depending on an auxiliary norm and the denominators clearly depend only on $\theta \bmod \mathbb{Z}^d$.

Another definition is the one given in Brentjes' book ([Br]). Let $h : \mathbb{R}^{d+1} \to \mathbb{R}$ be a linear form such that $h(\theta,1) > 0$ and let N be a norm on $\mathbb{R}^d$. A nonzero vector $(P,q) \in \mathbb{Z}^{d+1}$ is a best approximation vector of θ if for all nonzero (A,b) in $\mathbb{Z}^{d+1}$,

$$h(A,b) < h(P,q) \Rightarrow N(b\theta - A) > N(q\theta - P)$$

and

$$h(A,b) = h(P,q) \Rightarrow N(b\theta - A) \geq N(q\theta - P).$$

On the one hand, Definition 1 corresponds to Brentjes' definition with $h(X,y) = y$. On the other hand it is easy to show that there exists a positive real number α such that both definition are equivalent for the vectors (P,q) in $\mathbb{Z}^{d+1}$ with $N(q\theta - P) \leq \alpha$. Just use the equality $h(X,y) = h(X - y\theta, 0) + yh(\theta,1)$ together with the same way of reasoning as above. Observe that now α depends on $\frac{1}{h(\theta,1)}$.

2.1.2 A few historical points

To our knowledge, C.A. Rogers in 1951 [Rog] was the first to define best Diophantine approximations associated with the sup norm; he noticed that two consecutive remainder vectors $q_n\theta - P_n$ and $q_{n+1}\theta - P_{n+1}$ cannot lie in the same quadrant. This initial work on remainder vectors has been continued by V. T. Sós and G. Szekeres [SóSz], and by N. Moshchevitin [Mosh2].

In "Introduction to Diophantine Approximation" [Cas], J. W. S. Cassels defines the continued fraction expansion of a real number starting with the best approximation property. Then, he derives the unimodularity and constructs the Gauss map using only the best approximation property. As we will see in the next section, this program cannot be realized in dimension ≥ 2.

The study of best Diophantine approximations in higher dimensions actually begins in 1979 with the works of J. C. Lagarias [Lag1,2,3,4,5]. He defines best Diophantine approximations for any norm and studies the unimodularity property, the growth rate of the denominators and their computational complexity. Besides these works, he also defines best Diophantine approximations to a set of linear forms. Later in 1994, he proposes a geodesic multidimensional continued fraction expansion.

Negative results about unimodularity are due both to Lagarias and N. Moshchevitin [Mosh3,4] who disproved a conjecture of Lagarias (see also the survey [Mosh1]).

2.1.3 Best approximations are good approximations

Many authors implicitly use best Diophantine approximations, especially through the following lemma which shows that best Diophantine approximations are indeed good approximations. They are at least of the quality provided by the Dirichlet pigeonhole principle. The inequality in the lemma may be seen as an alternative to the Dirichlet theorem.

Lemma 3 *There exists a constant C_N depending only on the norm N such that for all $\theta \in \mathbb{R}^d$, and for all $n \in \mathbb{N}$,*

$$q_{n+1} r_n^d \leq C_N.$$

Proof. We prove the lemma for the sup norm $N = N_\infty$; the general case is easily deduced from the norms equivalence. In the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, the open balls $B_\infty(k\theta, \frac{r_n}{2})$, $k = 0, ..., q_{n+1} - 1$ are disjoint, therefore the sum of their volumes is less than 1. Since $r_n \leq \frac{1}{2}$, the volume of each of these balls is r_n^d and therefore $q_{n+1} r_n^d \leq 1$. □

Remark. When n is large enough, *i.e.* such that $r_n < d(0, \mathbb{Z}^d \backslash \{0\})$, using the Minkowski convex body theorem, it can be proven that the constant C_N depends only on the volume of the unit ball associated with the norm N.

2.2 Unimodularity and primitiveness

The multidimensional continued fraction expansions based either on a generalized Gauss map or on the unimodularity property are closely related. On the one hand, the classical d-dimensional continued fraction expansions admit geometric definitions using bases of $\mathbb{Z}^{d+1}$. On the other hand, the generalized Gauss maps are piecewise unimodular Möbius transformations and hence their iterates define sequences of bases of $\mathbb{Z}^{d+1}$. Let us see that the relations with best simultaneous Diophantine approximations cannot be as simple. Fix a norm N on $\mathbb{R}^d$. For $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$ and $n \in \mathbb{N}$, set

$$D_n = \begin{pmatrix} p_{n,1} & \cdots & p_{n+d,1} \\ \vdots & \vdots & \vdots \\ p_{n,d} & \cdots & p_{n+d,d} \\ q_n & \cdots & q_{n+d} \end{pmatrix}$$

where the columns of D_n are $d+1$ consecutive best approximation vectors of θ. We would like to know whether $\det D_n = \pm 1$. Note that $\det D_n = \pm 1$ if and only the $d+1$ consecutive best approximation vectors $(P_n, q_n), ..., (P_{n+d}, q_{n+d})$ form a basis of $GL(d+1, \mathbb{Z})$. One can wonder if for $k \leq d+1$, k consecutive best approximation vectors are primitive, *i.e.*, they are $\mathbb{R}$-linearly independent and they form a $\mathbb{Z}$-basis of the intersection of $\mathbb{Z}^{d+1}$ with the vector subspace they generate. It is clear that one best approximation vector is primitive and that two best approximation vectors are never collinear. The only other positive result is:

Lemma 4 *A pair of two consecutive best approximation vectors (P_n, q_n) and (P_{n+1}, q_{n+1}) is always primitive.*

Proof. Suppose there exists $(P, q) = a(P_n, q_n) + b(P_{n+1}, q_{n+1})$ in $\mathbb{Z}^{d+1}$ such that $(a, b) \in]0, 1[^2$. Considering the two points (P, q) and $(P_n, q_n) + (P_{n+1}, q_{n+1}) - (P, q)$, we can suppose that $a + b \leq 1$. Therefore, $0 < q < q_{n+1}$ and $N(q\theta - P) \leq a r_n + b r_{n+1} < r_n$ which shows that q_n and q_{n+1} are not consecutive. □

This lemma has been used inside proofs by several authors, see for instance [Cheu] or [Roy]. It may happen that three consecutive best approximation vectors are linearly dependant, see Lagarias' theorem below. It follows that the above lemma cannot be extended to more than two consecutive best approximation vectors.

In the 2-dimensional case, if $\dim_{\mathbb{Q}}[1, \theta_1, \theta_2] = 3$, there always exist infinitely many integers n such that $\operatorname{rank} D_n = 3$. Indeed, suppose $\det D_n = 0$ for all n large enough. Since two best approximation vectors are never collinear, the subspace spanned by two consecutive best approximation vectors is independent of n for large n. The vector $(\theta, 1)$ is in this subspace F for $(\theta, 1) = \lim_{n\to\infty} \frac{1}{q_n}(p_{n,1}, p_{n,2}, q_n)$. Since F contains two linearly independent integer vectors, $\dim_{\mathbb{Q}}[1, \theta_1, \theta_2] = 2$.

The key argument of the previous way of reasoning is that two best approximation vectors are not collinear. In the 3-dimensional case this argument is not strong enough to prove that if $\det D_n = 0$ for all n large enough, then the space spanned by three consecutive best approximation vectors is independent of n. There is no way to circumvent this problem as shown by the following two results.

Theorem 1 *([Lag4]). For any norm, there exists $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$ such that $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$ and for all integers n there exists an integer k such that*

$$\det D_k = \det D_{k+1} = ... = \det D_{k+n} = 0.$$

Theorem 2 *([Mosh3,4]). Assume N is the sup norm and $d \geq 3$. There exists an uncountable family of $\theta = (\theta_1, ..., \theta_d)$ in $\mathbb{R}^d$ such that $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$ and*

$$\operatorname{rank}(D_n) \leq 3$$

for all n large enough.

The Moshchevitin theorem disproves the following conjecture due to Lagarias.
For all $\theta \in \mathbb{R}^d \backslash \mathbb{Q}^d$ the two properties are equivalent:
- $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] \leq r$,
- *there exists an integer $k_0 = k_0(\theta, N)$ such that for all $k \geq k_0$, $\operatorname{rank}(D_k) \leq r$.*

Lagarias proved that these two properties are equivalent for $r = 2$.

These two negative results show that best simultaneous Diophantine approximations do not define an unimodular multidimensional continued fraction expansion. It is necessary to add intermediate approximations or to delete some of them.

After these bad news, we continue with positive results.

2.3 Periodic expansions

Let $p \geq 1$ be an integer. The positive solution of the equation $x^2 + px - 1 = 0$ is in the interval $[0, 1[$ and it is readily seen that

$$x = \frac{1}{p + x} = \frac{1}{p + \frac{1}{p+x}} ...$$

Thus, $x = [0, p, ..., p, ...]$ and the sequence $(q_n)_{n\geq 0}$ of denominators of the continued fraction expansion of x is such that

$$q_0 = 1,\ q_1 = p,\ q_{n+1} = pq_n + q_{n-1}$$

for all $n \geq 1$. An analogous result holds for best Diophantine approximations in the two-dimensional case.

Theorem 3 *([Hu,Me]). Let $P(x) = x^3 + bx^2 + ax - 1$ be an integer polynomial. Suppose that P has a unique real root β and that ($a \geq 0$ and $0 \leq b \leq a + 1$) or ($b = -1$ and $a \geq 2$). Then there exists a Euclidean norm on $\mathbb{R}^2$ (Rauzy's norm) such that the sequence of best Diophantine approximations of $\theta = (\beta, \beta^2)$ satisfies*

$$q_0 = 1,\ q_1 = a,\ q_2 = a^2 + 1,\ q_{n+3} = aq_{n+2} + bq_{n+1} + q_n.$$

The Lagrange theorem about periodic expansions can also be partially extended to best Diophantine approximations in $\mathbb{R}^2$. We want to find a formulation of periodicity without using the partial quotients. Observe that if the real number x in $[0,1[$ has a periodic expansion $x = [0, a_1, ..., a_n, ...] = [0, \overline{a_1, ..., a_k}]$, then for all positive integers of the form $n = kl+i$ with $i \in \{1, ..., k\}$, one has

$$\begin{aligned} \begin{pmatrix} p_n \\ q_n \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= A^l \begin{pmatrix} p_i \\ q_i \end{pmatrix} \end{aligned}$$

where $\frac{p_n}{q_n}$ are the convergents of x. Thus, the whole sequence of best approximation vectors of x is given by the first k best approximation vectors and the powers of the integer matrix A.

Theorem 4 *([Chev2]). Let $P(x) = x^3 + bx^2 + ax - 1$ be an integer polynomial. Suppose that P has an unique real root β and that $a, b \geq 0$. Then there exists a Euclidean norm on $\mathbb{R}^2$ and a finite number of best approximation vectors of $\theta = (\beta, \beta^2)$, $X_i = (P_i, q_i)$, $i = 1, ..., m$, such that the set*

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ a & 1 & b \end{pmatrix}^n \begin{pmatrix} P_i \\ q_i \end{pmatrix} : n \in \mathbb{N}, i = 1, ..., m \right\}$$

is included in the set of best approximations of θ and is equal to this set up to a finite number of elements.

It is very likely that some similar results hold for some polynomials of higher degrees.

2.4 Growth rate of best Diophantine approximations

2.4.1 Lower bound.

Let θ be in $\mathbb{R}^d \backslash \mathbb{Q}^d$. In the one dimensional case, the recurrence relation $q_{n+1} = a_{n+1} q_n + q_{n-1}$ implies that $q_{n+1} \geq q_n + q_{n-1} \geq 2q_{n-1}$. It follows that the denominator q_n increase at a geometric rate.

In higher dimension, Lagarias has proved that the inequality $q_{n+2^d} \geq q_{n+1} + q_n$ holds for best approximation associated with the sup norm ([Lag 3]). The weaker inequality $q_{n+2^d} \geq 2q_n$ is easy to obtain: by the pigeonhole principle, one of the 2^d "quadrants" of $\mathbb{R}^d$ contains at least two of the remainder vectors $q_{n+k}\theta - P_{n+k}$, $k = 0, ..., 2^d$. The distance between these two vectors $q_{n+k_1}\theta - P_{n+k_1}$ and $q_{n+k_2}\theta - P_{n+k_2}$, is at most r_n, therefore, by definition of the best Diophantine approximations, $q_{n+k_2} - q_{n+k_1} \geq q_n$. □

The following inequality and its nice proof are due to Lagarias.

