GAUSS LATTICES AND COMPLEX CONTINUED FRACTIONS

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Abstract. Our aim is to construct a complex continued fraction algorithm finding all the best Diophantine approximations to a complex number. Using the sequence of minimal vectors in a two-dimensional lattice over the ring of Gaussian integers, we obtain an algorithm defined on a submanifold of the space of unimodular two-dimensional Gauss lattices. This submanifold is transverse to the diagonal flow. The correspondence between the minimal vectors and the best Diophantine approximations ensures that our algorithm reaches its goal. A byproduct of the algorithm is the best constant for the complex version of Dirichlet's theorem about approximations of complex numbers by quotients of two Gaussian integers.

1. Introduction

Let us start with a very brief and partial account of the history of complex continued fractions (see [28], [29] or [30] for detailed historical accounts). Since the pioneering works [27], [17] and [18] of N. Michelangeli in 1887, Adolf Hurwitz in 1888 and Julius Hurwitz in 1895, complex continued fractions have been considered by many authors during the 20th century and at the beginning of the 21st century. Adolf Hurwitz considered continued fractions with partial quotients in a "system" $S \subset \mathbb{C}$ and the work of Julius Hurwitz used the $(1+i)\mathbb{Z}[i]$ subring of the ring of Gaussian integers, see also [25]. An important contribution of Adolf Hurwitz concerned the continued fractions associated with the ring of Gaussian integers using the nearest Gaussian integer (we refer to it as Adolf Hurwitz continued fraction). In his 1888 article, Adolf Hurwitz showed the non-trivial fact that the sequence of moduli of the denominators of such a continued fraction is increasing. In 1973, R. Lakein [23] studied complex continued fractions associated with the rings of integers of the quadratic number fields $\mathbb{Q}[\sqrt{-1}]$, $\mathbb{Q}[\sqrt{-3}]$, $\mathbb{Q}[\sqrt{-7}]$ and $\mathbb{Q}[\sqrt{-11}]$ (the imaginary quadratic fields with Euclidean rings of integers). For instance, he proved that the convergents associated with the Adolf Hurwitz continued fraction algorithm are best approximations for all complex numbers not in a countable family of lines and circles. At about the same time, in 1975, A. Schmidt proposed an algorithm based on the concept of Farey sets, very different from the A. Hurwitz continued fraction algorithm, see [31]. In 1985, A. Tanaka proposed a complex continued fraction algorithm ([32]) which turned out to be a new version of the Julius Hurwitz continued fraction algorithm, see [28]. More recently, D. Hensley produced complex numbers, solutions of irreducible quartic equations over $\mathbb{Q}[i]$, with a bounded, not ultimately periodic sequence of "Adolf Hurwitz" partial quotients, see [14]. In 2014, S. G. Dani and A. Nogueira, [7], proposed a general approach to complex continued fractions associated with the ring of Gaussian integers. Their approach has been taken up by other authors. In 2019, H. Ei, S. Ito, H. Nakada and R. Natsui studied the construction of the natural extension of the Hurwitz complex continued fraction map, see [10]. Beside they proved a "Legendre's theorem" for Adolf Hurwitz continued fractions. Also, in 2018, the PHD thesis of G. G. Robert [30] gave an almost complete overview of known results and many interesting new results.

In this work we propose a lattice approach to complex continued fractions associated with the ring of Gaussian integers. In particular, we address the question of finding all the best approximations of a complex number.

The lattice approach goes back to C. Hermite ([15]) and G. Voronoï ([33]). In a sequence of works, among which [19, 20, 21], J. C. Lagarias studied the best simultaneous Diophantine approximations and clearly stated the connections with the shortest vectors in lattices [19], see also [5].

Our starting point comes from the ordinary real continued fractions. In the space of dimension two unimodular lattices, $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$, let us consider the subset of lattices whose two minima with respect to the sup norm are equal. It is known that the first return map on this subset induced by the left action of the diagonal flow $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $t \in \mathbb{R}$, is a two-fold extension of the natural extension of the Gauss map $x \to \{1/x\}$ (see [6], see also [11] where another version of the natural extension is given). Observe that the ergodicity of the diagonal flow implies the ergodicity of the first return map. In the complex case, we shall use the exact same idea where the space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ is replaced by the space $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i])$ of unimodular lattices in $\mathbb{C}^2$. Like in the real case, we shall exploit two basic correspondences:

- The correspondence between pairs of consecutive minimal vectors in a lattice and the intersection of the orbits of the flow g_t with the transversal
$$T = \{\Lambda \in \mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i]) : \lambda_1(\Lambda) = \lambda_2(\Lambda)\}$$
where the minima are associated with the sup norm in $\mathbb{C}^2$ (see Lemma 26, notice that $\lambda_i(\Lambda)$, $i = 1, 2$ are the complex minima of the lattice Λ, see definition 38 in the appendix). Actually, we shall use a slightly smaller transversal (see section 6).
- The correspondence between best approximations and minimal vectors (see Proposition 9).

More precisely, for each lattice Λ over the ring of Gaussian integers in $\mathbb{C}^2$, let us consider the set of *minimal vectors* in Λ, i.e., the nonzero vectors $u = (u_1, u_2) \in \Lambda$ such that for any nonzero $z = (z_1, z_2) \in \Lambda$,
$$|z_1| \le |u_1| \text{ and } |z_2| \le |u_2| \Rightarrow |z_1| = |u_1| \text{ and } |z_2| = |u_2|.$$
Let us order these minimal vectors according to the moduli of their second coordinate. It is not difficult to prove that when $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two consecutive minimal vectors in this sequence, then the interior of the cylinder
$$C(u, v) = \{(z_1, z_2) : |z_1| \le \max(|u_1|, |v_1|),\ |z_2| \le \max(|u_2|, |v_2|)|\}$$
does not contain any nonzero element of Λ (see Lemma 4). Then, we can find a real number t such that the action of g_t transforms the cylinder $C(u, v)$ into a cylinder of the same width and height. For this value t, the action of g_t on u and v gives two new vectors u' and v' with sup norms $|u'|_\infty = |v'|_\infty = \lambda_1(g_t\Lambda) = \lambda_2(g_t\Lambda)$. Thus, the new lattice $g_t\Lambda$ is in the transversal T.

For a lattice of the shape
$$\Lambda_\theta = \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \mathbb{Z}[i]^2,$$
the sequence of minimal vectors gives all the best approximation vectors of the complex number θ, see Proposition 9. Thanks to the aforementioned work of R. Lakein, we know that the convergents associated with θ by Adolf Hurwitz's continued fraction expansion are best approximations for almost all $\theta \in \mathbb{C}$. Thus, the sequence of convergents of

Adolf Hurwitz's continued fraction expansion of θ is given by a subsequence of the sequence of minimal vectors of the lattice Λ_θ. So we can consider the transversal together with the first return map like a complex continued fraction map in the space of lattices $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i])$.

An important difference with the real case is that two consecutive minimal vectors are no longer necessarily primitive. Our first significant result is

Theorem 1. *If u and v are two consecutive minimal vectors of a lattice over the ring of Gaussian integers Λ in $\mathbb{C}^2$, then the sublattice $\mathbb{Z}[i]u+\mathbb{Z}[i]v$ is of index 1 or 2 in Λ. Furthermore, when $\mathbb{Z}[i]u+\mathbb{Z}[i]v$ is of index two,*

$$\Lambda = \langle u,v\rangle_J \overset{def}{=} \{gu+hv : (g,h)\in \mathbb{Z}[i]^2 \cup J^2\}$$

where $J=\frac{1}{1+i}\mathbb{Z}[i]\setminus\mathbb{Z}[i]$.

Observe that, in the case of index two, the lattice Λ is like a centered cubic lattice, where the index two ideal $2\mathbb{Z}$ in $\mathbb{Z}$ is replaced by the index two ideal $(1+i)\mathbb{Z}[i]$ in $\mathbb{Z}[i]$.

Our second result is about the geometry of numbers for two-dimensional lattices over the ring of Gaussian integers. It is the counterpart in the complex case of the easy result:

If $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are two linearly independent vectors in $\mathbb{R}^2$ then the interior of the rectangle
$R(u,v)=\{(x_1,x_2) : |x_1|\le \max(|u_1|,|v_1|),\ |x_2|\le \max(|u_2|,|v_2|)|\}$
contains no nonzero vector of the lattice $\mathbb{Z}u+\mathbb{Z}v$ iff u,v and $u\pm v$ are not in the interior of $R(u,v)$.

Theorem 2. *Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be two vectors in $\mathbb{C}^2$ such that $|u_1|>0$, $|u_1|\ge|v_1|$, $|v_2|>0$ and $|v_2|\ge|u_2|$.*

(1) *Zero is the only element of $\mathbb{Z}[i]u+\mathbb{Z}[i]v$ in the interior of the cylinder*

$$C(u,v)=\{(z_1,z_2) : |z_1|\le|u_1|,\ |z_2|\le|v_2|\}$$

iff $gu+hv\notin \overset{o}{C}(u,v)$ for all nonzero g, $h\in\mathbb{Z}[i]^2$ with $|g|\times|h|\le\sqrt{2}$.