Theorem 5 *([Lag3]). For any norm on $\mathbb{R}^d$, for all $\theta \in \mathbb{R}^d \backslash \mathbb{Q}^d$, and for all $n \in \mathbb{N}$,*

$$q_{n+2^{d+1}} \geq 2q_{n+1} + q_n.$$

Proof. Assume on the contrary that $q_{n+2^{d+1}} < 2q_{n+1} + q_n$. Among the integers $0, 1, ..., 2^{d+1}$, there are at least two of them, $i < j$, such that

$$(P_{n+i}, q_{n+i}) = (P_{n+j}, q_{n+j}) \bmod 2.$$

The vector $(P, q) = \frac{1}{2}(P_{n+j} - P_{n+i}, q_{n+j} - q_{n+i})$ has integer coordinates and

$$0 < q \leq \frac{1}{2}(q_{n+2^{d+1}} - q_n) < q_{n+1}.$$

But,

$$\begin{aligned} N(P - q\theta) &\leq \frac{1}{2}(N(P_{n+j} - q_{n+j}\theta) + N(P_{n+i} - q_{n+i}\theta)) \\ &\leq \frac{1}{2}(r_{n+1} + r_n) < r_n \end{aligned}$$

which contradicts the definition of q_{n+1}. □

The inequality $q_{n+2^{d+1}} \geq 2q_{n+1} + q_n$ implies that

$$\liminf_{n\to\infty} q_n^{1/n} \geq t$$

where t is the maximal root of the equation $t^{2^{d+1}} = 2t + 1$. So the sequence $(q_n)_n$ increase at least at the geometric rate t^n. Note that the lower bound t does not depend on the norm. The optimal value for this lower bound is known only for $d = 1$. See [Mosh5] for some improvements of the value of t.

There are similar results about r_n whose proofs are easy:

Proposition 5 *([Chev5], [Lag1]). For the sup norm, for all $\theta \in \mathbb{R}^d \backslash \mathbb{Q}^d$, and for all $n \in \mathbb{N}$,*

$$r_{n+3^d} \leq \frac{1}{3} r_n.$$

2. For any norm on $\mathbb{R}^d$, for all $\theta \in \mathbb{R}^d \backslash \mathbb{Q}^d$, and for all $n \in \mathbb{N}$,

$$r_{n+3^d} \leq \frac{1}{2} r_n.$$

This proposition allows to compare the growth rate of the sequences of best Diophantine approximations associated with two norms.

Corollary 6 *([Chev5]). Suppose that $\mathbb{R}^d$ is endowed with two norms N and N'. Denote by $(q_n)_{n\geq 0}$ and $(q'_n)_{n\geq 0}$ the best Diophantine approximations associated with the norms N and N'. There exists a constant k depending only on the norms N and N' such that for all $\theta \in \mathbb{R}^d \backslash \mathbb{Q}^d$, and for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that*

$$q_n \leq q'_m \leq q_{n+k}.$$

2.4.2 Upper bound.

In the following, "almost all" always refers to the Lebesgue measure on $\mathbb{R}^d$.

In the one dimensional case, the following theorem due to Levy shows that almost surely, best Diophantine approximations grow at most at the rate of a geometric progression (independently from Levy, Khinchin proved an inequality strong enough to ensure the same geometric growth rate).

Theorem 6 *For almost all θ in $\mathbb{R}$, $\lim_{n\to\infty} \frac{1}{n} \ln q_n = \frac{\pi^2}{12 \ln 2}$.*

In higher dimensions, the following weaker result holds.

Theorem 7 *([Chev3,4]). There exists a constant C_N depending only on the norm N such that, for almost all $\theta \in \mathbb{R}^d$,*

$$\limsup_{n\to\infty} \frac{1}{n} \ln q_n \leq C_N.$$

Actually, this result has been proved for the sup norm or the Euclidean norm, but by Corollary 6, given two norms N_1 and N_2, there exists a constant $C = C(N_1, N_2)$ such that the number of best Diophantine approximations associated with N_1 between two consecutive best Diophantine approximations associated with N_2, is at most C. Thus the geometric growth rates for the norm N_1 and N_2 are equivalent.

In [Chev3], the above theorem is derived from an asymptotic estimate by W. M. Schmidt [Schm] of the number of solutions of some Diophantine inequalities (actually, a less general result due to Susz is enough). In [Chev1] the result is proved for best Diophantine approximations to a set of linear forms. The proof follows a different way because it seems that there is no appropriate generalization of Schmidt's result to simultaneous approximations to a set of linear forms. As in many works, the proof uses the action of the diagonal matrices

$$\begin{pmatrix} e^{(n-d)t} I_d & 0 \\ 0 & e^{-dt} I_{n-d} \end{pmatrix}$$

on the homogeneous space $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ ($n = d+$ the number of linear forms) and some ergodic theory. Up to a renormalization, this diagonal action is the same as the diagonal action used by Lagarias to define his multidimensional expansion (see below).

2.5 Extension of the Borel-Bernstein Theorem

Theorem 8 ***(Borel-Bernstein)****. Let (α_n) be a sequence of positive integers.*
1. If $\sum_{n\geq 1} \frac{1}{\alpha_n} < +\infty$, then for almost all real number $\theta = [a_0, a_1, ..., a_n, ...]$, there are finitely many integers n such that $a_n \geq \alpha_n$.
2. If $\sum_{n\geq 1} \frac{1}{\alpha_n} = +\infty$, then for almost all real number $\theta = [a_0, a_1, ..., a_n, ...]$, there are infinitely many integers n such that $a_n \geq \alpha_n$.

In order to generalize this theorem to higher dimensions, we need to define the partial quotients associated with the best Diophantine approximations. In the one dimensional case, it is well-known that the partial quotients $a_0, ..., a_n, ...$ of a real number can be recovered from the denominators or from the remainders, $r_n(x) = |q_n(x)x - p_n(x)|$:

$$a_n = \lfloor \frac{q_n(x)}{q_{n-1}(x)} \rfloor = \lfloor \frac{r_{n-2}(x)}{r_{n-1}(x)} \rfloor$$

($\lfloor x \rfloor$ denote the integer part of the real number x). This suggests two definitions of the partial quotients of θ in $\mathbb{R}^d$

$$a_n(\theta) = \lfloor \frac{q_n(\theta)}{q_{n-1}(\theta)} \rfloor \text{ or } b_n(\theta) = \lfloor \frac{r_{n-2}(\theta)}{r_{n-1}(\theta)} \rfloor^d$$

(the integer part is not really important). Therefore, we have two natural definitions of partial quotients. The only simple (known) relation between a_n and b_n is:

$$\frac{q_{n+1}}{q_n} \geq \lfloor \frac{r_{n-1}}{r_n} \rfloor$$

for the sup norm. However, the coefficients b_n seem to have a stronger geometrical meaning than the coefficients a_n: each b_n is the quotient of the volume of two balls in the torus $\mathbb{T}^d$. Furthermore, the Borel-Bernstein theorem can be stated in all dimensions with b_n.

Theorem 9 *([Chev3]). Let (α_n) be a nondecreasing sequence of positive real numbers.*
1. If $\sum_{n\geq 1} \frac{1}{\alpha_n} < +\infty$, then for almost all θ in $\mathbb{R}^d$, there are finitely many integers n such that $\left(\frac{r_{n-1}(\theta)}{r_n(\theta)}\right)^d \geq \alpha_n$.
2. If $\sum_{n\geq 1} \frac{1}{\alpha_n} = +\infty$, then for almost all θ in $\mathbb{R}^d$, there are infinitely many integers n such that $\left(\frac{r_{n-1}(\theta)}{r_n(\theta)}\right)^d \geq \alpha_n$.

Note that in the previous theorem it is necessary to assume that the sequence (α_n) is nondecreasing while this assumption is not needed in the Borel-Bernstein Theorem.

2.6 Badly approximable and singular vectors, and lower bound of $q_{n+1}r_n^d$

Recall that a vector θ in $\mathbb{R}^d$ is *badly approximable* if

$$\lim \inf_{q\to\infty} q^{\frac{1}{d}} d(q\theta, \mathbb{Z}^d) > 0,$$

and that θ is *singular* (Khinchin) if

$$\lim_{N\to\infty} N^{1/d} \min\{d(k\theta, \mathbb{Z}^d) : k = 1, ..., N\} = 0.$$

These two definition are easy to formulate with best Diophantine approximations:
- θ is badly approximable if and only if $\liminf_{n\to\infty} q_n r_n^d > 0$.
- θ is singular if and only if $\lim_{n\to\infty} q_{n+1} r_n^d = 0$.

The well-known inequality $q_{n+1} r_n \geq \frac{1}{2}$ shows that singular vectors do not exist when $d = 1$. Their existence in dimension ≥ 2 was proven by Khinchin. Y. Cheung shows that the Hausdorff dimension of singular pairs is $\frac{4}{3}$ ([Cheu]) and one important ingredient of its proof is the above characterization of singular pairs.

In the one dimensional case it is well-known that badly approximable numbers are exactly the real number with bounded partial quotients. The transcription using the partial quotients a_n and b_n introduced in the previous subsection is less clear. If θ is badly approximable, then the sequences of partial quotients (a_n) and (b_n) are bounded but the converse does not hold. For instance one can take a badly approximable real number x and take the pair $\theta = (x, 0)$ which is not badly approximable.

One of the striking differences between the higher dimensions and the dimension one is the existence of singular vectors. As shown by Khinchin, singular vectors $\theta = (\theta_1, ..., \theta_d)$ exist even if we require $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$. This phenomenon implies that an inequality of the form $q_{n+1} r_n^d \geq c > 0$ can hold for all θ with $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$, only if $d = 1$. In fact, most θ in $\mathbb{R}^d$, $d \geq 2$, are both "regular" and "singular":

Theorem 10 *Suppose that $\mathbb{R}^d$ is equipped with the sup norm.*
1. ([Chev3]) If $d \geq 2$, then for almost all θ in $\mathbb{R}^d$,

$$\liminf_{n\to\infty} q_{n+1} r_n^d = 0.$$

2. There exists a constant $c = c(d) > 0$ such that for almost all θ in $\mathbb{R}^d$,

$$\limsup_{n\to\infty} q_{n+1} r_n^d \geq c.$$

One can wonder whether there are other ways to extend the one-dimensional inequality $q_{n+1} r_n \geq \frac{1}{2}$ to higher dimensions. Cheung found such an extension (see next subsection) however the geometrical meaning of this extension is not as clear as a lower bound on $q_{n+1} r_n^d$. In the two-dimensional case, a positive lower bound on

$$q_{n+1} r_n r_{n-1}$$

would have a quite clear geometrical meaning. In [Chev4] it is proven that, for the sup norm and for all $\theta = (\theta_1, \theta_2)$ in $\mathbb{R}^2$ such that $\dim_{\mathbb{Q}}[1, \theta_1, \theta_2] = 3$,

$$q_{n+1} r_n r_{n-1} \geq \frac{1}{100}$$

for infinitely many integers n. Nevertheless, the set of θ in $\mathbb{R}^2$ such that

$$\liminf_{n\to\infty} q_{n+1} r_n r_{n-1} = 0$$

contains a countable intersection of dense open subsets in $\mathbb{R}^2$. In the three dimensional case the situation is even worse, for there exists $\theta = (\theta_1, \theta_2, \theta_3)$ in $\mathbb{R}^d$ with $\dim_{\mathbb{Q}}[1, \theta_1, \theta_2, \theta_3] = 4$ and

$$\lim_{n\to\infty} q_{n+1} r_n r_{n-1} r_{n-2} = 0$$

(see [Chev3]).

2.7 Lattice in $\mathbb{R}^d$ and subgroup associated with a best Diophantine approximation

Let N be a norm on $\mathbb{R}^d$ and let θ be in $\mathbb{R}^d$. In this subsection, we give a few easy properties which give some geometric information about the set

$$E_n = \{0, \theta, ..., (q_n - 1)\theta\} + \mathbb{Z}^d.$$

Let (P_n, q_n) be the best approximation vector associated with q_n and let $\varepsilon_n = q_n\theta - P_n$ be the remainder vector. The rational approximation associated with q_n is

$$\theta_n = \frac{1}{q_n}P_n = \theta - \frac{\varepsilon_n}{q_n}.$$

In the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, we have $q_n\theta_n = 0$, hence the subgroup $\langle\theta_n\rangle$ generated by θ_n is finite. The lifting of this subgroup in $\mathbb{R}^d$ is the lattice

$$\Lambda_n = \mathbb{Z}\theta_n + \mathbb{Z}^d = \{0, \theta_n, ..., (q_n - 1)\theta_n\} + \mathbb{Z}^d.$$

Since q_n is a best Diophantine approximation of θ, θ_n is a good approximation of θ! Therefore the lattice Λ_n is closed to the set E_n, and the study of the geometrical properties of Λ_n should enlighten those of E_n. It is worth noting that in the one-dimensional case, the situation is crystal clear : there is only one subgroup in $\mathbb{T}^1$ with a given cardinality or equivalently, only one lattice in $\mathbb{R}$ with a given determinant. In higher dimensions, the geometry of a lattice is no longer determined by its determinant, the successive minima are needed to know quantitative informations about the geometry of a lattice. The existence of singular θ in dimension ≥ 2 is strongly related to this observation.

The following properties give the connections between Λ_n and E_n. They are easy to prove (see below) and are often used inside proofs. Recall that the *first minimum* $\lambda_1(\Lambda)$ of a lattice Λ is the minimum of the lengths of the nonzero vectors of Λ. Note that $\lambda_1(\Lambda)$ depends on the norm N used to compute the length of the vectors.

P1. *In the torus $\mathbb{T}^d$, the set $\{0, \theta, ..., (q_n - 1)\theta\}$ and the subgroup $\langle\theta_n\rangle$ generated by θ_n are close:*

$$\forall k \in \{0, ..., q_n - 1\},\ d_{\mathbb{T}^d}(k\theta, k\theta_n) \leq r_n(\theta)$$

(hence, the Hausdorff distance between $\{0, \theta, ..., (q_n - 1)\theta\}$ and $\langle\theta_n\rangle$ is smaller than r_n).

P2. *The minimal distance between two points of the subgroup $\langle\theta_n\rangle$, or equivalently between two points in the lattice Λ_n, is of the same order of size as the distances between two points of the set E_n:*

$$\frac{1}{2}r_{n-1}(\theta) \leq \lambda_1(\Lambda_n) \leq 2r_{n-1}(\theta)$$

where $\lambda_1(\Lambda_n)$ is the first minimum of Λ_n. Moreover

$$\lambda_1(\Lambda_n) \leq N(q_{n-1}\theta_n - P_{n-1}) \leq 4\lambda_1(\Lambda_n)$$

P3. $\forall k \in \{1, ..., q_n - 1\},\ k\theta_n \neq 0 \bmod \mathbb{Z}^d$, *hence the cardinality of the subgroup $\langle\theta_n\rangle$ is q_n.*

P4. $\det \Lambda_n = \frac{1}{q_n}$.

P5. ([Cheu]) *Let $\Delta_n = (\delta_{n,1}, ..., \delta_{n,d})$ be the vector of $\mathbb{R}^d$ whose coordinates are the determinants* $\delta_{n,i} = \det\begin{pmatrix} p_{n-1,i} & p_{n,i} \\ q_{n-1} & q_n \end{pmatrix}$ $i = 1, ..., d$. *Then*

$$\frac{1}{2}N(\Delta_n) \leq q_n r_{n-1} \leq 2N(\Delta_n).$$

Proof. **P1**. In the torus $\mathbb{T}^d$, for $k \leq q_n$,

$$d_{\mathbb{T}^d}(k\theta_n, k\theta) = d_{\mathbb{R}^d}(k(\theta + \frac{\varepsilon_n}{q_n}), k\theta + \mathbb{Z}^d) \leq \frac{k}{q_n}N(\varepsilon_n) \leq N(\varepsilon_n) = r_n.$$

P2 and **P3**. Let k be in $\{1, ..., q_n - 1\}$. We want to bound below $d_{\mathbb{T}^d}(k\theta_n, 0)$. We have

$$\begin{aligned} d_{\mathbb{T}^d}(k\theta_n, 0) &= d_{\mathbb{T}^d}(k(\theta - \frac{\varepsilon_n}{q_n}), 0) = d_{\mathbb{R}^d}(k(\theta - \frac{\varepsilon_n}{q_n}), \mathbb{Z}^d) \\ &\geq d_{\mathbb{R}^d}(k\theta, \mathbb{Z}^d) - d_{\mathbb{R}^d}(k\theta, k\theta - k\frac{\varepsilon_n}{q_n}) \\ &\geq r_{n-1} - \frac{k}{q_n}N(\varepsilon_n). \end{aligned}$$

Since in the torus $q_n\theta_n = 0$, the distances $d_{\mathbb{T}^d}((q_n-k)\theta_n, 0)$ and $d_{\mathbb{T}^d}(k\theta_n, 0)$ are equal, hence we can assume that $k \leq \frac{q_n}{2}$. We have then

$$d_{\mathbb{T}^d}(k\theta_n, 0) \geq r_{n-1} - \frac{k}{q_n}N(\varepsilon_n) \geq r_{n-1} - \frac{r_n}{2} \geq \frac{r_{n-1}}{2}.$$

It follows both that $k\theta_n \neq 0 \bmod \mathbb{Z}^d$ for all $k \in \{1, ..., q_n - 1\}$ and that

$$\begin{aligned}\lambda_1(\Lambda_n) &= \min\{N(u) : u \in \Lambda_n \backslash \{0\}\} \\ &= d_{\mathbb{T}^d}(0, \{\theta_n, 2\theta_n, ..., (q_n-1)\theta_n\}) \\ &\geq \frac{r_{n-1}}{2}.\end{aligned}$$

Since $1 \leq q_{n-1} < q_n$, $q_{n-1}\theta_n - P_{n-1}$ is a nonzero vector of Λ_n. Therefore

$$\begin{aligned}\lambda_1(\Lambda_n) &\leq N(q_{n-1}\theta_n - P_{n-1}) \\ &\leq N(q_{n-1}(\theta_n - \theta)) + N(q_{n-1}\theta - P_{n-1}) \\ &\leq r_n + r_{n-1} \leq 2r_{n-1},\end{aligned}$$

which also implies that, $N(q_{n-1}\theta_n - P_{n-1}) \leq 2r_{n-1} \leq 4\lambda_1(\Lambda_n)$.
P4. $\det \Lambda_n = \frac{1}{q_n}$ for $\operatorname{card} \Lambda_n/\mathbb{Z}^d = \operatorname{card}\langle\theta_n\rangle = q_n$.
P5. We have

$$\lambda_1(\Lambda_n) \leq N(q_{n-1}\theta_n - P_{n-1}) = N(q_{n-1}\frac{P_n}{q_n} - P_{n-1}) = \frac{1}{q_n}N(\Delta_n)$$

hence by **P2**, $q_n r_{n-1} \leq 2N(\Delta_n)$.
Moreover

$$\begin{aligned}N(\Delta_n) &= N(q_n P_{n-1} - q_{n-1}P_n) \\ &= q_n q_{n-1} N(\theta_{n-1} - \theta_n) \\ &\leq q_n q_{n-1}(N(\theta_{n-1} - \theta) + N(\theta_n - \theta)) \\ &= q_n q_{n-1}(\frac{r_{n-1}}{q_{n-1}} + \frac{r_n}{q_n}) \leq 2q_n r_{n-1}.\end{aligned}$$

□

Remark. In dimension one, since $\Lambda_n = \frac{1}{q_n}\mathbb{Z}$, the inequalities

$$\lambda_1(\Lambda_n) \leq N(q_{n-1}\theta_n - P_{n-1}) \leq 4\lambda_1(\Lambda_n)$$

become

$$1 \leq |q_n P_{n-1} - q_{n-1}P_n| \leq 4.$$

Therefore, these inequalities may be seen as an extension of the well-known relation $q_n p_{n-1} - q_{n-1}p_n = \pm 1$.
Property **P5** also extends the well-known one dimensional inequality $q_n r_{n-1} \geq \frac{1}{2}$, for in that case, $\Delta_n = \pm 1$. Actually, Cheung gives the slightly better lower bound $(q_n + q_{n-1})r_{n-1} \geq \Delta_n$ which is easily deduced from the above proof.

2.8 Extension of the Legendre theorem

The Legendre theorem asserts that if x is a real number and if $\frac{p}{q}$ is a fraction such that $\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is a convergent of x. Using the lattices introduced in the previous section, this result can be generalized.

Proposition 7 *([Cheu]) Suppose that $\mathbb{R}^d$ is endowed with any norm N. Let $X = (P, q)$ be a primitive element in $\mathbb{Z}^{d+1}$ with $q > 0$. Call Λ_X the lattice*

$$\mathbb{Z}\frac{P}{q} + \mathbb{Z}^d.$$

1. *If θ is in the closed ball $B'(\frac{P}{q}, \frac{\lambda_1(\Lambda_X)}{2q})$ then X is a best approximation vector of θ.*
2. *Conversely, if X is a best approximation vector of θ then θ is in the ball $B'(\frac{P}{q}, \frac{2\lambda_1(\Lambda_X)}{q})$.*

Note that, if $d=1$, then $\lambda_1(\Lambda_X)=\frac{1}{q}$, hence the first point of the proposition is exactly the Legendre theorem. We give the proof of this result which relies only on the triangle inequality.

Proof. 1. Suppose that θ is in the ball $B(\frac{P}{q},\frac{\lambda_1(\Lambda_X)}{2q})$. Then for all $a\in\{1,...,q-1\}$ and all b in $\mathbb{Z}^d$,

$$N(a\theta-b)\geq N(a\frac{P}{q}-b)-aN(\theta-\frac{P}{q}).$$

Since X is primitive, $a\frac{P}{q}-b\neq 0$, and therefore $N(a\frac{P}{q}-b)\geq\lambda_1(\Lambda_X)$. It follows that

$$N(a\theta-b)\geq\lambda_1(\Lambda_X)-a\frac{\lambda_1(\Lambda_X)}{2q}>\frac{\lambda_1(\Lambda_X)}{2}.$$

Using again that $N(\theta-\frac{P}{q})\leq\frac{\lambda_1(\Lambda_X)}{2q}$, we get

$$N(a\theta-b)>N(q\theta-P)$$

therefore q is a best approximation of θ. To see that (P,q) is best approximation vector it remains to see that, if P' is in $\mathbb{Z}^d$, then $N(q\theta-P')\geq N(q\theta-P)$. Now

$$\begin{aligned}N(q\theta-P')&\geq N(q\frac{P}{q}-P')-qN(\theta-\frac{P}{q})\\&\geq N(P'-P)-\frac{\lambda_1(\Lambda_X)}{2}\end{aligned}$$

and if $P'\neq P$ then $N(P'-P)\geq\lambda_1(\Lambda_X)$, thus $N(q\theta-P')\geq\frac{\lambda_1(\Lambda_X)}{2}\geq N(q\theta-P)$.
Conversely, if $(P,q)=(P_n,q_n)$ is a best approximation vector of θ, then by Property **P2**, $N(\theta-\frac{P_n}{q_n})=\frac{1}{q_n}r_n\leq\frac{1}{q_n}r_{n-1}\leq\frac{1}{q_n}2\lambda_1(\Lambda_n)$. □

2.9 Dual lattice

All the result of this subsection can be found in [Chev5]. We assume that $\mathbb{R}^d$ is endowed with the standard Euclidean norm $\|.\|$.

In Subsection 2.7 we have seen that the lattices Λ_n associated with the sequence of best approximations (q_n) of an element θ in $\mathbb{R}^d$ are well-suited to study the sets

$$E_n=\{0,\theta,...,(q_n-1)\theta\}+\mathbb{Z}^d.$$

If we want to study the transition between the sets E_n and E_{n+1}, *i.e.*, the sets

$$\{0,\theta,...,q\theta\}+\mathbb{Z}^d$$

with $q_n\leq q<q_{n+1}$, the dual lattice

$$\Lambda_n^*=\{x\in\mathbb{R}^d:\forall y\in\Lambda_n,\ x.y\in\mathbb{Z}\}$$

(the dot denote the scalar product) provides an important geometric information. Let x_n^* be the shortest vector of Λ_n. The net of hyperplanes $\mathcal{H}_n=\{x\in\mathbb{R}^d:x.x_n\in\mathbb{Z}\}$ is the tightest net of hyperplanes that contains Λ_n. By the property **P1**, the set E_n is close to the net of hyperplanes $\mathcal{H}_n$. The distance d_n between two consecutive hyperplanes of $\mathcal{H}_n$ is $\frac{1}{\|x_n^*\|}$, and we can use Minkowski theorem on minima of a lattice to bound above $\|x_n^*\|$. Recall that the k-th minimum $\lambda_k(\Lambda)$ of a lattice Λ is the infimum of the set of $r\geq 0$ such that the ball $B(0,r)$ contains k linearly independent vectors of Λ. By the Minkowski theorem,

$$\|x_n^*\|^d=\lambda_1(\Lambda_n^*)^d\leq\lambda_1(\Lambda_n^*)...\lambda_d(\Lambda_n^*)\ll\det\Lambda_n^*=q_n,$$

hence

$$\|x_n^*\|\ll q_n^{1/d},\ d_n\gg\frac{1}{q_n^{1/d}}$$

(we have use the standard notation $a \ll b$ meaning that $a \leq Cb$ where C is a constant depending only on certain parameters, here the dimension). Property **P1** shows that the closeness of E_n to $\mathcal{H}_n$ compared with d_n is bounded above by $q_n^{1/d} r_n$ (up to a multiplicative constant).

The vector x_n^* allows to regroup best approximations : either $x_n^* \in \Lambda_{n+1}^*$, or $x_n^* \notin \Lambda_{n+1}^*$. In the two cases described below, we assume that $q_n r_n^d$ is small.

Case 1: $x_n^* \in \Lambda_{n+1}^*$. E_{n+1} is still close to $\mathcal{H}_n$, hence all the points $q\theta$, $q < q_{n+1}$ are close to $\mathcal{H}_n$. More precisely, for all $q < q_{n+1}$,

$$d(q\theta.x_n^*, \mathbb{Z}) \ll q_n^{1/d} r_{n+1}.$$

Case 2: $x_n^* \notin \Lambda_{n+1}^*$. At the time $q = q_{n-1}$ the trajectory $\{0, \theta, ..., q\theta\} + \mathbb{Z}^d$ is close to $\mathcal{H}_n$. It can be proved that when q increases from q_n to q_{n+1}, the points $q\theta + \mathbb{Z}^d$ fill the gap between the hyperplanes of $\mathcal{H}_n$. This gap between the hyperplanes is filled in a very simple way: the trajectory moves away from E_n by small jumps. Indeed, for all $a < q_n$, and $k = 0, ..., [\frac{q_{n+1}}{q_n}]$,

$$\begin{aligned}(kq_n + a)\theta &= a\theta + kq_n(\theta - \theta_n) \\ &\equiv a\theta + k\varepsilon_n \mod \mathbb{Z}^d\end{aligned}$$

where $\varepsilon_n = q_n\theta - P_n$ and $\|\varepsilon_n\| = r_n(\theta)$.