(2) *Zero is the only element of $\langle u,v\rangle_J$ in the interior of the cylinder $C(u,v)$ iff $gu+hv\notin \overset{o}{C}(u,v)$ for all $(g,h)\in J^2$ with $|g|=|h|=\frac{1}{\sqrt{2}}$.*

The proof of this theorem depends only on elementary geometry, but is not as simple as in the real case. We use a computer to rule out many cases. We shall also give a variant of this result with strict inequality and a slightly more precise corollary, see section 4.

Next theorem explains how to compute inductively the sequence of minimal vectors of a lattice over the ring of Gaussian integers in $\mathbb{C}^2$. Let us equip $\mathbb{C}^2$ with the *lexicographic preorder*

$$(x_1,x_2)\prec(y_1,y_2)$$

iff $|x_2|<|y_2|$ or $|x_2|=|y_2|$ and $|x_1|\le|y_1|$.

Theorem 3 (Continued fraction algorithm)**.** *Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be two consecutive minimal vectors in a unimodular lattice Λ with $|u_2|<|v_2|$. Let $w_1=\frac{v_1}{u_1}$ and $w_2=\frac{u_2}{v_2}$. If $w_1\neq 0$ then there exists $v'\in\Lambda$ a minimal vector such that v and v' are two consecutive minimal vectors and*

- *if $\det_{\mathbb{C}}(u,v)=1$, then v' is any vector that is minimal for the preoder $\prec$ in the set*

$$E_1=\left\{z=-au+gv : a\in\{1,1+i\},\ g\in\mathbb{Z}[i],\ |\tfrac{a}{w_1}-g|<1\right\}.$$

Moreover with $u' = v = (u'_1, v'_2 w'_2)$ and $v' = -au + gv = (u'_1 w'_1, v'_2)$, we have

$$w'_1 = g - \frac{a}{w_1}, \qquad w'_2 = \frac{1}{g - aw_2}. \tag{1}$$

- *if $\det_{\mathbb{C}}(u, v) = 1 + i$, then v' is any vector that is minimal for the preoder $\prec$ in the set*

$$E_2 = \left\{ z = -\tfrac{1}{1+i}(u+v) + gv : g \in \mathbb{Z}[i],\ |\tfrac{1}{(1+i)w_1} + \tfrac{1}{(1+i)} - g| < 1 \right\}.$$

Moreover with $u' = v = (u'_1, v'_2 w'_2)$ and $v' = -au + gv = (u'_1 w'_1, v'_2)$, we have

$$w'_1 = g - \tfrac{1}{(1+i)w_1} - \tfrac{1}{(1+i)}, \qquad w'_2 = \frac{1}{g - \tfrac{1}{(1+i)} w_2 - \tfrac{1}{(1+i)}}. \tag{2}$$

The set E_1 in the above theorem has eight elements at most and E_2 has four elements at most because there are at most four Gaussian integers g such that $|g - w| < 1$ for a given complex number w. Therefore, the map

$$T_G : (w_1, w_2) \to (w'_1, w'_2)$$

is easy to compute. This map is the core of the first return map in the transversal, see Theorem 9. In fact, the minimal vectors u' and v' can be easily computed because $u'_1 = u_1 w_1$, $v'_2 = v_2 / w'_2$ and as explained before, it is possible to bring the lattice Λ in the transversal using the flow g_t and the two consecutive minimal vectors u' and v'.

In the first case of Theorem 3, the new consecutive minimal vectors u', v' have index 1 or 2 (determinant 1 or $1 + i$) because by Theorem 1 two consecutive minimal vectors have index 1 or 2. In the second case of Theorem 3, the new consecutive minimal vectors u', v' have index 1 because u, v have index 2 and it is not possible for two consecutive pairs of minimal vectors u, v and $u' = v, v'$ to have both index 2, see Proposition 20. It is worth noticing that the proof of this latter proposition uses Theorem 2.

The map T_G might have some links with the natural extension of the Adolf Hurwitz map studied in [10]. Indeed, T_G could be the natural extension of the unknown algorithm that computes all the best approximations of complex numbers while the map defined by H. Ei, S. Ito, H. Nakada and R. Natsui is the natural extension of the Hurwitz map which gives only a subsequence of the sequence of best approximations (see again [23]).

In the first case of Theorem 3 with $a = 1$, the condition $|\frac{a}{w_1} - g| < 1$ is the condition considered by Dani and Nogueira to define an approximation sequence, see [7]. In Theorem 3, the second variable controls the choice among the possible Gaussian integers g.

It is not that easy to have an explicit description of the transversal or of the domain of definition of T_G. However, with a good choice of the parametrization, this domain becomes a finite union of products of subsets in the complex plane whose boundaries are arcs of circle, see Figure 3 in subsection 4.3. The domain of definition can be found thanks to Theorem 2, see section 4 where a description of the domain is given. We also give the invariant measure of the first return map of the flow g_t in the transversal. The open transversal is parametrized with three parameters θ, w_1, w_2 where $\theta \in [0, \pi/2]$ and (w_1, w_2) is in an open set included in $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}^2$.

Theorem 4 (Invariant measure)**.** *Using the parametrization of the transversal (see section 6), the measure ν induced by the Haar measure in $\mathrm{SL}(2, \mathbb{C}) / \mathrm{SL}(2, \mathbb{Z}[i])$ and the flow g_t, has the density*

$$h(\theta, w_1, w_2) = \frac{32}{|1 - w_1 w_2|^4}$$

with respect to the Lebesgue measure of $[0, \pi/2] \times \mathbb{D}^2$.

The constant 32 depends on our choice of the normalization of the Haar measure.

A byproduct of our work is the exact value of the one-dimensional complex Dirichlet constant. To the best of our knowledge this constant was unknown.

Theorem 5 (Complex Dirichlet constant)**.** *For every complex number z and for every real number $Q \geq 1$, there exist two Gaussian integers p and q such that*

$$\begin{cases} 0 < |q| < Q, \\ |qz - p| \leq \frac{\sqrt{2}}{3-\sqrt{3}} \times \frac{1}{Q}, \end{cases}$$

where $\frac{\sqrt{2}}{3-\sqrt{3}} = \frac{1}{\sqrt{6-3\sqrt{3}}} = 1.115355\ldots$. Furthermore the set of complex numbers z for which the constant $\frac{\sqrt{2}}{3-\sqrt{3}}$ can be improved, is of zero Lebesgue measure.

In fact, the optimality of the constant is slightly stronger.

Theorem 5 bis. *For almost all $\theta \in \mathbb{C}$, all $C < \frac{\sqrt{2}}{3-\sqrt{3}}$ and all $T \geq 1$, there exists $Q \geq T$ such that the system*

$$\begin{cases} 0 < |q| < Q, \\ |qz - p| \leq C \times \frac{1}{Q}, \end{cases}$$

has no solution with $p, q \in \mathbb{Z}[i]$.

The essential ingredient of the proof of these latter theorems is Corollary 19 of Theorem 2 about the geometry of numbers (or the explicit description of the transversal). With this description we can show that the best Dirichlet constant is bounded above by $\frac{\sqrt{2}}{3-\sqrt{3}}$. To see that this constant is the best possible constant for almost all $\theta \in \mathbb{C}$ we use an additional tool, the ergodicity of the diagonal flow g_t.

It should be noticed that the complex version of Hurwitz best constant $\frac{1}{\sqrt{5}}$ is known. In 1925, Lester Ford [12] proved that for all irrational complex numbers z there exist infinitely many Gaussian integers p and $q \neq 0$ such that $|z - \frac{p}{q}| < \frac{1}{\sqrt{3}|q|^2}$. The constant $\frac{1}{\sqrt{3}}$ is the best possible. Ford's proof did not use continued fractions and in 1975 R. Lakein gave a new proof of this result using complex continued fractions (see [24]).

The paper is organized as follows. We begin by some preliminaries on lattices over the ring of Gaussian integers, minimal vectors, the sequence of minimal vectors associated with a lattice and the relation between minimal vectors and best Diophantine approximations. Next, we prove the theorem about the index of consecutive minimal vectors. In the next section, we prove the geometry of numbers' result, a more explicit version of this result (see Corollary 19) and an example showing that two linearly independent minimal vectors can both be successors of the same minimal vector. Thanks to Theorem 2, we prove that two consecutive pairs of consecutive minimal vectors cannot have both index 2, see Proposition 20.

Next, we define the transversal and a parametrization of the transversal, then we give explicit formulas for the first return map in the transversal, see Theorem 3 and 9. Then, we prove Theorem 4 about the density of the measure induced by the flow. Finally, we prove Dirichlet's theorem. We finish the paper by two more small sections and an appendix. In the first of these sections we explain how the Gauss reduction algorithm of basis in two-dimensional lattices can be used to find two consecutive minimal vectors. The second section is devoted to a few open questions. The appendix is devoted to some basic facts about lattices over the ring of Gaussian integers.

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2. Preliminaries

2.1. **Notations.** We collect the notations that we shall use.