We have seen in the Subsection 2.6 that the one-dimensional inequality $q_{n+1} r_n \geq \frac{1}{2}$ is difficult to extend to higher dimensions. However the use of the shortest vector x_n^* enables to prove a partial extension of this inequality.

Theorem 11 *There exists a positive constant $c(d)$ such that, for all $\theta \in \mathbb{R}^d$ such that $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$, then either $q_n r_n^d(\theta) \geq c(d)$ or $q_{n+1} d(x_n^*.\theta, \mathbb{Z}) \geq c(d)$ for infinitely many n.*

3 Lattices in $\mathbb{R}^{d+1}$ associated with θ in $\mathbb{R}^d$

For each θ in $\mathbb{R}^d$, consider the lattice

$$\Lambda_\theta = M_\theta \mathbb{Z}^{d+1}$$

where

$$M_\theta = \begin{pmatrix} I & -\theta^T \\ 0 & 1 \end{pmatrix} \in SL(d+1, \mathbb{R}).$$

The flow $(g_t)_{t \in \mathbb{R}}$ defined by

$$g_t = \begin{pmatrix} e^t I_d & 0 \\ 0 & e^{-dt} \end{pmatrix} \in SL(d+1, \mathbb{R}),$$

acts on the lattice Λ_θ by left multiplication. Since for $(P, q) \in \mathbb{Z}^d \times \mathbb{Z}$, $g_t M_\theta \begin{pmatrix} P \\ q \end{pmatrix} = \begin{pmatrix} e^t(P - q\theta) \\ e^{-dt} q \end{pmatrix}$, a short vector of $g_t \Lambda_\theta$ provides a good rational approximation of θ.

Some authors consider the action of the matrices

$$h_s = \begin{pmatrix} I_d & 0 \\ 0 & s \end{pmatrix},$$

$s > 0$, instead of the matrices g_t. However, since with $s = e^{-(d+1)t}$, we have $h_s = e^{-t} g_t$, the two lattices $h_s \Lambda_\theta$ and $g_t \Lambda_\theta$ are homothetic. Thus finding short vectors in one of these lattices is equivalent to finding short vectors in the other. The advantage of g_t over h_s is that all the lattices $g_t \Lambda_\theta$, $t \in \mathbb{R}$, are in the space of unimodular lattices $SL(d+1, \mathbb{R})/SL(d+1, \mathbb{Z})$, and that this space has finite Haar measure which allows to use tools from ergodic theory. On the other hand, some calculations are slightly simpler with h_s than with g_t and the map $s > 0 \longmapsto h_s M_\theta M_\theta^T h_s$, can be interpreted as a geodesic in the space of positive definite quadratic forms (see [Lag1]). This is the reason why we shall use both h_s and g_t in what follows.

3.1 Euclidean norms

Assume that $\|.\|_{\mathbb{R}^d}$ and $\|.\|_{\mathbb{R}^{d+1}}$ are the standard Euclidean norms on $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$. The Next lemma connects best approximation vectors of θ with shortest vectors of the lattices $h_s\Lambda_\theta$; it is due to Lagarias [Lag 1].

Lemma 8 *([Lag1]). If $v_s = (P_s - q_s\theta, sq_s)$ is a shortest vector of $h_s\Lambda_\theta$ and if $q_s > 0$, then q_s is a best Diophantine approximation of θ.*

Proof. If q is an integer between 1 and $q_s - 1$ then for all $K \in \mathbb{Z}^d$,

$$\|(k\theta - q, sq)\|_{\mathbb{R}^{d+1}} \geq \|v_s\|_{\mathbb{R}^{d+1}}$$

hence

$$\|K - q\theta\|^2_{\mathbb{R}^d} + q^2s^2 \geq \|P_s - q_s\theta\|^2_{\mathbb{R}^d} + q_s^2s^2$$

and therefore

$$\|K - q\theta\|^2_{\mathbb{R}^d} > \|P_s - q_s\theta\|^2_{\mathbb{R}^d}\,.$$

□

With the hypothesis of the previous lemma, it is easy to prove that if $s > t > 0$ then $q_s \leq q_t$. This lemma is the main observation leading to a weak form of Lagarias multidimensional expansion:

For each $s > 0$, compute the shortest vector (P_s, q_s) of $h_s\Lambda_\theta$. The set of q_s is a subsequence of the sequence of all best Diophantine approximations of θ.

These best Diophantine approximations are called *Hermite* best Diophantine approximations ([Lag1]). We denote by $(h_m)_{m\geq 0}$ the increasing sequence of Hermite best Diophantine approximations and by H_m the corresponding best Diophantine approximation vectors. The subsequence $(h_m)_{m\geq 0}$ is generally a strict subsequence of the sequence of all best Diophantine approximations $(q_n)_{n\geq 0}$, even for $d = 1$ ([Hum]). However, the following properties proved in [Chev1] show that the sequence of Hermite best Diophantine approximations is not a too sparse subsequence of $(q_n)_{n\geq 0}$.

1. *There exists a constant m_0 depending only on the dimension d such that for all θ in $\mathbb{R}^d$ and all integers m, the cardinality of the set of n with*

$$h_m < q_n < h_{m+1}$$

is at most m_0.

2. *There exists a constant C depending only on the dimension d such that for θ in $\mathbb{R}^d$ and all integers m,*

$$h_{m+1}d(h_m\theta, \mathbb{Z}^d)^d \leq C.$$

3. *There exists a positive constant c depending only on the dimension d such that for θ in $\mathbb{R}^d$ and all best Diophantine approximation q_n of θ, there exists a Hermite best Diophantine approximation h_m such that*

$$c\,d(h_m\theta, \mathbb{Z}^d) \leq d(q_n\theta, \mathbb{Z}^d) \leq d(h_m\theta, \mathbb{Z}^d).$$

These three properties can be deduced from a lemma due to Cheung (see the next subsection) together with inequalities that connect best Diophantine approximations associated with two different norms.

If $d = 1$, Hermite proved that for all integers n

$$\operatorname{rank}(H_n, H_{n+1}) = 2$$

(two best approximation vectors are never collinear!).

If $d = 2$, it can happen that

$$\operatorname{rank}(H_n, H_{n+1}, H_{n+2}) < 3$$

([Lag1]). Consequently, the unimodularity property does not hold for the sequence $(H_n)_{n\geq 0}$.

3.2 Cheung's norm.

Y. Cheung introduces a notion of best approximants in a lattice (or even in a discrete set) that is very naturally connected to best approximations vectors. His idea is to work with a sup norm instead of the Euclidean norm. With this norm, the shortest vectors of the lattices $h_s\Lambda_\theta$, $s \in \mathbb{R}$, are more closely related to best approximations vectors of θ than with Euclidean norms. As seen in the previous subsection, with the standard Euclidean norms, shortest vectors of $h_s\Lambda_\theta$ are always associated with a best approximation vectors of θ while best approximations vectors are not always associated with a shortest vectors. With the sup norm introduced by Cheung, both direction works. The only point is to choose adequately the shortest vector of $h_s\Lambda_\theta$ in case of a nonunique shortest vector. This can be done with the help of minimal vectors defined below.

We suppose that $\mathbb{R}^d$ is endowed with a norm N.
Notations. Let $X = (U, v)$ be in $\mathbb{R}^d \times \mathbb{R}$. The *height* of X is $|X| = |v|$ and the box $B'_{\mathbb{R}^d}(0, N(U)) \times [-|X|, |X|]$ is denoted by $B(X)$.

Definition 9 Let Λ be a lattice in $\mathbb{R}^{d+1}$. A vector X in Λ is a *minimal* vector of Λ (with respect to the norm N) if $X = (U, v) \neq 0$ and if the only nonzero vectors in $\Lambda \cap B(X)$ are in the two $(d-1)$-spheres $\{(A, b) \in \mathbb{R}^{d+1} : N(A) = N(U) \text{ and } v = \pm |X|\}$.

When $\mathbb{R}^{d+1}$ is endowed with the norm N_c defined by

$$N_c(U, v) = \max(N(U), |v|),$$

the connection between minimal vectors and shortest vector of a lattice in $\mathbb{R}^{d+1}$ is simple: in a given lattice there is at least one shortest vector of the lattice that is minimal, and if $X = (U, v)$ is minimal in a lattice Λ, then X is a shortest vector of the lattice $g_t\Lambda$ with $t(X) = \frac{1}{d+1} \ln \frac{|v|}{N(U)}$ (if $v = 0$ or $N(U) = 0$, it is to understand that X is a shortest vector of $g_t\Lambda$ when t is in a neighborhood of $-\infty$ or $+\infty$). Moreover if a vector $X = (U, v)$ in Λ is a "robust" shortest, *i.e.* if g_tX is a shortest in $g_t\Lambda$ for all t in a neighborhood of $t(X)$, then X is a minimal vector.

The connection between minimal vectors and best approximations is also simple:

Lemma 10 *For θ in $\mathbb{R}^d$, the set of minimal vectors with nonzero height of the lattice Λ_θ is exactly the set of all $M_\theta V^T$ where*

$$V = (P, q)$$

ranges in the set of all best approximation vectors of θ such that

$$N(q\theta - P) < d(0, \mathbb{Z}^d \backslash \{0\}).$$

Proof. Let $V = (P, q)$ be a non zero vector in $\mathbb{Z}^{d+1}$ and $X = M_\theta V^T$.
Suppose that V is a best approximation vector of θ such that $N(q\theta - P) < d(0, \mathbb{Z}^d \backslash \{0\})$. Let $W = (U, v)$ be a non zero vector in $\mathbb{Z}^{d+1}$. If $Y = M_\theta W^T$ is in $B(X)$ then $|v| \leq q$ and $N(v\theta - U) \leq N(q\theta - P)$, hence $v \neq 0$. Since V is a best approximation vector the two previous inequalities must be two equalities which shows X is minimal with non zero height. Conversely suppose that X is minimal vector of Λ_θ with non zero height. If $W = (U, v)$ is in $\mathbb{Z}^{d+1}$ is such that $|v| < q$ then by minimality, $N(v\theta - U) > N(q\theta - P)$, which shows that X is a best approximation vector of θ with $N(q\theta - P) < d(0, \mathbb{Z}^d \backslash \{0\})$. □

It is worth noting that the inequality $q_{n+1} r_n^d \leq c$ may be stated with minimal vectors:

Lemma 11 *Let Λ be a lattice in $\mathbb{R}^{d+1}$ and $X = (U, v)$ and X' two minimal vectors of Λ. Suppose that $|X| < |X'|$ and that X and X' are two consecutive minimal vectors, i.e., there is no minimal vector the height of which is in the interval $]\,|X|, |X'|\,[$. Then*

$$|X'|\, N(U)^d \leq 2^{d+1}/v_d$$

where v_d is the volume of the ball $B_{\mathbb{R}^d}(0, 1)$ associated with the norm N.

Proof. Since X and X' are consecutive minimal vectors the interior of the box

$$B(X,X') = B_{\mathbb{R}^d}(0, N(U)) \times [-|X'|, |X'|]$$

contains no nonzero point of Λ. Consequently, by the Minkowski convex body theorem, the volume of this box is $\leq 2^{d+1}$. □

Y. Cheung has also found the following nice dynamical interpretation of the minimal vectors of a lattice (see also W.M. Schmidt and L. Summerer, [Schm, Sum]) . Let Λ be a lattice in $\mathbb{R}^{d+1}$ and consider the function

$$\delta_\Lambda(t) = \ln \lambda_1(g_t\Lambda),\ t \in \mathbb{R},$$

where λ_1 is the first minimum with respect to the norm N_c, *i.e.* the length of a shortest vector. The function δ_Λ is piecewise affine with two slopes and is entirely given by the minimal vectors:

Proposition 12 *Let Λ be a lattice in $\mathbb{R}^{d+1}$. For all minimal vector $X = (U, v)$ in Λ there exists an interval $I(X)$ that contains the time*

$$t(X) = \frac{1}{d+1} \ln \frac{|v|}{N(U)}$$

in its interior such that:
- two intervals $I(X)$ are either equal or have disjoint interiors,
- the union of all the intervals $I(X)$ is $\mathbb{R}$, and
- for all minimal vector $X = (U, v)$ and all t in $I(X)$,

$$\delta_\Lambda(t) = \ln N_c(g_t X) = \begin{cases} -dt + \ln |v|, & \text{if } t \leq t(X) \\ t + \ln N(U), & \text{if } t \geq t(X) \end{cases}.$$

3.3 Computation of best Diophantine approximations with the lattice $h_s\Lambda_\theta$.

J. C. Lagarias studied ([Lag2]) the complexity of the computation of best Diophantine approximations. Here we only explain one method to compute them.

By the Lagarias lemma, a shortest vector of $h_s\Lambda_\theta$ gives rise to a best Diophantine approximation of θ. The LLL algorithm (sees Section 6) is likely to be the most efficient way to compute such a vector (see *e.g.* [G,L,S] for LLL algorithm). Assume that $\mathbb{R}^d$ is endowed with the Euclidean norm $\|.\|$. Use the LLL algorithm with the lattice $h_s\Lambda_\theta$ as input, the output is a "reduced" basis $(e_1, ..., e_d)$ of $h_s\Lambda_\theta$ whose first vector is almost a shortest vector of $h_s\Lambda_\theta$:

$$\|e_1\| \leq 2^{(d-1)/2}\lambda_1$$

where λ_1 is the first minimum of $h_s\Lambda_\theta$. It seems that, in practice, the length of the vector e_1 is often very close to λ_1 which means we have a very good Diophantine approximation of θ. In order to obtain a shortest vector of $h_s\Lambda_\theta$ it is possible to use the following result due to Babai ([Ba]):
for all $k \in \{1, ..., d\}$, the sinus of the angle between e_k and the subspace generated by the other vectors of the basis, is $\geq (\sqrt{3}/2)^d$.
It follows that, if the absolute value of one of the coordinates of a vector X in $h_s\Lambda_\theta$ is $> (\frac{2}{\sqrt{3}})^d$, then its norm is $> \lambda_1$. Therefore, the shortest vector of $h_s\Lambda_\theta$ is among the vectors $X = \sum_{i=1}^d x_i e_i$ with $|x_i| \leq (\frac{2}{\sqrt{3}})^d$, $i = 1, ..., d$. So, $(2 \times \frac{2}{\sqrt{3}})^{d^2}$ computations are enough to find this shortest vector.

4 Multidimensional expansions and lattices reduction

4.1 General definition

Interpreting the lattices in $\mathbb{R}^n$ as points in $GL(n,\mathbb{R})/GL(n,\mathbb{Z})$, a reduction theory for lattices is given by a subset $\mathcal{B}_n$ in $GL(n,\mathbb{R})$ which contains a fundamental domain for the right action of $GL(n,\mathbb{Z})$; that is, for all matrix M in $GL(n,\mathbb{R})$ there exists P in $GL(n,\mathbb{Z})$ such that $MP \in \mathcal{B}_n$.

A matrix in $\mathcal{B}_n$ has to be seen as a good basis of the underlying lattice. Matrices in $\mathcal{B}_n$ must enjoy some geometrical properties depending on the aim of the reduction theory, *e.g.,* the vectors of the basis must be as short as possible. To such a set of reduced matrices, one can associated a multidimensional expansion :

Definition 13 Let $\mathcal{B}_{d+1}$ be a subset of $GL(d+1,\mathbb{R})$ which contains a fundamental domain for the right action of $SL(d+1,\mathbb{Z})$. Let $\theta \in \mathbb{R}^d$. A $\mathcal{B}_{d+1}$-expansion of θ is a map $Q = Q_\theta : s \in]0,+\infty[\to Q(s) \in GL(d+1,\mathbb{Z})$ such that $h_s M_\theta Q(s) \in \mathcal{B}_{d+1}$ for all s.

Expansions are associated with any classical sets of reduced bases, *e.g.* Minkowski reduced bases, Korkine-Zolotarev reduced bases, Lovasz reduced bases, Siegel domains, *etc...*
The definition of the matrices $h_s M_\theta$ shows that the elements of the last row of the matrices $Q(s)$ are the denominators of the expansion.

The desired properties of such an expansion are:
E1. Uniqueness or finiteness : for each $s > 0$, there is only one possible choice for $Q(s)$ or finitely many possible choices.
E2. Convexity : for a given matrix P in $GL(d+1,\mathbb{Z})$, the set of s such that $P = Q(s)$ is as simple as possible, *e.g.,* one interval.
E3. Convergents may be associated with the expansion.
E4. $s \longmapsto Q(s)$ provides a strongly convergent expansion of θ : denoting $\begin{pmatrix} P_i(s) \\ q_i(s) \end{pmatrix}$ the columns of $Q(s)$,

$$\lim_{s\to 0} \max\{\|q_i(s)\theta - P_i(s)\| : 1 \leq i \leq d+1\} = 0$$

for all $\theta = (\theta_1, ..., \theta_d)$ such that $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d+1$.
E4bis. The first column of $Q(s)$ is a best approximation vector of θ.
E5. Positivity: if $s > s'$, then the positive cone spanned by the columns of $Q(s)$ contains the positive cone spanned by the columns of $Q(s')$.

We will see that the Lagarias expansion conciliates all these properties except the last one, while it is known that classical multidimensional continued fraction expansions such as Jacobi-Perron algorithm, Brun algorithm,... are not strongly convergent but are positive.

4.2 Lexicographically reduced bases

Lagarias expansion is defined with a set reduced matrices slightly smaller that the set $\mathcal{M}_n$ of Minskowski reduced bases. The reduced bases are called lexicographically reduced bases. In fact lexicographically reduced bases correspond to Hermite reduced quadratic forms (see [Tam1]).

$\mathbb{R}^n$ is endowed with the Euclidean norm $\|.\|$.

Definition 14 A basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$ is *lexicographically* reduced if the vector of norms

$$(\|e_1\|, ..., \|e_n\|)$$

is minimal for the lexicographical order among all vectors of norms associated with the bases of $\Lambda = \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$.

It is not difficult to show by induction that each lattice Λ admits a lexicographically reduced basis. Consequently, the set $\mathcal{L}_n$ of lexicographically reduced bases contains a fundamental domain.

The aim of this definition is to obtain bases with vectors as short as possible. Since a shortest vector of a lattice may be extended into a basis, the first vector of a lexicographically reduced basis is a shortest vector of Λ. Recall that a basis $e_1, ..., e_n$ of a lattice Λ in $\mathbb{R}^n$ is *Minkowski reduced* if for $i = 1, ..., n$, e_i is a vector of minimal length among the vectors x in Λ such that

$$(e_1, ..., e_{i-1}, x)$$

may be extended into a basis of Λ. Clearly, lexicographically reduced bases are Minkowski reduced. The above definition is not the one given by Lagarias who considers the minimum for the lexicographical order only among Minkowski reduced bases. The next lemma is easy and connects the above definition with Lagarias' definition.

Lemma 15 *For each basis $(f_1, ..., f_n)$ of the lattice Λ, there exists a Minkowski reduced basis $(e_1, ..., e_n)$ of Λ such that*

$$(\|e_1\|, ..., \|e_n\|) \preceq (\|f_1\|, ..., \|f_n\|)$$

for the lexicographical order.

In all dimensions, the interior $\mathcal{M}_n^\circ$ of $\mathcal{M}_n$ is included in $\mathcal{L}_n$ which in turn is included in $\mathcal{M}_n$ and it is known that $\mathcal{L}_n = \mathcal{M}_n$ for $n \leq 6$ while $\mathcal{L}_n \neq \mathcal{M}_n$ for $n > 6$ (see [Tam]). A quite simple example due to H.W. Lenstra shows that there exist Minkowski reduced bases that are not lexicographically reduced in dimension $d = 13$ (see [Lag1]).

4.3 Definition of the Lagarias expansion

Definition 16 Suppose that $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$ are endowed with the Euclidean norms $\|.\|_{\mathbb{R}^d}$ and $\|.\|_{\mathbb{R}^{d+1}}$. Let θ be in $\mathbb{R}^d$. The *Lagarias expansion* of θ is the expansion $s \to Q_\theta(s)$ associated with the set $\mathcal{L}_{d+1}$ of lexicographically reduced bases.

By definition, for all $s > 0$, the columns of $h_s M_\theta Q_\theta(s)$ form a lexicographically reduced basis of $\mathbb{R}^{d+1}$, hence the first column of $h_s M_\theta Q_\theta(s)$ is a shortest vector of the lattice $h_s \Lambda_\theta$ and by Lagarias lemma this first column is a best approximation vector of θ.
Actually, Lagarias gives a more precise definition based on the main theorem Section 4.6.

4.4 Convexity properties of the Lagarias expansion

To see that **E2** holds for Lagarias expansion, it is necessary to move $\mathcal{L}_{n=d+1}$ in the space $\mathcal{S}_n^+$ of symmetric positive definite matrices with the map

$$\begin{aligned}\varphi : GL(n, \mathbb{R}) &\to \mathcal{S}_n^+ \\ : M &\to M^t\, M\end{aligned}$$

The next statement is proved in [Lag 1], and is a folklore result when stated with $\mathcal{M}_n$.

Theorem 12 *$\mathcal{Q}_n = \varphi(\mathcal{L}_n)$ is a convex set.*

Remark. It is clear that if q is quadratic form in $\mathcal{Q}_n$ then for all $\lambda > 0$, λq is in $\mathcal{Q}_n$, hence $\mathcal{Q}_n$ is a convex cone. Thanks to the fact that the set of Minskowski reduced quadratic forms can be defined by finitely many linear inequalities (see [Waer]), we see that the convex cone $\mathcal{Q}_n$ has finitely many faces. When $n \leq 6$, Tammela [Tam1] gives all the inequalities defining the faces of $\mathcal{Q}_n$.

Notation. Let θ be in $\mathbb{R}^d$. For each Q in $GL(d+1, \mathbb{Z})$, let $I(Q)$ denote the set of real numbers $s > 0$ such that $h_s M_\theta Q \in \mathcal{L}_{d+1}$.

The next result is proved in [Lag1]. We give its proof which is simple.

Theorem 13 *1. For all matrix Q in $GL(d+1, \mathbb{Z})$, $I(Q)$ is an interval.*
2. Let Q and Q' be in $GL(d+1, \mathbb{Z})$.
a. Then either $I(Q) = I(Q')$ or $I(Q) \cap I(Q')$ contains at most one element.
b. If $I(Q) \cap I(Q')$ contains at least 2 elements, then Q and Q' have the same last row up to signs.

Proof. 1. For $s > 0$, denote by B_s the diagonal matrix $(1, ..., 1, s)$. For all θ in $\mathbb{R}^d$ and all $s > 0$, we have $h_s M_\theta = B_s M_1(\theta) \in GL(d+1, \mathbb{R})$. Now, $s \in I(Q)$ means that the quadratic form

$$\varphi(h_s M_\theta Q) = Q^t M_1^t(\theta) B_{s^2} M_1(\theta) Q$$

is in $\mathcal{Q}_{d+1}$. Since $\varphi(h_s M_\theta Q)$ is an affine function of s^2, and since $\mathcal{Q}_{d+1}$ is convex, the set of positive real numbers s^2 such that $\varphi(h_s M_\theta Q) \in \mathcal{Q}_{d+1}$ is an interval. Thus $I(Q)$ is an interval.

2. Let $Q, Q' \in GL(d+1, \mathbb{Z})$ such that $I(Q) \cap I(Q')$ contains at least two elements $s_1 \neq s_2$. Denote by (P_i, q_i) the i-th column of Q and by (P'_i, q'_i) the i-th column of Q'. By definition of lexicographically reduced bases, the length of these two columns are equal, hence for $s \in \{s_1, s_2\}$,

$$\|P_i - q_i\theta\|^2_{\mathbb{R}^d} + s^2 q_i^2 = \|P'_i - q'_i\theta\|^2_{\mathbb{R}^d} + s^2 q_i'^2.$$

Therefore $|q_i| = |q'_i|$ and $\|P_i - q_i\theta\|^2_{\mathbb{R}^d} = \|P'_i - q'_i\theta\|^2_{\mathbb{R}^d}$. It follows that $I(Q) = I(Q')$. □

4.5 Finiteness property of expansions

Following Lagarias, we will show that the finiteness and the convergence of an expansion depend only on the following property of the reduction set :

Definition 17 A subset $\mathcal{R}$ of $GL(n,\mathbb{R})$ is a *Hermite domain* if there exists a constant C such that, for any matrix M in $\mathcal{R}$ with columns $(e_1, ..., e_n)$,

$$\|e_1\|_{\mathbb{R}^n} ... \|e_n\|_{\mathbb{R}^n} \leq C\,|\det(e_1, ..., e_n)|\,.$$

The matrices in $GL(n,\mathbb{R})$ will be identified with the bases of $\mathbb{R}^n$ by taking the columns of the matrices. The sets $\mathcal{M}_n$ of Minkowski reduced matrices, $\mathcal{L}_n$ of the lexicographically reduced matrices, of Lovász reduced matrices, Siegel domain, *etc...* are Hermite domain. A proof that $\mathcal{M}_n$ is a Hermite domain can be found in [Waer]. If a basis $(e_1, ..., e_n)$ in a Hermite domain is such that

$$\|e_1\|_{\mathbb{R}^n} \leq \|e_2\|_{\mathbb{R}^n} \leq ... \leq \|e_n\|_{\mathbb{R}^n}$$

then the Minkowski minima theorem implies that

$$\|e_k\|_{\mathbb{R}^n} \leq C'\lambda_k(\Lambda)$$

where Λ is the lattice spanned by the vectors $e_1, ..., e_n$ and $\lambda_k(\Lambda)$ is the k-th minimum of Λ. The constant C' depend only on C and the constant involved in Minkowski's theorem. The same conclusion remains true if one assume that

$$\|e_{k+1}\|_{\mathbb{R}^n} \geq \lambda\,\|e_k\|_{\mathbb{R}^n}$$

where λ is a fixed positive real number.
Remark. The union of the images of a Hermite domain by the maps induced by all the permutations of the rows is another Hermite domain with the same constant C.

Assumption about Hermite domain: In what follows we will always assume that the columns of a matrix in a Hermite domain are reordered in such a way that their norms increase. This is not a real restriction and this leads to simpler statements.