- $|z|$ is the modulus of the complex number z and $\arg z \in [0, 2\pi)$ its argument.
- If E is a subset in $\mathbb{C}$, $\overline{E}$ and $\overset{o}{E}$ denote its closure and its interior. Although we are working with complex numbers there should not be any confusion between closure and conjugate. Most of the time "the bar" will be used for the closure.
- $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. $D(a, r)$ denote the closed disk of center $a \in \mathbb{C}$ and radius r and $\mathbb{D}(a, r)$ the open disk of center a and radius r.
- $\mathbf{C}(a, r)$ denote the circle of center $a \in \mathbb{C}$ and radius r.
- $|(z_1, z_2)|_\infty = \max(|z_1|, |z_2|)$ is the sup norm on $\mathbb{C}^2$ and $B_\infty(x, r)$ is the closed ball of radius r and center x in $\mathbb{C}^2$ associated with the sup norm.
- Let a and b be two non-negative real numbers and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in $\mathbb{C}^2$. We define the cylinders
$$\begin{aligned} C(a,b) &= \{(x,y) \in \mathbb{C}^2 : |x| \le |a|, |y| \le |b|\}, \\ C(u) &= C(|u_1|, |u_2|), \\ C(u,v) &= C(\max(|u_1|,|v_1|), \max(|u_2|,|v_2|)), \\ C_1(a) &= \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| \le a\}, \\ C_2(a) &= \{(z_1,z_2) \in \mathbb{C}^2 : |z_2| \le a\}. \end{aligned}$$
- When $C(u, v)$ has nonempty interior, $C(u, v)$ is the unit ball of a norm $|.|_{u,v}$ defined on $\mathbb{C}^2$. Observe that for any $x = (x_1, x_2) \in \mathbb{C}^2$,
$$|x|_{u,v} = \max\left(\tfrac{|x_1|}{\max(|u_1|,|v_1|)}, \tfrac{|x_2|}{\max(|u_2|,|v_2|)}\right).$$
- $\mathbb{U}_n = \{z \in \mathbb{C} : z^n = 1\}$ is the group n-th roots of unity in $\mathbb{C}$.
- $\mathbb{D}_8$ is the group of isometries acting on $\mathbb{C}$ generated by the multiplications by elements in $\mathbb{U}_4$ and by conjugation.
- $(x_1, x_2) \prec (y_1, y_2)$ iff $|x_2| < |y_2|$ or $|x_2| = |y_2|$ and $|x_1| \le |y_1|$ is the lexicographic preorder on $\mathbb{C}^2$.
- When A is a subset of $\mathbb{C}$ or $\mathbb{C}^2$, $A^* = A \setminus \{0\}$.
- $\mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$, $I = (1 + i)\mathbb{Z}[i]$ and $J = \frac{1}{1+i}(\mathbb{Z}[i] \setminus I)$.
- For $u, v \in \mathbb{C}^2$, $\langle u, v\rangle_J \overset{def}{=} \{gu + hv : (g, h) \in \mathbb{Z}[i]^2 \cup J^2\}$
- We shall use also the following sets
$$\begin{aligned} \mathcal{C} =& \{z \in \mathbb{C} : |z| < 1, \arg z \in [0, \tfrac{\pi}{4}]\} \\ \mathcal{D} =& \{w \in \mathbb{C} : |z| < 1, \mathrm{d}(w_2, 1) > 1, \mathrm{d}(w_2, 1-i) > 1\}, \\ \mathcal{T} =& \{w \in \mathbb{C} : |z| < 1, \mathrm{d}(w_2, 1) > \sqrt{2}, \mathrm{d}(w_2, -i) > \sqrt{2}\}, \\ F =& \{(1,1), (1,-i), (1, 1-i), (1, 1+i), (1+i, 1)\}. \\ S =& [-\tfrac{1}{2}, \tfrac{1}{2}) + [-\tfrac{1}{2}, \tfrac{1}{2})i \end{aligned}$$
- For $\theta \in \mathbb{C}$,
$$M_\theta = \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix}, \qquad \Lambda_\theta = M_\theta \mathbb{Z}[i]^2.$$

- When A is a commutative ring with unit 1_A, $\mathrm{SL}(2, A)$ is the set of 2×2 matrices with entries in A and determinant 1_A.
- $\lambda_1(\Lambda, \|.\|, \mathbb{C})$ and $\lambda_2(\Lambda, \|.\|, \mathbb{C})$ are the two complex minima of a Gauss lattice Λ in $\mathbb{C}^2$ associated with the norm $\|.\|$, see definition 38.
- The space of unimodular lattices in $\mathbb{C}^2$
$$\Omega_2 = \mathrm{SL}(2, \mathbb{C}) / \mathrm{SL}(2, \mathbb{Z}[i]).$$
- The transversal T is defined in subsection 6.1 and T', T_1, T_2 are defined in subsection 6.2.
- The negligible set $\mathcal{N}$ is defined in subsection 6.2.
- The parametrizations $\Psi_k(\theta, w_1, w_2)$ are defined in Proposition 27 in subsection 8.1.
- The sets W_1 and W_2 are defined in subsection 8.3.
- The sets W_1' and W_2', the map T_G and the coefficients $a_k(w_1, w_2)$ are defined in subsection 8.4.

2.2. **The set of unimodular Gauss lattices in $\mathbb{C}^2$.**

Definition 1. Let E be a finite dimensional $\mathbb{C}$-vector space. A subset Λ in E is a Gauss lattice if it is a $\mathbb{Z}[i]$-submodule of E, if it is a discrete subset of E and if it generates the vector space E.

Let Ω_2 be the set of Gauss lattices Λ in $\mathbb{C}^2$ that admits a basis (u, v) with determinant in $\mathbb{U}_4 = \{\pm 1, \pm i\}$. By definition, $\Lambda = M\mathbb{Z}[i]^2$ where M is the matrix with columns u and v. Changing u to $\pm u$ or to $\pm iu$, we can assume that $M \in \mathrm{SL}(2, \mathbb{C})$. Next proposition is clear.

Proposition 2. *The map*
$$M\,\mathrm{SL}(2, \mathbb{Z}[i]) \in \mathrm{SL}(2, \mathbb{C}) / \mathrm{SL}(2, \mathbb{Z}[i]) \to M\mathbb{Z}[i]^2 \in \Omega_2$$
is well defined and is bijective.

Thanks to the proposition, we can identify Ω_2 and $\mathrm{SL}(2, \mathbb{C}) / \mathrm{SL}(2, \mathbb{Z}[i])$ and use results from ergodic theory. For $t \in \mathbb{R}$, consider the matrices
$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$
The flow $(g_t)_{t \in \mathbb{R}}$ acts on Ω_2 by left multiplication :
$$g_t \Lambda = \{g_t x : x \in \Lambda\} = g_t M\mathbb{Z}[i]^2 \cong g_t M\,\mathrm{SL}(2, \mathbb{Z}[i]).$$

2.3. **Minimal vectors.** The notion of minimal vector goes back to Voronoï, see [33]. He used minimal vectors to find units in cubic fields. The Voronoï algorithm has been generalized by Buchmann to find units in some quartic and quintic fields, see [2, 3]

Definition 3. Let Λ be a Gauss lattice in $\mathbb{C}^2$.

- A nonzero vector $u = (u_1, u_2) \in \Lambda$ is a *minimal* vector in Λ if for every nonzero $v \in \Lambda$, $v \in C(u) = \{(z_1, z_2) : |z_1| \leq |u_1|, |z_2| \leq |u_2|\} \Rightarrow |v_1| = |u_1|$ and $|v_2| = |u_2|$.
- Two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are *equivalent* if $C(u) = C(v)$.
- Two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are *consecutive* iff $|u_2| < |v_2|$ and there is no minimal vector $w = (w_1, w_2)$ such $|u_2| < |w_2| < |v_2|$.

Remark 1. Following Buchmann ([2, 3]), we could have define the minimal vectors using the preoder $u \ll v$ iff $|u_1| \leq |v_1|$ and $|u_2| \leq |v_2|$ for u, v be in $\mathbb{C}^2$. With this preorder, the minimal vectors of a Gauss lattice Λ in $\mathbb{C}^2$ are the minimal elements in $(\Lambda \setminus \{0\}, \ll)$. Observe that the lexicographic order $\prec$ is also used by Buchmann in the same papers.

Remark 2. If $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two minimal vectors in a lattice $\Lambda \subset \mathbb{C}^2$ and if $|v_2| > |u_2|$, then by definition, $|u_1| > |v_1|$. Therefore, there exist complex numbers w_1 and w_2 unique such that $u = (u_1, v_2 w_2)$ and $v = (u_1 w_1, v_2)$. Moreover $|w_1|, |w_2| < 1$.

We collect a few easy lemmas about minimal vectors.

Lemma 4. *Two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in a Gauss lattice $\Lambda \subset \mathbb{C}^2$ are consecutive iff $|u_2| < |v_2|$ and the only lattice point in the interior of $C(u, v)$ is zero.*

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two minimal vectors with $|u_2| < |v_2|$. If the set $\overset{o}{C}(u, v) \cap \Lambda \setminus \{0\}$ is nonempty, then it is finite and there is a $w = (w_1, w_2)$ minimal in this set for the lexicographic preorder $\prec$. On the one hand, w is minimal in Λ. On the other hand, $|w_1| < |u_1|$ and $|w_2| < |v_2|$ and since u is a minimal vector we have $|w_2| > |u_2|$. Hence u and v are not consecutive.