Theorem 14 *Let $\mathcal{R}$ be a Hermite domain in $GL(d+1,\mathbb{R})$. Let θ be in $\mathbb{R}^d$. For all $b > a > 0$, there exist finitely many matrices Q in $GL(d+1,\mathbb{Z})$ such that the basis $h_s M_\theta Q$ is in $\mathcal{R}$ for at least one $s \in [a,b]$.*

Proof. Consider a matrix Q in $GL(n,\mathbb{Z})$ such that $h_s M_\theta Q$ is in $\mathcal{R}$ for some $s \in [a,b]$. Let $e_1, ..., e_{d+1}$ be the columns of $h_s M_\theta Q$. Since $\mathcal{R}$ is a Hermite domain,

$$\|e_i\|_{\mathbb{R}^{d+1}} \leq 2C\lambda_i(h_s\Lambda_\theta) \leq C\lambda_{d+1}(h_s\Lambda_\theta)$$

for $i = 1, ..., d+1$, where $\lambda_i(h_s\Lambda_\theta)$ is the i-th minimum of $h_s\Lambda_\theta$. Now for all $t \leq b$, the last minimum $\lambda_{d+1}(h_s\Lambda_\theta)$ is $\leq 1 + b + \|\theta\|_{\mathbb{R}^d}$. Thus, with $e_i = (V_i, v_i)$, we get $\|h_s M_\theta e_i\|^2_{\mathbb{R}^{d+1}} = \|V_i - v_i\theta\|^2_{\mathbb{R}^d} + s^2 v_i^2 \leq C^2 \max^2(1,b)$. Since $s \geq a$, $|v_i| \leq C\max(1,b)/a$ and V is in the union of balls $\cup_{|v|\leq \max(1,b)C/a} B(v\theta, \max(1,b)C)$. Therefore the number of matrices Q such that $h_s M_\theta Q$ is in $\mathcal{R}$ for at least one $s \in [a,b]$, is finite. □

Theorem 15 *Let θ be in $\mathbb{R}^d$. For all $s_0 > 0$, there exist finitely many matrices Q in $GL(n,\mathbb{Z})$ such that the basis $h_s M_\theta Q$ is lexicographically reduced for at least one $s \geq s_0$.*

Proof. By the previous theorem, it is enough to consider the case $s > s_0 = 1$. For every nonzero V in $\mathbb{Z}^{d+1}$, we have $\|h_s M_\theta V\|_{\mathbb{R}^{d+1}} \geq 1$ and, if the last coordinate of V is nonzero then $\|h_s M_\theta V\|_{\mathbb{R}^{d+1}} > 1$. Therefore all the lexicographically reduced bases are of the form:
- the first d vectors are $\pm h_s M_\theta e_i = \pm e_i$ where $i \leq d$ and e_i is the i-th vector of the canonical basis of $\mathbb{R}^{d+1}$,
- the last vector is $h_s M_\theta V$ where $V = \pm(p_1, ..., p_n, 1)$ and p_i is such that $|p_i - \theta_i|$ is minimal.
As a result, the number of such bases is finite. □

4.6 Main theorem for the Lagarias expansion

From the previous Theorems, it follows immediately that:

Theorem 16 *(Main theorem) If $\theta \notin \mathbb{Q}^d$, there exists an infinite sequence $\infty = s_{-1} > s_0 > s_1 > ... > s_n > ...$ going to 0 and a sequence of matrices $Q_0, ..., Q_n, ... \in GL(d+1, \mathbb{Z})$ such that $]s_n, s_{n-1}[= I(Q_n)^o$.*

The sequence of matrices $(Q_n)_{n\geq 0}$ is not uniquely defined but, by Theorem 13, the last row is unique up to signs. The sequence of matrices $(Q_n)_{n\geq 0}$ is the Lagarias expansion of θ. The partial quotients associated with the sequence $(Q_n)_{n\geq 0}$ are defined by $Q_{n-1}^{-1}Q_n$. Since $\mathcal{M}_{d+1}Q_{n-1} \cap \mathcal{M}_{d+1}Q_n \neq \emptyset$, there are only finitely many possible partial quotients (see [Waer]). Thus it is an additive expansion. To obtain a multiplicative expansion it suffices to keep only the matrices Q_n at the times n where the first column of Q_n changes. It should be noticed that there are Hermite domains $\mathcal{R}$ such that there exist infinitely many matrices Q in $SL(n\mathbb{Z})$ with $\mathcal{R}Q \cap \mathcal{R} \neq \emptyset$.

5 Convergence

Let $\mathcal{R}$ be a subset of $GL(d+1, \mathbb{R})$ containing a fundamental domain. For all $\theta \in \mathbb{R}^d$ denote for $s > 0$

$$Q_\theta(s) = \begin{pmatrix} P_1(s) & \cdots & P_{d+1}(s) \\ q_1(s) & \cdots & q_{d+1}(s) \end{pmatrix}$$

an $\mathcal{R}$-expansion of θ.

5.1 Strong convergence

The classical multidimensional continued fraction expansions are weakly convergent but most of them have been shown to be not strongly convergent (see [Br] for the case of Jacobi-Perron's algorithm or Brun's algorithm). However, the Lagarias expansion is strongly convergent and Lagarias proof allows to show the more general result:

Theorem 17 *Let $\mathcal{R}$ be a Hermite domain in $GL(d+1, \mathbb{R})$ containing a fundamental domain. Let θ be in $\mathbb{R}^d$, let $s \to Q(s)$ be an $\mathcal{R}$-expansion associated with θ in $\mathbb{R}^d$. If $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] \geq r$, then the first r columns of the matrix $Q_\theta(s)$ strongly converge to θ:*

$$\lim_{s \to 0} \max\{\|q_i(s)\theta - P_i(s)\| : 1 \leq i \leq r\} = 0.$$

Proof. The i-th column of $h_s M_\theta Q_\theta(s)$ is $(P_i(s) - q_i(s)\theta, sq_i(s))^T$. Since $\mathcal{R}$ is a Hermite domain, one has

$$\|(P_i(s) - q_i(s)\theta, sq_i(s))\|_{\mathbb{R}^{d+1}} \leq C\lambda_{s,i}(\theta)$$

where $\lambda_{s,i}(\theta)$ is the i-th minimum of the lattice $h_s\Lambda_\theta = h_s M_\theta \mathbb{Z}^{d+1}$. Thus, it is enough to prove that

$$\lim_{s \to 0} \lambda_{s,r}(\theta) = 0.$$

We will use the dual lattice $\Lambda_\theta^*(s) = \{Y \in \mathbb{R}^{d+1} : \forall X \in h_s\Lambda_\theta,\ X.Y \in \mathbb{Z}\}$. The following lemma is standard (see *e.g.* [Schm]). If Λ be a lattice in $\mathbb{R}^n$ and Λ^* its dual lattice, then the minima of both lattices satisfy

$$1 \leq \lambda_i \lambda_{n+1-i}^* \leq (n+1)!$$

Consequently, it is enough to prove that the minimum $\lambda_{s,d+2-r}^*$ of $\Lambda_\theta^*(s)$ goes to infinity when s goes to zero. Suppose on the contrary that, there exists a sequence $(s_n)_n$ going to 0 such that for all n, $\lambda_{s_n,d+2-r}^* \leq K$. The lattice $\Lambda_\theta^*(s)$ is spanned by the rows of the matrix

$$(h_s M_\theta)^{-1} = \begin{pmatrix} I & s^{-1}\theta \\ 0 & s^{-1} \end{pmatrix},$$

hence for all n, there exist $(A_{n,i}, b_{n,i}) \in \mathbb{Z}^{d+1}$, $i = 1, ..., d+2-r$, linearly independent such that the vector

$$v_{n,i} = (A_{n,i}, b_{n,i})(h_{s_n} M_\theta)^{-1} = (A_{n,i}, \frac{1}{s_n}(b_{n,i} + A_{n,i}.\theta))$$

has a norm $\leq K$. If $s_n \leq 1$, then

$$\|A_{n,i}\|_{\mathbb{R}^d} \leq K$$

and

$$|b_{n,i}| \leq K + |A_{n,i}.\theta| \leq K'$$

where K' does not depend on n. Extracting a subsequence, we can assume $A_{n,i} = A_i$ and $b_{n,i} = b_i$ for all n with (A_i, b_i) linearly independent, $i = 1, ..., d+2-r$. Therefore,

$$|b_i + A_i.\theta| \leq K s_n.$$

Now, $s_n \to 0$ hence $b_i + A_i.\theta = 0$, $i = 1, ..., d+2-r$ which contradicts the assumption $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] \geq r$. □

In the Lagarias expansion, the first column of the matrix $Q(s)$ is always a good approximation because it is a best approximation vector of θ. In the more general case of an a expansion associated with a Hermite domain, the first column of $Q(s)$ is also a good approximation.

Proposition 18 *If $\mathcal{R}$ is a Hermite domain in $GL(d+1, \mathbb{R})$ containing a fundamental domain, then there exists a constant C such that for all θ in $\mathbb{R}^d$ and all $s > 0$,*

$$\|q_1(s)\theta - P_1(s)\| \, |q_1(s)|^{1/d} \leq C.$$

Proof. Let s be a positive real number. By definition of a Hermite domain there exists a constant c such that

$$\|(P_1(s) - q_1(s)\theta, sq_1(s))\|_{\mathbb{R}^{d+1}} \leq c\lambda_1(h_s\Lambda_\theta).$$

By the Minkowski minima theorem, $\lambda_1(h_s\Lambda_\theta) \ll \det(h_s\Lambda_\theta)^{\frac{1}{d+1}} = s^{\frac{1}{d+1}}$. It follows that

$$s\,|q_1(s)| \ll s^{\frac{1}{d+1}}$$

and

$$\|P_1(s) - q_1(s)\theta\|_{\mathbb{R}^d} \ll s^{\frac{1}{d+1}}.$$

Suppose first that $\|P_1(s) - q_1(s)\theta\|_{\mathbb{R}^d} \leq s\,|q_1(s)|$. We then have

$$\|P_1(s) - q_1(s)\theta\|_{\mathbb{R}^d} |q_1(s)|^{1/d} \leq s\,|q_1(s)|^{1+1/d} \ll 1.$$

Suppose now that $\|P_1(s) - q_1(s)\theta\|_{\mathbb{R}^d} > s\,|q_1(s)|$. Set $t = \frac{\|P_1(s)-q_1(s)\theta\|_{\mathbb{R}^d}}{|q_1(s)|}$. There exists a vector (P_0, q_0) in $\mathbb{Z}^d \times \mathbb{N}$ such that

$$\|(P_0 - q_0\theta, tq_0)\|_{\mathbb{R}^{d+1}} = \lambda_1(h_t\Lambda_\theta).$$

Let us prove that

$$\|(P_1(s) - q_1(s)\theta, tq_1(s))\|_{\mathbb{R}^{d+1}} \leq 2c\,\|(P_0 - q_0\theta, tq_0)\|_{\mathbb{R}^{d+1}}.$$

Now by definition of t,

$$\|(P_1(s) - q_1(s)\theta, tq_1(s))\|_{\mathbb{R}^{d+1}} = \sqrt{2}\,\|P_1(s) - q_1(s)\theta\|_{\mathbb{R}^d}$$

hence the converse inequality

$$\|(P_1(s) - q_1(s)\theta, tq_1(s))\|_{\mathbb{R}^{d+1}} > 2c\,\|(P_0 - q_0\theta, tq_0)\|_{\mathbb{R}^{d+1}}$$

would implies that

$$\max(\|P_0 - q_0\theta\|_{\mathbb{R}^d}, tq_0) < \frac{1}{2c}\sqrt{2}\,\|P_1(s) - q_1(s)\theta\|_{\mathbb{R}^d},$$

and since $s < t$, we would have

$$\begin{aligned}\|(q_0\theta - P_0, sq_0)\|_{\mathbb{R}^{d+1}} &\leq \sqrt{2}\max(\|q_0\theta - P_0\|_{\mathbb{R}^d}, sq_o) \\ &\leq \sqrt{2}\max(\|q_0\theta - P_0\|_{\mathbb{R}^d}, tq_o) \\ &< \frac{1}{c}\|q_1(s)\theta - P_1(s)\|_{\mathbb{R}^d} \leq \lambda_1(h_s\Lambda_\theta)\end{aligned}$$

which is impossible.
The calculation of the first case with t instead of s allows to conclude. □

5.2 A general negative result

Some general negative results about convergence are known. For instance, Grabiner proved that no sequence of refined triangulations made of Farey simplices can be strongly convergent (see [Gr]). Here we prove that, for general expansions, strong convergence can be arbitrarily slow when the rate is measured with the maximum of the denominators involved in the matrices of the expansion. In fact this rate does not depend on the expansion.

Proposition 19 *Let $\varphi(1), \varphi(2), ..., \varphi(q), ...$ be a positive function going to zero. Then there exists θ in $\mathbb{R}^d$ such that*
- $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$,
- for all sequence of matrices $(Q_n)_n$ in $GL(d+1, \mathbb{Z})$ with $|Q_n| = \max_{i=1}^{d+1} |q_{i,n}| \to \infty$,

$$\max\{\|q_{i,n}\theta - P_{i,n}\| : 1 \leq i \leq d+1\} \geq \varphi(|Q_n|)$$

for n large enough.