Conversely, if u and v are not consecutive there is a minimal vector w with $|u_2| < |w_2| < |v_2|$. Since w is minimal $|u_1| > |w_1|$, hence $w \in \overset{o}{C}(u, v) \cap \Lambda$. □

Next lemma is clear.

Lemma 5. *Let Λ be a Gauss lattice in $\mathbb{C}^2$ and let u be a minimal vector in Λ.*

- *All minimal vectors $v \in \Lambda$ such that u and v are consecutive, are equivalent.*
- *If u' and v are minimal vectors such that u is equivalent to u', and u and v are consecutive, then u' and v are consecutive.*

Next lemma is useful to construct minimal vector in lattice.

Lemma 6. *Let Λ be a Gauss lattice in $\mathbb{C}^2$ and let r be a positive real number. Let C be the infinite cylinder $C_1(r) = \{(z_1, z_2) : |z_1| \leq r\}$ or its interior.*

- *The set $C \cap \Lambda \setminus \{0\}$ is nonempty and admits a minimal element for the lexicographic order.*
- *If u is a minimal element for the lexicographic order in the set $C \cap \Lambda \setminus \{0\}$, then u is minimal in Λ.*

Proof. Since $r > 0$, by Minkowski convex body theorem $C \cap \Lambda \setminus \{0\}$ is nonempty. Let $C_2(\rho) = \{(z_1, z_2) : |z_2| \leq \rho\}$. If $v = (v_1, v_2)$ is in $C \cap \Lambda \setminus \{0\}$, then $C \cap \Lambda \setminus \{0\} \cap C_2(|v_2|)$ is finite and nonempty and so $C \cap \Lambda \setminus \{0\} \cap C_2(|v_2|)$ must contain a minimal element u for the lexicographic preorder. This element u is also minimal in $C \cap \Lambda \setminus \{0\}$ for the lexicographic preorder.

If $w = (w_1, w_2) \in C(u) \cap \Lambda \setminus \{0\}$ then $w \prec u$ and $w \in C$. Since u is minimal for the lexicographic order we also have $u \prec w$ which implies $|u_2| = |w_2|$ and $|w_1| = |v_1|$, hence u is minimal in Λ □

2.4. **The sequence of minimal vectors.** Given a Gauss lattice Λ in $\mathbb{C}^2$, the set of minimal vectors can be arranged in a sequence $(X_n(\Lambda))_{n \in I_\Lambda} = (z_{1,n}, z_{2,n})_{n \in I_\Lambda}$ where I_Λ is an interval in $\mathbb{Z}$ such that the sequence $(|z_{2,n}|)_{n \in I_\Lambda}$ is increasing and each minimal vector is equivalent to a minimal vector of the sequence. This sequence might be finite, infinite one sided or two sided. Two minimal vectors are consecutive if and only if they are equivalent to two consecutive terms of the sequence $(X_n(\Lambda))_{n \in I_\Lambda}$. For all $n \in I_\Lambda$, let denote $r_n(\Lambda) = |z_{1,n}|$ and $q_n(\Lambda) = |z_{2,n}|$. The three following results are standard in the frame work of best Diophantine approximations and continued fractions. The second inequality of the first item gives an upper bound of the Dirichlet complex constant. The lemma will not be used in the sequel.

Lemma 7. *Let Λ be a lattice in $\mathbb{C}^2$ and let $(X_n(\Lambda))_{n\in I_\Lambda}$ be the sequence of minimal vectors of Λ.*

(1) *If n and $n+1 \in I_\Lambda$, then $\frac{1}{2}|\det_{\mathbb{C}}(\Lambda)| \le q_{n+1}(\Lambda)r_n(\Lambda) \le \frac{4}{\pi}|\det_{\mathbb{C}}(\Lambda)|$.*
(2) *If n and $n+14 \in I_\Lambda$, then $q_{n+14}(\Lambda) \ge Cq_n(\Lambda)$ where $C = \frac{1}{2}(1+\cos(\frac{2\pi}{7})) > 1.1234$*
(3) *If n and $n+70 \in I_\Lambda$, then $r_{n+70}(\Lambda) \le \frac{1}{2}r_n(\Lambda)$.*

Proof. 1. Making use of Minkowski convex body theorem with the cylinder $C(X_n(\Lambda), X_{n+1}(\Lambda))$ and the lattice Λ, we obtain that $(\pi q_{n+1}(\Lambda)r_n(\Lambda))^2 \le 16|\det_{\mathbb{R}}(\Lambda)|$, thus $q_{n+1}(\Lambda)r_n(\Lambda) \le \frac{4}{\pi}|\det_{\mathbb{C}}(\Lambda)|$. Since the minimal vectors $X_n(\Lambda) = (z_{1,n}, z_{2,n})$ and $X_{n+1}(\Lambda) = (z_{1,n+1}, z_{2,n+1})$ are linearly independent, $|\det_{\mathbb{R}}(X_n(\Lambda), X_{n+1}(\Lambda))|$ is a positive integer multiple of $|\det_{\mathbb{R}}(\Lambda)|$. It follows that $|\det_{\mathbb{C}}(X_n(\Lambda), X_{n+1}(\Lambda))| \ge |\det_{\mathbb{C}}(\Lambda)|$ and then

$$\begin{aligned}2q_{n+1}(\Lambda)r_n(\Lambda) &\ge |z_{1n}z_{2,n+1}| + |z_{2n}z_{1,n+1}| \\ &\ge |\det_{\mathbb{C}}(X_n(\Lambda), X_{n+1}(\Lambda))| \ge |\det_{\mathbb{C}}(\Lambda)|.\end{aligned}$$

2. This is a standard application of the pigeonhole principle. Given $r > 0$ and $C' < C$, 7 closed disks of radius $\frac{1}{2}r$ are enough to cover a disk of radius r and 8 open disks of radius $\frac{1}{2}r$ are enough to cover a closed disk of radius $C'r$. The first covering result is very well known and easy, the second is due to G. Fejes Toth, [9]. It follows that $7\times 8 = 56$ translates of the semi-open box $B_1 = D(0, \frac{1}{2}r_n(\Lambda)) \times \overset{o}{D}(0, \frac{1}{2}q_n(\Lambda))$ can cover the box $B_2 = C(r_n(\Lambda), C'q_n(\Lambda))$ for any $C' < C$. Now if $q_{n+14}(\Lambda) < Cq_n(\Lambda)$ then all the $4\times 15 = 60$ points of the set $\mathbb{U}_4\{X_n(\Lambda), \dots, X_{n+14}(\Lambda)\}$ are in the box $B_2 = C(r_n(\Lambda), C'q_n(\Lambda))$ with $C' = \frac{q_{n+14}(\Lambda)}{q_n(\Lambda)}$, so at least two of them are in the same translate of the box B_1. It follows that their difference is in the box $2B_1$ which contradicts that $X_n(\Lambda)$ is a minimal vector.
3. We use twice the pigeonhole principle. We can split $\mathbb{C}$ in eight angular sector $C_1, \dots, C_8$ such that if z and z' are in the same angular sector then $|z - z'| \le \max(|z|, |z'|)$. Consider the 57 minimal vectors $X_n(\Lambda) = (z_{1n}, z_{2n}), \dots, X_{n+56}(\Lambda) = (z_{1,n+56}, z_{2,n+56})$. There is a sector C_i that contains at least seven of the z_{1j}, say for the $j \in J$. Since $r_j(\Lambda) \le r = r_n(\Lambda)$ for $j \in J$ and $\operatorname{card} J \ge 7$, there exists $k \ne j$ in J such that $|z_{1k} - z_{1j}| \le \frac{1}{2}r$. Therefore, the vector $X = X_k(\Lambda) - X_j(\Lambda) = (x_1, x_2)$ is such that $|x_1| \le \frac{1}{2}r$ and $|x_2| = |z_{2k} - z_{2j}| \le 2\max(|z_{2k}|, |z_{2j}|) \le 2q_{n+56}(\Lambda)$. The cylinder $C(X)$ contains a minimal vector X_i which is one of $X_n(\Lambda), \dots, X_{n+56+14}(\Lambda)$ so we are done. □

2.5. **Minimal vectors and Diophantine approximations.**

Definition 8. Let θ be a complex number. A pair $(p, q) \in \mathbb{Z}[i]$ is a best approximation vector of θ if $|q| > 0$ and for all $(a, b) \in \mathbb{Z}[i]^2$,

$$\begin{cases} 0 < |b| < |q| \Rightarrow |p - q\theta| < |a - b\theta| \\ 0 < |b| \le |q| \Rightarrow |p - q\theta| \le |a - b\theta| \end{cases}.$$

Proposition 9. *Let θ be a complex number and consider the lattice Λ_θ defined by*

$$\Lambda_\theta = \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \mathbb{Z}[i]^2 = M_\theta \mathbb{Z}[i]^2.$$

Then $X = \begin{pmatrix} x \\ y \end{pmatrix} = M_\theta \begin{pmatrix} p \\ q \end{pmatrix} \in \Lambda_\theta$ is a minimal vector with $y \ne 0$ iff (p, q) is a best Diophantine approximation vector of θ.

In the multidimensional real setting, Lagarias proved that a shortest vector of the lattice $g_t\Lambda_\theta$ is associated with a best Diophantine approximation of θ, see [19]. His result was stated for the Euclidean norm instead of the sup norm. That is why some best approximations are not associated with a shortest vector even in the one-dimensional case.

Proof. Suppose that $X = \begin{pmatrix} x \\ y \end{pmatrix}$ is a minimal vector with $y \neq 0$. If a and b are two Gaussian integers with $0 < |b| < |y = q|$, then $Y = \begin{pmatrix} a - b\theta \\ b \end{pmatrix} \notin C(X)$ which implies $|a - b\theta| > |p - q\theta|$. If $|b| = |q|$ and if $Y \in C(X)$ then $|a - b\theta| = |p - q\theta|$.

Conversely, if (p, q) is a best Diophantine approximation vector of θ, then for any nonzero $(a, b) \in \mathbb{Z}[i]^2$, $Y = \begin{pmatrix} a - b\theta \\ b \end{pmatrix} \in C(X)$ implies

$$\begin{cases} |a - b\theta| \leq |p - q\theta| \\ |b| \leq |q| \end{cases}.$$

If $b \neq 0$ this in turn implies $|a - b\theta| = |p - q\theta|$ and $|b| = |q|$ by definition of best approximation vectors. If $b = 0$ and $a \neq 0$ then $|a| \geq 1 > \frac{\sqrt{2}}{2} \geq |p - q\theta|$, hence $Y \notin C(X)$. $\square$

3. Proof of Theorem 1, index of lattices spanned by two consecutive minimal vectors

Let I be the ideal in $\mathbb{Z}[i]$ generated by $1+i$, i.e. $I = (1+i)\mathbb{Z}[i]$ and let $J = \frac{1}{1+i}(\mathbb{Z}[i] \setminus I)$. Theorem 1 is a consequence of the following proposition.

Proposition 10. *Let Λ be a Gauss lattice in $\mathbb{C}^2$. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two linearly independent minimal vectors in Λ and such that $\overset{o}{C}(u, v) \cap \Lambda = \{0\}$. Call L the lattice spanned by u and v. Then*

1. $\frac{1}{4}\det_{\mathbb{R}}(\Lambda) \leq |u_1|^2|v_2|^2 \leq \frac{16}{\pi^2}\det_{\mathbb{R}}(\Lambda)$,
2. *L has index 1 or 2:* $[\Lambda : L] = \frac{|\det_{\mathbb{R}}(L)|}{|\det_{\mathbb{R}}(\Lambda)|} = 1$ *or* 2.
3. *If L has index 2, then*

$$\Lambda = \{au + bv : (a, b) \in \mathbb{Z}[i]^2 \cup J^2\}$$

and $(U = u, V = \frac{1}{1+i}(u + v))$ *and* $(U' = \frac{1}{1+i}(u + v), V' = v)$ *are two bases of* Λ.

When u and v are two consecutive minimal vectors, we shall say that $[L : \Lambda]$ is the index of the two consecutive minimal vectors u and v.

Proof. Since u and v are minimal vectors, we can suppose $|u_2| \leq |v_2|$ and $|v_1| \leq |u_1|$ w.l.o.g.. By Minkowski convex body theorem,

$$\mathrm{Vol}(C(u, v)) = \pi^2|u_1|^2|v_2|^2 \leq 2^4|\det_{\mathbb{R}}(\Lambda)| = 2^4|\det_{\mathbb{C}}(\Lambda)|^2.$$

Now $|\det_{\mathbb{C}}(\Lambda)| \leq |\det_{\mathbb{C}}(L)| \leq 2|u_1||v_2|$, hence

$$|\det_{\mathbb{R}}(L)| \leq 4|u_1|^2|v_2|^2 = 4\frac{\mathrm{Vol}(C(u, v))}{\pi^2} \leq \frac{64|\det_{\mathbb{R}}(\Lambda)|}{\pi^2}.$$

Therefore,

$$\frac{|\det_{\mathbb{R}}(L)|}{|\det_{\mathbb{R}}(\Lambda)|} \leq \frac{64}{\pi^2} = 6.48\ldots$$

Therefore, $[\Lambda : L] \leq 6$. Since this index is the square of the modulus of a Gaussian integer, it is the sum of two squares and cannot be 3 or 6.

By Theorem 11 about basis in $\mathbb{Z}[i]$-modules, there exist a basis U, V of Λ and Gaussian integers a, b and c such that

$$\begin{cases} u = aU \\ v = bU + cV. \end{cases}$$

We have $V = -\frac{b}{c}U + \frac{1}{c}v$. Since u is primitive in Λ, a must be a unit in $\mathbb{Z}[i]$. By changing U to $a^{-1}U$, we can suppose $a = 1$.

Since $[\Lambda : L] = |c|^2$, the only possible values for $|c|^2$ are $1, 2, 4$ or 5. We have to exclude the values 4 and 5.

Suppose that $|c| = 2$. Again by changing V to zV where z is a unit, we can suppose $c = 2$ w.l.o.g. There exists a Gaussian integer g such that $|g - \frac{b}{c}| \le \frac{1}{\sqrt{2}}$. Since $|cg - b| \le \sqrt{2}$, $|cg - b| = 0, 1$ or $\sqrt{2}$. If $cg - b = 0$ then $V + gU = \frac{1}{c}v \in \Lambda$, but this is not possible for v is primitive. If $|cg - b| = 1$, consider the vector $w = V + gU = \frac{cg-b}{c}u + \frac{1}{c}v \in \Lambda$. Since u and v are minimal, either $|u_1| > |v_1|$ and $|v_2| > |u_2|$ or $|u_1| = |v_1|$ and $|v_2| = |u_2|$. In the first case, by convexity, w would be in the interior of the cylinder $C(u, v)$ which is not possible by assumption. In the second case, the linear independence implies $(cg - b)u \ne v$, so that one of the coordinates of $(cg - b)u$ and of v are not equal, and therefore the corresponding coordinate of w would be strictly smaller which contradicts the minimality of u and v. If $|cg - b| = \sqrt{2}$, then the inverse z of $\frac{cg-b}{c}$ is a Gaussian integer and the vector $w' = zw - u = \frac{z}{c}v$ is in Λ. But this is impossible for $|\frac{z}{c}| < 1$ and v is primitive.

Suppose that $|c| = \sqrt{5}$. There is 8 possible values for c. By changing V to zV where z is a unit, or by considering the image of Λ by the map $(z_1, z_2) \to (\overline{z_1}, \overline{z_2})$, we can suppose that $c = 2 - i$. We can also suppose that $|b| \le \frac{1}{\sqrt{2}}|c|$ by changing V to $V + gU$ where g is a Gaussian integer such that $|\frac{b}{c} - g| \le \frac{1}{\sqrt{2}}$. So $|b| \le \frac{\sqrt{5}}{\sqrt{2}}$. Now $|b|^2$ is an integer, hence $|b|^2 = 0, 1$ or 2. The case $b = 0$ is not possible for v is minimal. If $|b| = 1$, then $|\frac{b}{c}| + |\frac{1}{c}| < 1$ and $V = -\frac{b}{c}u + \frac{1}{c}v$ would be in the interior of $C(u, v)$.

It remains to consider the cases $b = 1 + i, 1 - i, -1 - i$ and $-1 + i$. Since $b = z(1 + i)$ with $z \in \mathbb{U}_4$, the vector

$$w = V + ziu = -z(1+i)\frac{2+i}{5}u + \frac{2+i}{5}v + ziu = z\frac{-1+2i}{5}u + \frac{2+i}{5}v$$

is in Λ and in the interior of $C(u, v)$ for the sum of the moduli of the coefficients of u and v is < 1. So $|c| = \sqrt{5}$ is not possible and we conclude that $|c| = 1$ or $\sqrt{2}$.

If $|c| = 1$, $L = \Lambda$.

Suppose that $|c| = \sqrt{2}$. We have

$$\begin{cases} u = U \\ v = bU + cV \end{cases}$$

and by changing V to zV for some $z \in \mathbb{U}_4$, we can suppose that $c = 1 + i$. There is a Gaussian integer g such that $b = g(1 + i)$ or $g(1 + i) + 1$. Changing V to $V + gU$, we can suppose that $b = 0$ or 1. Again $b \ne 0$ since v is primitive, hence $b = 1$. Solving in U, V, we obtain

$$\begin{cases} U = u \\ V = \frac{1}{c}(-u + v) \end{cases}$$

and for all $g, h \in \mathbb{Z}[i]$

$$gU + hV = \frac{cg - h}{c}u + \frac{h}{c}v.$$

On the other hand, $c \in I$, hence either $cg-h$ and h are both in I or $cg-h$ and h are both in $\mathbb{Z}[i] \setminus I$, which implies that

$$\Lambda = \{gU + hV : (g,h) \in \mathbb{Z}[i]^2\} \subset \{g'u + h'v : (g',h') \in \mathbb{Z}[i]^2 \cup J^2\}.$$

The reverse inclusion also holds because if $(g',h') = \frac{1}{c}(p,q)$ with $p,q \in \mathbb{Z}[i] \setminus I$, then $g'u + h'v = \frac{1}{c}(p+q)U + qV \in \Lambda$. □

4. Geometry of numbers, proof of Theorem 2

Our aim is to prove Theorem 2. In fact we shall prove the following two theorems, the first is just a reformulation of Theorem 2 using the norm $|.|_{u,v}$ instead of the cylinder $C(u,v)$. The norm is defined by
$|x|_{u,v} = \max(\frac{|x_1|}{\max(|u_1|,|v_1|)}, \frac{|x_2|}{\max(|u_2|,|v_2|)})$ for $x = (x_1,x_2) \in \mathbb{C}^2$.