Proof. Let $\omega : [0, +\infty[\to]0, +\infty[$ be a positive function such that $\lim_{t\to\infty} \omega(t) = 0$ and $\omega(\frac{1}{4\varphi(q)}) \leq \frac{1}{4q}$ for all integers $q \geq 1$. Such a function exists: just assume that φ is nonincreasing by replacing $\varphi(q)$ by $\max_{k\geq q} \varphi(k)$ and set

$$\omega(t) = \frac{1}{4q}$$

for $t \in]\frac{1}{4\varphi(q-1)}, \frac{1}{4\varphi(q)}]$. By the Khinchin theorem on singular linear forms there exists θ in $\mathbb{R}^d$ such that $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$ and such that for all T large enough there exist a nonzero $q^* = (q_1^*, ..., q_d^*)$ in $\mathbb{Z}^d$ and p^* in $\mathbb{Z}$ such that

$$\|q^*\|_{\mathbb{R}^d} \leq T$$

and

$$|q^*.\theta + p^*| \leq \omega(T).$$

Fix n and $T = \frac{1}{4\varphi(|Q_n|)}$. Choose (q^*, p^*) associated with T by Khinchin's theorem. Let s be the positive real number defined by

$$\|q^*\|_{\mathbb{R}^d} = \frac{1}{s}|q^*.\theta + p^*|.$$

The vectors $(P_i - q_i\theta, sq_i) = (P_{i,n} - q_{i,n}\theta, sq_{i,n})$, $i = 1, ..., d+1$, form a basis of the lattice $h_s\Lambda_\theta$ hence one of them, say $(P_{i_0} - q_{i_0}\theta, sq_{i_0})$, is not in the hyperplane defined by the linear form

$$l^*(x, y) = q^*.x + \frac{1}{s}(q^*.\theta + p^*)y.$$

Since l^* is in the dual lattice $(h_s\Lambda_\theta)^*$, we have

$$|l^*((P_{i_0} - q_{i_0}\theta, sq_{i_0}))| \geq 1.$$

Therefore (by Cauchy Schwarz inequality)

$$\begin{aligned}\|(P_{i_0} - q_{i_0}\theta, sq_{i_0})\|_{\mathbb{R}^{d+1}} &\geq \frac{1}{\left\|(q^*, \frac{1}{s}(q^*.\theta + p^*))\right\|_{\mathbb{R}^{d+1}}} \\ &\geq \frac{1}{2\|q^*\|_{\mathbb{R}^d}}.\end{aligned}$$

Now, $|sq_{i_0}| = \frac{|q^*.\theta+p^*|}{\|q^*\|_{\mathbb{R}^d}} |q_{i_0}| \leq \frac{\omega(T)|Q_n|}{\|q^*\|}$ and $\omega(T) = \omega(\frac{1}{4\varphi(|Q_n|)}) \leq \frac{1}{4|Q_n|}$, hence

$$|sq_{i_0}| \leq \frac{1}{4\|q^*\|_{\mathbb{R}^d}}$$

which implies

$$\|P_{i_0} - q_{i_0}\theta\|_{\mathbb{R}^d} \geq \frac{1}{4\|q^*\|_{\mathbb{R}^d}} \geq \frac{1}{4T} = \varphi(|Q_n|).$$

□

A natural question arises :
Let $\varphi(1), \varphi(2), ..., \varphi(q), ...$ be a positive function going to zero. Does there exist θ in $\mathbb{R}^d$ such that
- $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d+1$,
- for all sequence of matrices $(Q_n)_n$ in $GL(d+1,\mathbb{Z})$ with $\min_{i=1}^{d+1} |q_{i,n}| \to \infty$,

$$\max\{\frac{\|q_{i,n}\theta - P_{i,n}\|}{\varphi(|q_{i,n}|)} : 1 \leq i \leq d+1\} \geq 1$$

for n large enough?

5.3 Almost everywhere convergence

By the previous proposition it is not possible to improve the convergence rate for all θ, but what about the convergence rate for almost all θ in $\mathbb{R}^d$? For classical multidimensional expansions such as Jacobi Perron, there are improvements, but not of the convergence rate provided by the Dirichlet theorem. For instance, in their deep paper [Br, Gu], A. Broise and Y. Guivarc'h prove that the second characteristic exponent λ_2 of the two-dimensional Jacobi-Perron algorithm is negative. Denoting by (P_n, q_n) the n-th integer vectors computed by the Jacobi-Perron algorithm, A. Broise and Y. Guivarc'h's result implies that for almost all θ in $\mathbb{R}^2$,

$$\lim_{n\to\infty} \frac{1}{n} \ln \|q_n\theta - P_n\| = \lambda_2 < 0$$

(see [Lag6]). Since the sequence $\frac{1}{n}\ln q_n$ also converges to the first characteristic exponent λ_1 for almost all θ in $\mathbb{R}^2$, we obtain

$$\lim_{n\to\infty} \|q_n\theta - P_n\| \, q_n^{\alpha} = 0$$

almost everywhere, for any $\alpha < -\frac{\lambda_2}{\lambda_1}$. On the other hand, A. Broise and Y. Guivarc'h prove that the characteristic exponents are distinct for all $d \geq 2$. Since the sum of the exponents is 0, this implies that $\frac{\lambda_2}{\lambda_1} > -\frac{1}{d}$. It follows that there exists an exponent $\beta < \frac{1}{d}$ such that

$$\lim_{n\to\infty} \|q_n\theta - P_n\| \, q_n^{\beta} = +\infty$$

for almost all θ in $\mathbb{R}^d$. A. Broise and Y.Guivarc'h prove that the same result holds for Brun's algorithm and they claim that their proof can be adapted to other multidimensional algorithms.

In the case of an expansion associated with a Hermite domain, the rate of convergence is very close to the optimal rate: there is only a logarithmic extra factor.

Theorem 18 *If $\mathcal{R}$ is a Hermite domain in $GL(d+1,\mathbb{R})$ containing a fundamental domain, then for almost all θ in $\mathbb{R}^d$*

$$\limsup_{s\to 0} \max_{i=1}^{d+1} \|q_i(s)\theta - P_i(s)\|_{\mathbb{R}^d} \frac{|q_i(s)|^{1/d}}{\ln^{2/d}|q_i(s)|} \leq 1.$$

Proof. In this proof, the minima of lattices in $\mathbb{R}^{d+1}$are defined with respect to the Cheung's norm associated with the Euclidean norm:

$$\|(x,y)\|_c = \max(\|x\|_{\mathbb{R}^d}, |y|)$$

By the Khinchin-Groshev theorem, for almost all θ in $\mathbb{R}^d$ and all $c \geq 0$, there exist only finitely many nonzero vectors (q^*, p^*) in $\mathbb{Z}^d \times \mathbb{Z}$ such that

$$|q^*.\theta + p^*| \leq \frac{c}{\|q^*\|_{\mathbb{R}^d}^d \ln^2 \|q^*\|_{\mathbb{R}^d}}.$$

Note that it is the easy part of the theorem depending only on the Borel-Cantelli lemma.
We prove the theorem by contradiction. We assume that we can find θ in $\mathbb{R}^d$ for which the following three properties hold:
- $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$,
- for all $c \geq 0$, there exist only finitely many nonzero vectors (q^*, p^*) in $\mathbb{Z}^d \times \mathbb{Z}$ for which the above inequality holds, and
- there are a sequence $(s_n)_n$ of positive real numbers going to zero and a sequence $(i_n)_n$ in $\{1, ..., d+1\}$ such that

$$\|q_{i_n}(s_n)\theta - P_{i_n}(s_n)\|_{\mathbb{R}^d} \frac{|q_{i_n}(s_n)|^{1/d}}{\ln^{2/d}|q_{i_n}(s_n)|} > 1.$$

Extracting a subsequence, we can suppose that $i_n = k$ for all n (it is just for sake of simplicity).
To avoid too heavy notations, we drop the index k and set $P_n = P_k(s_n)$ and $q_n = q_k(s_n)$, and we assume $q_n \geq 0$.
Case 1: $\|P_n - q_n\theta\|_{\mathbb{R}^d} \geq s_n q_n$.
Since $\lambda_{d+1}(h_{s_n}\Lambda_\theta) \geq \lambda_k(h_{s_n}\Lambda_\theta)$, we have

$$\lambda_1((h_{s_n}\Lambda_\theta)^*) \ll \frac{1}{\lambda_k(h_{s_n}\Lambda_\theta)}.$$

Choose (q_n^*, p_n^*) in $\mathbb{Z}^d \times \mathbb{Z}$ such that $(q_n^*, p_n^*)(h_{s_n}M_\theta)^{-1}$ is a shortest vector of the dual lattice $(h_{s_n}\Lambda_\theta)^*$. We have

$$\max(\|q_n^*\|_{\mathbb{R}^d}, \frac{1}{s_n}|q_n^*.\theta + p_n^*|) \ll \frac{1}{\lambda_k(h_{s_n}\Lambda_\theta)}.$$

It follows that

$$\begin{aligned}&|q_n^*.\theta + p_n^*|\,\|q_n^*\|_{\mathbb{R}^d}^d \ln^2 \|q_n^*\|_{\mathbb{R}^d}^d \\ &\ll \frac{s_n}{\lambda_k(h_{s_n}\Lambda_\theta)} \times \frac{1}{\lambda_k^d(h_{s_n}\Lambda_\theta)} \times (-\ln\lambda_k(h_{s_n}\Lambda_\theta) + C)^2\end{aligned}$$

where C is a constant independent of n. Making use of the assumption $\|P_n - q_n\theta\|_{\mathbb{R}^d} \geq s_n q_n$, we obtain

$$|q_n^*.\theta + p_n^*|\,\|q_n^*\|_{\mathbb{R}^d}^d \ln^2 \|q_n^*\|_{\mathbb{R}^d}^d \ll \frac{\|q_n\theta - P_n\|_{\mathbb{R}^d}(-\ln\lambda_k(h_{s_n}\Lambda_\theta) + C)^2}{q_n\lambda_k^{d+1}(h_{s_n}\Lambda_\theta)}.$$

Since $\mathcal{R}$ is a Hermite domain, $\lambda_k(h_{s_n}\Lambda_\theta) \gg \|(q_n\theta - P_n, s_n q_n)\|_c = \|q_n\theta - P_n\|_{\mathbb{R}^d}$. Moreover by the strong convergence theorem, $\|q_n\theta - P_n\|_{\mathbb{R}^d}$ goes to zero when n goes to infinity, hence

$$|q_n^*.\theta + p_n^*|\,\|q_n^*\|_{\mathbb{R}^d}^d \ln \|q_n^*\|_{\mathbb{R}^d}^d \ll \frac{\|q_n\theta - P_n\|_{\mathbb{R}^d}(-\ln\|q_n\theta - P_n\|_{\mathbb{R}^d})^2}{q_n\|q_n\theta - P_n\|_{\mathbb{R}^d}^{d+1}}$$

for n large enough. By definition of the sequence $(s_n)_n$, $q_n\|q_n\theta - P_n\|_{\mathbb{R}^d}^d \geq \ln^2 q_n$, hence

$$|q_n^*.\theta + p_n^*|\,\|q_n^*\|_{\mathbb{R}^d}^d \ln^2 \|q_n^*\|_{\mathbb{R}^d}^d \ll \frac{1}{\ln^2 q_n}(-\ln\frac{\ln^2 q_n}{q_n^{1/d}})^2 \ll 1.$$

Case 2: $\|q_n\theta - P_n\|_{\mathbb{R}^d} < s_n q_n$.
Set $t_n = \frac{\|P_n - q_n\theta\|_{\mathbb{R}^d}}{q_n}$. Let us see how the k-th minimum of $h_s\Lambda_\theta$ behave when s decrease from s_n to t_n. Consider a vector Y in $\mathbb{R}^{d+1}$ and two times $s \geq t > 0$. The following property is easy to check:

$$\frac{\|h_s Y\|_c}{\|h_t Y\|_c} \leq \frac{s}{t}.$$

We use this property with $s = s_n > t = t_n$. Let $Y_1, ..., Y_k$ be k linearly independent vectors in $M_\theta \mathbb{Z}^{d+1}$ such that

$$\max_{i=1}^{k} \|h_{t_n} Y_i\|_c = \lambda_k(h_{t_n}\Lambda_\theta)$$

and set $X = M_\theta(P_n, q_n)^T$.
If $\|h_{t_n} X\|_c \geq a\lambda_k(h_{t_n}\Lambda_\theta)$ for some $a \geq 1$, then

$$\begin{aligned}\|h_{s_n} X\|_c &= \frac{s_n}{t_n} \|h_{t_n} X\|_c \geq \frac{s_n}{t_n} a\lambda_k(h_{t_n}\Lambda_\theta) \\ &= \frac{s_n}{t_n} a \max_{i=1}^{k} \|h_{t_n} Y_i\|_c .\end{aligned}$$

Now by the above property, $\|h_{t_n} Y_i\|_c \geq \frac{t_n}{s_n} \|h_{s_n} Y_i\|_c$, hence

$$\begin{aligned}\|h_{s_n} X\|_c &\geq \frac{s_n}{t_n} a \max_{i=1}^{k} \|h_{t_n} Y_i\|_c \\ &\geq a \max_{i=1}^{k} \|h_{s_n} Y_i\|_c \geq a\lambda_k(h_{s_n}\Lambda_\theta).\end{aligned}$$

Since $\|h_{s_n} X\|_c \ll \lambda_k(h_{s_n}\Lambda_\theta)$, it follows that $\lambda_k(h_{t_n}\Lambda_\theta) \gg \|h_{t_n} X\|_c = \|P_n - q_n\theta\|_{\mathbb{R}^d}$ which enables to use the calculation of the first case with t_n instead of s_n. So the same conclusion holds:

$$|q_n^*.\theta + p_n^*| \, \|q_n^*\|_{\mathbb{R}^d}^d \ln^2 \|q_n^*\|_{\mathbb{R}^d}^d \ll 1.$$

The constant involved in $\ll$ depends only on the constant of the Hermite domain $\mathcal{R}$.