Theorem 6 (Theorem 2a)**.** *Let $u = (u_1, v_2w_2)$ and $v = (u_1w_1, v_2)$ be two vectors in $\mathbb{C}^2$ with $|u_1|, |v_2| > 0$ and $|w_1|, |w_2| \leq 1$.*

1. *If $|gu + hv|_{u,v} \geq 1$ for all nonzero g, $h \in \mathbb{Z}[i]$ with $|g| \times |h| \leq \sqrt{2}$, then $|z|_{u,v} \geq 1$ for all nonzero $z \in \mathbb{Z}[i]u + \mathbb{Z}[i]v$.*
2. *If $|gu + hv|_{u,v} \geq 1$ for $(g,h) \in J^2$ with $|g| = |h| = \frac{1}{\sqrt{2}}$, then $|z|_{u,v} \geq 1$ for all nonzero $z \in \langle u,v\rangle_J$.*

The next theorem deals with strict inequality and is useful to determine the open transversal.

Theorem 7 (Theorem 2b)**.** *Let $u = (u_1, v_2w_2)$ and $v = (u_1w_1, v_2)$ be two vectors in $\mathbb{C}^2$ with $|u_1|, |v_2| > 0$ and $|w_1|, |w_2| < 1$.*

1. *If $|gu + hv|_{u,v} > 1$ for all nonzero $g, h \in \mathbb{Z}[i]$ with $|g| \times |h| \leq \sqrt{2}$ then $|z|_{u,v} > 1$ for all nonzero $z \in (\mathbb{Z}[i]u + \mathbb{Z}[i]v) \setminus \mathbb{U}_4u \cup \mathbb{U}_4v$.*
2. *If $|gu + hv|_{u,v} > 1$ for the four vectors $(g,h) = (\frac{1}{1+i}, \frac{\alpha}{1+i})$, $\alpha \in \mathbb{U}_4$, then $|z|_{u,v} > 1$ for all nonzero $z \in \langle u,v\rangle_J \setminus \mathbb{U}_4u \cup \mathbb{U}_4v$.*

The proof of these two theorems are very similar and based on many case distinctions. The first case distinction is made on the location of w_1 in the unit disk.

Let $\mathcal{C} = \{z \in \mathbb{C} : |z| < 1, \arg z \in [0, \frac{\pi}{4}]\}$. The first case distinction is $w_1 \in \bar{\mathcal{C}}$ (the closure of $\mathcal{C}$) or in $i\bar{\mathcal{C}}$ or in $-\bar{\mathcal{C}}$ or in $-i\bar{\mathcal{C}}$ or in the conjugates of one of these sets. Thanks to the following subsection about symmetries these eight cases reduce to the single case $w_1 \in \bar{\mathcal{C}}$.

The same reduction will also be helpful for computing the Dirichlet constant in the Theorem 5.

4.1. Symmetries, reduction to the case $w_1 \in \bar{\mathcal{C}}$.

Let denote $\mathbb{U}_n = \{z \in \mathbb{C} : z^n = 1\}$ the group n-th roots of unity in $\mathbb{C}$ and let denote $\mathbb{D}_8$ the group of isometries acting on $\mathbb{C}$ generated by the multiplications by elements in $\mathbb{U}_4$ and by conjugation.

Proposition 11. *Let $u = (u_1, v_2w_2)$ and $v = (u_1w_1, v_2)$ be in $\mathbb{C}^2$. Assume that $|w_1|, |w_2| \leq 1$ and $|u_1|, |v_2| > 0$. Let φ be in $\mathbb{D}_8$. Consider $u' = (u_1', v_2'\frac{1}{\varphi(1)^2}\varphi(w_2))$ and $v' = (u_1'\varphi(w_1), v_2')$ where $|u_1'|, |v_2'| > 0$. Then*

1. *For all nonzero complex numbers a and b,*

$$|au - bv|_{u,v} = |\varphi(1)\varphi(a)u' - \varphi(b)v'|_{u',v'}.$$

(2) *When* $|w_1|, |w_2| < 1$, *the vectors* u *and* v *are consecutive minimal vectors in* $\mathbb{Z}[i]u+\mathbb{Z}[i]v$ *(resp. in* $\langle u,v\rangle_J$*) iff* u' *and* v' *are consecutive minimal vectors in* $\mathbb{Z}[i]u'+\mathbb{Z}[i]v'$ *(resp. in* $\langle u',v'\rangle_J$*)*

Let us explain how the first assertion in the proposition allows us to reduces the proofs of Theorems 6 and 7 to the case $w_1 \in \overline{\mathcal{C}}$. When $\varphi \in \mathbb{D}_8$, the three maps φ, $\psi : z \in \mathbb{C} \to \psi(z) = \varphi(1)\varphi(z)$ and $\varphi' : z \in \mathbb{C} \to \varphi'(z) = \frac{\varphi(z)}{\varphi(1)^2}$ are isometries and bijection on the ring of Gaussian integers and on J. Using part 1 of the proposition, we see that, if for some vectors u, v and a subset F of $\mathcal{R}$ where $\mathcal{R} = (\mathbb{Z}[i]\setminus\{0\})^2$ or $(\mathbb{Z}[i]\setminus\{0\})^2 \cup J^2$, one has

$$\forall (g,h) \in F, |gu-hv|_{u,v} \geq 1 \Rightarrow \forall (g,h) \in \mathcal{R}^2 \text{ with } fg \neq 0, |gu-hv|_{u,v} \geq 1,$$

then one has the same implication with u', v' and

$$F' = \{(\psi(g), \varphi(h)) : (g,h) \in F\}$$

instead of F. Since the images of $\overline{\mathcal{C}}$ by the maps $\varphi \in \mathbb{D}_8$ cover the closed unit disk, we have only to deal with $w_1 \in \overline{\mathcal{C}}$.

Before proving the proposition, we need a simple formula.

Lemma 12. *For all* $\varphi \in \mathbb{D}_8$ *and all* $x, y \in \mathbb{C}$

$$\varphi(xy) = \frac{1}{\varphi(1)}\varphi(x)\varphi(y)$$

Proof. The formula is obvious since the maps $\varphi \in \mathbb{D}_8$ are of the shape $\varphi(z) = \alpha z$ or $\alpha\bar{z}$ with $\alpha \in \mathbb{U}_4$. $\square$

Proof of the proposition. 1. For all $a, b \in \mathbb{C}$ and all $\varphi \in \mathbb{D}_8$, we have

$$au - bv = (u_1(a - bw_1), v_2(aw_2 - b))$$

and using the above lemma, we obtain

$$\begin{aligned}\varphi(1)\varphi(a)u' - \varphi(b)v' &= (u_1'(\varphi(1)\varphi(a) - \varphi(b)\varphi(w_1)),\\ &\qquad v_2'(\varphi(1)\varphi(a)\frac{1}{\varphi(1)^2}\varphi(w_2) - \varphi(b)))\\ &= (u_1'(\varphi(1)\varphi(a) - \varphi(1)\varphi(bw_1)), v_2'(\varphi(aw_2) - \varphi(b)))\\ &= (u_1'\varphi(1)\varphi(a - bw_1), v_2'\varphi(aw_2 - b)).\end{aligned}$$

Therefore,

$$|au - bv|_{u,v} = |\psi(a)u' - \varphi(b)v'|_{u',v'}.$$

2. The vector u is minimal iff for all nonzero $au - bv \in \mathbb{Z}[i]u + \mathbb{Z}[i]v$ (resp. $\in \langle u,v\rangle_J$)

$$\left\{\begin{array}{l}|bw_1 - a| \leq 1\\ |aw_2 - b| \leq |w_2|\end{array}\right. \Rightarrow \left\{\begin{array}{l}|bw_1 - a| = 1\\ |aw_2 - b| = |w_2|\end{array}\right.$$

and u' is minimal iff for all nonzero $\varphi(1)\varphi(a)u' - \varphi(b)v' \in \mathbb{Z}[i]u' + \mathbb{Z}[i]v'$ (resp. $\in \langle u',v'\rangle_J$)

$$\left\{\begin{array}{l}|\varphi(1)\varphi(bw_1 - a)| \leq 1\\ |\varphi(aw_2 - b)| \leq |\frac{1}{\varphi(1)^2}\varphi(w_2)|\end{array}\right. \Rightarrow \left\{\begin{array}{l}|\varphi(1)\varphi(bw_1 - a)| = 1\\ |\varphi(aw_2 - b)| = |\frac{1}{\varphi(1)^2}\varphi(w_2)|\end{array}\right.$$