By our assumption, there are only finitely many different vectors $(q_n^*, p_n^*)_{n\geq 0}$. Since $\dim_{\mathbb{Q}}[1, \theta_1, ..., \theta_d] = d + 1$, there is a positive constant α such that for all n,

$$\frac{1}{s_n} |q_n^*.\theta + p_n^*|) \geq \frac{\alpha}{s_n}$$

hence $\lambda_1((h_{s_n}\Lambda_\theta)^*) \geq \frac{\alpha}{s_n}$, but by the Minkowski minima theorem, we have

$$\lambda_1((h_{s_n}\Lambda_\theta)^*) \ll (\det(h_{s_n}\Lambda_\theta)^*)^{1/(d+1)} = \frac{1}{s_n^{1/(d+1)}}$$

which contradicts the previous inequality when n is large enough. □

6 A multidimensional continued fraction expansion using the LLL algorithm

Consider the linear map

$$h = \begin{pmatrix} I_d & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in GL(d + 1, \mathbb{R}).$$

Note that $h = h_{1/2}$ where $(h_s)_{s>0}$ is the multiplicative group defined in Section 3. For θ in $\mathbb{R}^d$, consider the sequence of lattices

$$\Gamma_n = h^n\Lambda_\theta, \; n \in \mathbb{N}.$$

A continued fraction expansion $(Q_n)_{n\geq 0}$ of θ is deduced from the LLL algorithm as follows:

- *Start with a basis Q_0 of $\mathbb{Z}^{d+1}$ such that $M_\theta Q_0$ is an LLL-reduced basis of $\Gamma_0 = \Lambda_\theta$.*

- *For all integers n, the matrix Q_{n+1} is such that $h^{n+1}M_\theta Q_{n+1}$ is the LLL-reduced basis of Γ_{n+1} computed by the LLL algorithm with initial basis $h^{n+1}M_\theta Q_n$.*

This expansion is implicitly proposed by Lagarias in [Lag 1] and [Lag 2]. Since the domain of LLL-reduced basis is a Hermite domain (see the point 2 below), this expansion is strongly convergent and the previous theorem on almost everywhere convergence holds for this expansion. The only point we want to add is that this expansion is “truly an algorithm”.

Proposition 20 *The number of steps needed to compute the basis Q_{n+1} with the LLL algorithm starting with the basis Q_n, is bounded above by a constant C that depends only on d.*

Before proving this Proposition we need to recall some facts about the LLL algorithm. We follow [G,L,S] pages 139-146.

Let $(b_1, ..., b_k)$ be a basis of $\mathbb{R}^k$. Call $(b_1^*, ..., b_k^*)$ the Gram-Schmidt orthogonalization of $(b_1, ..., b_k)$. The basis $(b_1, ..., b_k)$ is *proper* if

$$b_j = b_j^* + \sum_{i<k} \mu_{j,i} b_i^*$$

with $|\mu_{j,i}| \leq \frac{1}{2}$, $1 \leq i < j \leq k$. The basis $(b_1, ..., b_k)$ is LLL-reduced if it is proper and if for all $j = 1, ..., n-1$,

$$\left\|b_{j+1}^* + \mu_{j+1,j} b_j^*\right\|_{\mathbb{R}^k}^2 \geq \frac{3}{4} \left\|b_j^*\right\|_{\mathbb{R}^k}^2.$$

(the constant $\frac{3}{4}$ might be change in any constant $\delta \in]0,1[$). Here $\|.\|_{\mathbb{R}^k}$ stands for the standard Euclidean norm on $\mathbb{R}^k$. Let $(b_1, ..., b_k)$ be an LLL-reduced basis and $\Lambda = \oplus_{j=1}^k \mathbb{Z} b_j$ the lattice spanned by the basis $(b_1, ..., b_k)$. The following two properties of LLL-reduced bases are the most important one:

1. $\|b_1\|_{\mathbb{R}^k} \leq 2^{(k-1)/2} \lambda_1(\Lambda)$,
2. $\|b_1\|_{\mathbb{R}^k} ... \|b_k\|_{\mathbb{R}^k} \leq 2^{k(k-1)/4} \det \Lambda$.

Besides, the following two facts are also known and are easily deduced from the definition of reduced bases:

3. $\left\|b_j^*\right\|_{\mathbb{R}^k} \leq \sqrt{2} \left\|b_{j+1}^*\right\|_{\mathbb{R}^k}$, $j = 1, ..., k-1$,
4. $\|b_j\|_{\mathbb{R}^k} \leq 2^{(j-1)/2} \left\|b_j^*\right\|_{\mathbb{R}^k}$, $j = 1, ..., k$.

We shall also need of the following easy lemma.

Lemma 21 *If $(b_1, ..., b_k)$ is a LLL reduced basis of $\mathbb{R}^k$, then for $j = 1, ..., k$,*

$$\|b_j\|_{\mathbb{R}^k} \leq 2^{k/2} \lambda_j(\Lambda)$$

where $\Lambda = \oplus_{j=1}^k \mathbb{Z} b_j$.

Proof. Let $x_1, ..., x_j$ be j linearly independent vectors in Λ. One at least of these vectors has a nonzero coordinate over $b_j, ..., b_k$, say $x_j = \sum_{i=1}^l a_i b_i$ with $l \geq j$ and $a_l \neq 0$. Since $x_j = \sum_{i=1}^l a_i^* b_i^*$ with $a_l^* = a_l$, we have

$$\begin{aligned} \|x_j\|_{\mathbb{R}^k}^2 &\geq |a_l|^2 \|b_l^*\|_{\mathbb{R}^k}^2 \\ &\geq \|b_l^*\|_{\mathbb{R}^k}^2 \geq 2^{-(k-j)} \left\|b_j^*\right\|_{\mathbb{R}^k}^2 \\ &\geq 2^{-(k-j)} 2^{-(j-1)} \|b_j\|_{\mathbb{R}^k}^2 \\ &\geq 2^{-k} \|b_j\|_{\mathbb{R}^k}^2. \end{aligned}$$

Therefore

$$\|b_j\|_{\mathbb{R}^k} \leq 2^{k/2} \lambda_j(\Lambda).$$

□

The LLL algorithm proceed as follows. Start with a basis $(b_1, ..., b_k)$ of a lattice Λ in $\mathbb{R}^k$.

Step I: *for $j = 2, ..., n$, and given j, for $i = 1, ..., j-1$, replace b_j by $b_j - \lceil \mu_{j,i} \rfloor b_i$ where $\lceil \mu_{j,i} \rfloor$ is the integer nearest to $\mu_{j,i}$.*

Step II: *if there is a subscript j such that $\left\|b_{j+1}^* + \mu_{j+1,j} b_j^*\right\|_{\mathbb{R}^k}^2 < \frac{3}{4} \left\|b_j^*\right\|_{\mathbb{R}^k}^2$ then interchange b_j and b_{j+1} and go to step I.*

The first step doesn't change the product of Gram determinants

$$\begin{aligned} P(b_1, ..., b_k) &= \prod_{p=1}^{k} \det(b_i.b_j)_{1 \le i,j \le p} \\ &= \prod_{j=1}^{k} \|b_j^*\|_{\mathbb{R}^k}^{2(n-j+1)} \end{aligned}$$

while each interchange decreases $P(b_1, ..., b_k)$ by a factor $< \frac{3}{4}$. This is the key observation to bound above the number of steps of the LLL algorithm.

The Gram determinants are naturally associated with the Grassmann algebra, it is the reason why it is convenient to use Malher's theory of compound sets. The result below is a particular case of the Malher theorem which can be found in [Schm]. Actually it is stated in terms of parallelepipeds and integer points, but the equivalence of norms, allows us to state it with the standard Euclidean norm and a general lattice in $\mathbb{R}^k$.

Let p be an integer in $\{1, ..., k\}$. The Grassmann algebra $\wedge^p \mathbb{R}^k$ is equipped with the Euclidean structure defined by the Gram determinants,

$$\|x_1 \wedge ... \wedge x_p\|_{\wedge^p \mathbb{R}^k}^2 = \det(x_i.x_j)_{1 \le i,j \le p}.$$

For a lattice Γ in $\mathbb{R}^k$, denote $\wedge^p \Gamma$ the lattice in $\wedge^p \mathbb{R}^{d+1}$ spanned by the vectors $x_1 \wedge ... \wedge x_p$, when $x_1, ..., x_p$ range in Γ. A particular case of the Malher theorem is:

There is a constant C_1 depending only on the dimension k such that for any lattice Γ in $\mathbb{R}^k$ and any $p \in \{1, ..., k\}$,

$$\lambda_1(\Gamma)...\lambda_p(\Gamma) \le C_1 \lambda_1(\wedge^p \Gamma).$$

We are now able to prove the proposition.

Proof. Suppose that $Q_n = (b_{1,n}, ..., b_{d+1,n})$ is a basis of $\mathbb{Z}^{d+1}$ such that the basis

$$(u_1, ..., u_{d+1}) = (h^n(M_\theta b_{1,n}), ..., h^n(M_\theta b_{d+1,n}))$$

is an LLL reduced basis of $\Gamma_n = h^n(\Lambda_\theta)$. The vectors

$$(v_1, ..., v_{d+1}) = (h(u_1), ..., h(u_{d+1}))$$

form a basis of $\Gamma_{n+1} = h(\Gamma_n)$. By definition of the expansion, the LLL algorithm is used to find a reduced basis of Γ_{n+1} starting with the basis $(v_1, ..., v_{d+1})$. By the key observation, to bound above the number of steps, it is enough to bound above the product of Gram determinants

$$P(v_1, ..., v_{d+1}) = \prod_{p=1}^{d+1} \det(v_i.v_j)_{1 \le i,j \le p}$$

and to compare it with the product

$$P_{\min}(\Gamma_{n+1}) = \prod_{p=1}^{d+1} (\lambda_1(\wedge^p \Gamma_{n+1}))^2$$

because this last product is a lower bound of $P(e_1, ..., e_{d+1})$ when $(e_1, ..., e_{d+1})$ ranges over all the basis of Γ_{n+1}. On the one hand, it is clear that the map $\wedge^p h$ does not increase the norms, and therefore

$$\begin{aligned} \det(v_i.v_j)_{1 \le i,j \le p} &= \|v_1 \wedge ... \wedge v_p\|_{\wedge^p \mathbb{R}^{d+1}} \\ &= \|\wedge^p h(u_1 \wedge ... \wedge u_p)\|_{\wedge^p \mathbb{R}^{d+1}} \\ &\le \|u_1 \wedge ... \wedge u_p\|_{\wedge^p \mathbb{R}^{d+1}} = \det(u_i.u_j)_{1 \le i,j \le p}, \end{aligned}$$

hence

$$P(v_1, ..., v_{d+1}) \le P(u_1, ..., u_{d+1}).$$

On the other hand, it is clear that the map $\wedge^p h^{-1}$ increases the norms by a factor of at most 2. It follows that

$$\lambda_1(\wedge^p \Gamma_n) = \lambda_1(\wedge^p h^{-1}(\wedge^p \Gamma_{n+1})) \leq 2\lambda_1(\wedge^p \Gamma_{n+1})$$

for $p = 1, ..., d+1$. Therefore

$$P_{\min}(\Gamma_{n+1}) \geq 2^{-2(d+1)} P_{\min}(\Gamma_n).$$

By the Malher theorem, we have

$$P_{\min}(\Gamma_n) \geq \prod_{p=1}^{d+1} (\frac{1}{C_1} \prod_{j=1}^{p} \lambda_j(\Gamma_n))^2,$$

together with the above lemma this imply that

$$\begin{aligned} P_{\min}(\Gamma_n) &\geq \prod_{p=1}^{d+1} (\frac{1}{C_1} \prod_{j=1}^{p} 2^{-(d+1)} \|u_j\|_{\mathbb{R}^{d+1}})^2 \\ &= \frac{1}{C_2} \prod_{p=1}^{d+1} (\prod_{j=1}^{p} \|u_j\|_{\mathbb{R}^{d+1}})^2 \\ &\geq \frac{1}{C_2} P(u_1, ..., u_{d+1}) \\ &\geq \frac{1}{C_2} P(v_1, ..., v_{d+1}) \end{aligned}$$

where the constant C_2 depends only on the dimension. By the key observation about the number of interchanges, it follows that the number of interchanges is smaller than

$$\frac{\ln \frac{P(v_1, ..., v_{d+1})}{P_{\min}(\Gamma_{n+1})}}{\ln \frac{4}{3}}$$

which is bounded above by a constant depending only on d. □

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Nicolas Chevallier
nicolas.chevallier@uha.fr
Université de Haute Alsace, Mulhouse, France