Therefore u is minimal iff u' is minimal. We see that v is minimal iff v' is minimal as well. Furthermore, by Lemma 4, u and v are consecutive iff $|au - bv|_{u,v} \geq 1$ for all nonzero $au - bv \in \mathbb{Z}[i]u + \mathbb{Z}[i]v$ (resp. $\in \langle u,v\rangle_J$). The formula $|au - bv|_{u,v} = |\psi(a)u' - \varphi(b)v'|_{u',v'}$ implies that u and v are consecutive iff u' and v' are. $\square$

4.2. **Proof of Theorem 6 and 7 when $w_1 \in \overline{\mathcal{C}}$.** We shall need the following sets

$$\begin{aligned}\mathcal{D} =& \{z \in \mathbb{C} : |z| < 1,\ \mathrm{d}(z,1) > 1,\ \mathrm{d}(z, 1-i) > 1\},\\ \mathcal{T} =& \{z \in \mathbb{C} : |z| < 1,\ \mathrm{d}(z,1) > \sqrt{2},\ \mathrm{d}(z,-i) > \sqrt{2}\},\\ F =& \{(1,1),(1,-i),(1,1-i),(1,1+i),(1+i,1)\}.\end{aligned}$$

Theorem 6 and 7 are obvious consequences of the following proposition where we assume $w_1 \in \overline{\mathcal{C}}$.

Proposition 13. *Let $u = (u_1, v_2w_2)$ and $v = (u_1w_1, v_2)$ in $\mathbb{C}^2$ be such that $w_1 \in \overline{\mathcal{C}}$ and $|w_1|, |w_2| \le 1$ (resp. < 1) and $|u_1|,\ |v_2| > 0$.*

0. *If $w_1 = 0$ then $|gu - hv|_{u,v} \ge 1$ for all nonzero Gaussian integers g, h, and $|u - hv|_{u,v} = 1$ for at least one $h \in \mathbb{U}_4$, and $|\frac{1}{1+i}u - \frac{b}{1+i}v| < 1$ for at least one $b \in \mathbb{U}_4$.*
1. *Suppose that $w_1 \neq 0$ and that $|gu - hv|_{u,v} \ge 1$ (resp. > 1) for all $(g,h) \in F$. Then $|gu - hv|_{u,v} \ge 1$ (resp. > 1) for all nonzero g, h in $\mathbb{Z}[i]$. If moreover, $w_1 \neq 1$, then $w_2 \in \overline{\mathcal{D}}$.*
2. *Suppose that $w_1 \neq 0$ and that $|gu - hv|_{u,v} \ge 1$ (resp. > 1) for all $(g,h) \in \{(\frac{1}{1+i}, \frac{\alpha}{1+i}) : \alpha \in \mathbb{U}_4\}$. Then $|gu - hv|_{u,v} \ge 1$ (resp. > 1) for all nonzero g, h both in $\mathbb{Z}[i]$ or both in J. If moreover, $w_1 \neq 1$ and $w_2 \neq -1$, then $w_1 \in \overline{\mathcal{C}} \setminus \mathbb{D}(-i, \sqrt{2})$ and $w_2 \in \overline{\mathcal{T}}$.*

The following simple formula will be useful.

Lemma 14 (Distance formula)**.** *Let $u = (u_1, v_2w_2)$ and $v = (u_1w_1, v_2)$ be in $\mathbb{C}^2$. Assume that $|w_1|, |w_2| \le 1$ and $|u_1|, |v_2| > 0$. Then for all nonzero complex numbers a and b,*

$$|au - bv|_{u,v} = \max(|b|\,\mathrm{d}(w_1, \tfrac{a}{b}), |a|\,\mathrm{d}(w_2, \tfrac{b}{a})).$$

Proof. Since $|w_1|, |w_2| \le 1$, for any $x = (x_1, x_2) \in \mathbb{C}^2$, $|x|_{u,v} = \max(\frac{|x_1|}{|u_1|}, \frac{|x_2|}{|v_2|})$. Therefore

$$\begin{aligned}|au - bv|_{u,v} &= \max(\tfrac{1}{|u_1|}|au_1 - bu_1w_1|, \tfrac{1}{|v_2|}|av_2w_2 - bv_2|)\\ &= \max(|b||w_1 - \tfrac{a}{b}|, |a||w_2 - \tfrac{b}{a}|).\end{aligned}$$

□

Proof of the proposition. The proof needs only elementary geometry but is rather long, the strategy works as follows. We assume that $w_1 \in \overline{\mathcal{C}}$, $|w_2| \le 1$ and $|gu - hv|_{u,v} \ge 1$ for all nonzero $(g,h) \in F$ or for all $(g,h) \in J^2$ with $|g| = |h| = \frac{1}{\sqrt{2}}$. We want to show that $|gu - hv|_{u,v} \ge 1$ for all nonzero Gauss integers or for all $(g,h) \in \mathbb{Z}[i]^2 \cup J^2$.

(1) We first get rid of the four particular cases $w_1 = 0$, $w_1 = 1$, $w_2 = -i$ and $w_2 = -1$.
(2) We show that $|gu - hv|_{u,v} \ge 1$ for $(g,h) = (1,1)$ and $(1, 1-i)$ implies that $w_2 \in \overline{\mathcal{D}}$
(3) We show that $|gu - hv|_{u,v} \ge 1$ for $(g,h) = (\frac{1}{1+i}, \frac{1}{1+i})$ and $(\frac{i}{1+i}, \frac{1}{1+i})$ implies that $w_2 \in \overline{\mathcal{T}}$.
(4) Let a be a positive real number. We show that, if g and h are two nonzero complex numbers such that $|g|, |h| \ge \frac{1}{a}$ and, $\frac{|g|}{|h|}$ or $\frac{|g|}{|h|} > 1 + a$, then $|gu - hv|_{u,v} > 1$.
(5) We show that if $|gu - hv|_{u,v} \ge 1$ for $(g,h) \in \{(1,1),(i,1),(1,1-i)\}$, then $|gu - hv|_{u,v} > 1$ for all complex numbers g and h such that $|g|$ and $|h| \ge 3$ (Lemma 15).
(6) Thanks to points (4) and (5), we shall see that we are reduced to deal with the pairs (g,h) with $|g|$ and $|h| \le 6$. Since g and h are in $\mathbb{Z}[i]$ or in J, we are left

with finitely many pairs (g,h). Then we are able to conclude the proof with a computer.

1) The four particular cases $w_1 = 0$, $w_1 = 1$ $w_2 = -i$ and $w_2 = -1$.

Case $w_1 = 0$. For all nonzero $a, b \in \mathbb{C}$,

$$|au - bv|_{u,v} = \max(|a|, |a||w_2 - \tfrac{b}{a}|).$$

Therefore, $|au - bv|_{u,v} \geq 1$ for all nonzero $a \in \mathbb{Z}[i]$. Furthermore, since the four closed disks $D(b,1)$, $b \in \mathbb{U}_4$, cover the closed disk $D(0,1)$, we have $|1 \times u - bv|_{u,v} = 1$ for at least one $b \in \mathbb{U}_4$, and $|\frac{1}{1+i}u - \frac{b}{1+i}v|_{u,v} < 1$ for at least one $b \in \mathbb{U}_4$.

In the three other cases we don't have to consider the strict inequalities because $|w_1|$ or $|w_2| = 1$.

Case $w_1 = 1$. For all nonzero $a, b \in \mathbb{C}$,

$$|au - bv|_{u,v} = \max(|a - b|, |a||w_2 - \tfrac{b}{a}|).$$

Therefore, $|au - bv|_{u,v} \geq 1$ for all $a \neq b$ both in $\mathbb{Z}[i]$ or both in J. If $a = b$, then $|au - bv|_{u,v} = |a||u - v|_{u,v}$, hence $|au - av|_{u,v} \geq 1$ for all nonzero $a \in \mathbb{Z}[i]$ iff $|u - v|_{u,v} \geq 1$, and $|au - av|_{u,v} \geq 1$ for all $a \in J$ iff $|\frac{1}{1+i}u - \frac{1}{1+i}v|_{u,v} \geq 1$. So the proposition holds when $w_1 = 1$.

Cases $w_2 = \alpha = -1$ *or* $-i$. For all nonzero $a, b \in \mathbb{C}$,

$$|au - bv|_{u,v} = \max(|b||w_1 - \tfrac{a}{b}|, |a\alpha - b|).$$

Therefore, $|au - bv|_{u,v} \geq 1$ for all $b \neq \alpha a$ both in $\mathbb{Z}[i]$ or both in J. If $b = \alpha a$, then $|au - bv|_{u,v} = |a||1 \times u - \alpha v|_{u,v}$, hence $|au - \alpha av|_{u,v} \geq 1$ for all nonzero $a \in \mathbb{Z}[i]$ iff $|1 \times u - \alpha v|_{u,v} \geq 1$ which always holds when $\alpha = -1$ because $|1 \times u - \alpha v|_{u,v} = |w_1 + 1|$ and $w_1 \in \overline{\mathcal{C}}$. Likewise, $|au - \alpha av|_{u,v} \geq 1$ for all $a \in J$ iff $|\frac{1}{1+i}u - \frac{\alpha}{1+i}v|_{u,v} \geq 1$. So the proposition holds when $w_2 = \alpha$.

We now suppose that $w_1 \neq 0$, $w_1 \neq 1$, $w_2 \neq -i$ and $w_2 \neq -1$.

2) By the distance formula (Lemma 14), for all nonzero g and h,

$$|gu - hv|_{u,v} = \max(|h| \,\mathrm{d}(w_1, \tfrac{g}{h}), |g| \,\mathrm{d}(w_2, \tfrac{h}{g})).$$

Now since $w_1 \in \overline{\mathcal{C}} \setminus \{0,1\}$, we have

$$|1| \,\mathrm{d}(w_1, \tfrac{1}{1}) < 1 \text{ and } |1 - i| \,\mathrm{d}(w_1, \tfrac{1}{1-i}) < 1.$$

Therefore, if $|gu - hv|_{u,v} \geq 1$ (resp. > 1) for $(g,h) = (1,1)$ and $(1, 1-i)$, then

$$|1| \,\mathrm{d}(w_2, \tfrac{1}{1}) \geq 1 \text{ and } |1| \,\mathrm{d}(w_2, \tfrac{1-i}{1}) \geq 1,$$

(resp. > 1) which in turn implies $w_2 \in \overline{\mathcal{D}}$ (resp. $w_2 \in \mathcal{D}$).

3) Since $w_1 \in \overline{\mathcal{C}} \setminus \{0,1\}$, we have

$$|\tfrac{1}{1+i}| \,\mathrm{d}(w_1, 1 = \tfrac{\frac{1}{1+i}}{\frac{1}{1+i}}) < 1 \text{ and } |\tfrac{1}{1+i}| \,\mathrm{d}(w_1, i = \tfrac{\frac{1}{1+i}}{\frac{-i}{1+i}}) < 1.$$

Therefore, if $|gu - hv|_{u,v} \geq 1$ (resp. > 1) for $(g,h) = (\frac{1}{1+i}, \frac{1}{1+i})$ and $(g,h) = (\frac{1}{1+i}, \frac{-i}{1+i})$, then

$$\mathrm{d}(w_2, 1) \geq \sqrt{2} \text{ and } \mathrm{d}(w_2, -i) \geq \sqrt{2}$$

(resp. $> \sqrt{2}$) which in turn implies $w_2 \in \overline{\mathcal{T}}$ (resp. $w_2 \in \mathcal{T}$).

4) Let a be a positive real number. Let g, h be two nonzero complex numbers with $|g|, |h| \geq \frac{1}{a}$. Since $|w_1|$ and $|w_2| \leq 1$, if $\frac{|g|}{|h|}$ or $\frac{|h|}{|g|} > 1 + a$, then by the distance formula (Lemma 14),

$$\begin{aligned} |gu - hv|_{u,v} &= \max(|h|\,\mathrm{d}(w_1, \tfrac{g}{h}), |g|\,\mathrm{d}(w_2, \tfrac{h}{g})) \\ &\geq \max(|h|(\tfrac{|g|}{|h|} - 1), |g|(\tfrac{|h|}{|g|} - 1)) > 1. \end{aligned}$$

5)

Lemma 15. *Suppose $w_1 \in \overline{\mathcal{C}} \setminus \{0, 1\}$ and $w_2 \in D(0, 1)$. If $|gu - hv|_{u,v} \geq 1$ for $(g, h) \in \{(1, 1), (i, 1), (1, 1 - i)\}$ then $|gu - hv|_{u,v} > 1$ for all complex numbers g and h such that $|g|$ and $|h| \geq 3$.*

Proof of Lemma 15. We proceed by contradiction and assume that $|gu - hv|_{u,v} \leq 1$ for some complex numbers g and h with $|g|$ and $|h| \geq 3$. Set $z = \frac{g}{h}$ and $z' = \frac{1}{z}$. By the distance formula (Lemma 14),

$$\mathrm{d}(w_1, z) \text{ and } \mathrm{d}(w_2, z') \leq \tfrac{1}{3}$$

hence $|z|, |z'| \leq \frac{4}{3}$. It follows that $|z|, |z'| \geq \frac{3}{4}$ and then that $|w_1|, |w_2| \geq \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$.

Also observe that since $w_1 \in \overline{\mathcal{C}}$, $\Re z$ and $\Im z$ are $\geq -\frac{1}{3}$ which implies that the inverse z' of z is neither in the open disk $\mathbb{D}(-\frac{3}{2}, \frac{3}{2})$ nor in the open disk $\mathbb{D}(\frac{3}{2}i, \frac{3}{2})$.

We divide the proof in two cases:

(1) $\mathrm{d}(w_1, i) \geq 1$,
(2) $\mathrm{d}(w_1, i) < 1$.

The first case uses the following intermediate lemma.

Lemma 16. *Let $w \in \mathbb{C}$ be such that $\Re w, \Im w \geq 0$, $\frac{5}{12} \leq |w| \leq 1$, and $\mathrm{d}(w, i) \geq 1$ then $\mathrm{d}(w, 1) < \frac{2}{3}$.*

Proof of the intermediate lemma. We want to show that the function $f(z) = |z - 1|^2 - \frac{4}{9}$ is < 0 when $|z| \leq 1$, $\Re z \geq 0$, $\Im z \geq 0$ and z is neither in the interiors of $D(i, 1)$ nor in the interior of $D(0, \frac{5}{12})$. It is easy to see that the maximum of f on this region is reached at a point which belongs to the circle C of radius 1 and center i. The circle C has polar equation $r = 2\sin\theta$. Since for $z = re^{i\theta} \in C$,

$$f(re^{i\theta}) = \tfrac{5}{9} + r^2 - 2r\cos\theta = \tfrac{5}{9} + 4\sin^2\theta - 2\sin 2\theta = g(\theta),$$

it is enough to show that $g(\theta) < 0$ for $\theta \in [\arcsin\frac{5}{24}, \frac{\pi}{6}]$. Now $g'(\theta) = 4\sin 2\theta - 4\cos 2\theta$ is negative if $\theta < \frac{\pi}{8}$ and non-negative otherwise, hence it is enough to check that g is < 0 at the extremities of the interval. Since

$$\begin{aligned} g(\tfrac{\pi}{6}) &= \tfrac{5}{9} + 1 - \sqrt{3} < 0, \\ g(\arcsin\tfrac{5}{24}) &= \tfrac{5}{9} + (\tfrac{5}{12})^2 - 4\tfrac{5}{24}\sqrt{1 - (\tfrac{5}{24})^2} \leq -0.08 \end{aligned}$$

we are done. □

End of proof of Lemma 15.

Case 1: $\mathrm{d}(w_1, i) \geq 1$. By the above lemma, $|z - 1| \leq |z - w_1| + |w_1 - 1| < \frac{1}{3} + \frac{2}{3} = 1$, hence its inverse z' has a real part $> 1/2$. We also already know that z' is not in the open disk $\mathbb{D}(\frac{3}{2}i, \frac{3}{2})$.

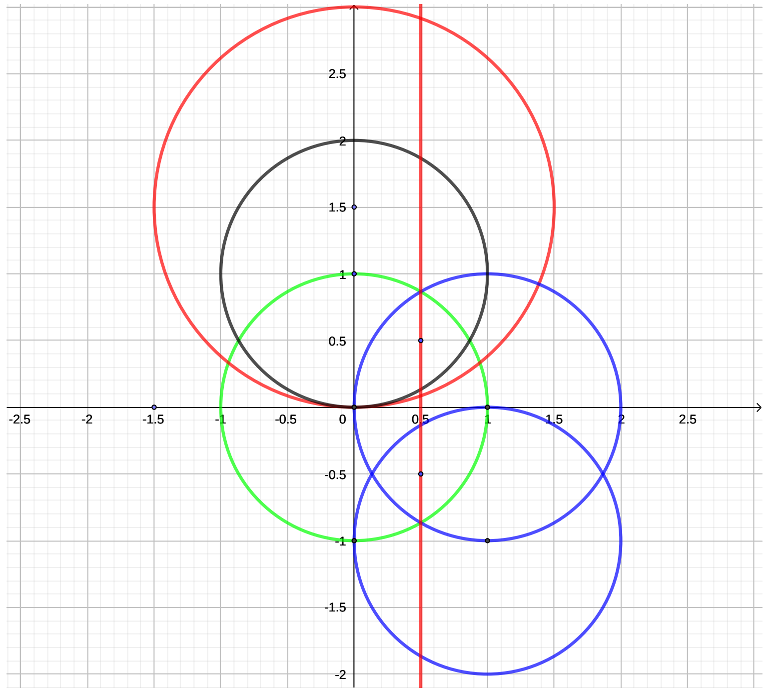

FIGURE 1. Proof of Lemma 15, case 1

Since w_1 is in $\overline{\mathcal{C}}$ and $w_1 \neq 0$, $\mathrm{d}(w_1, 1) < 1$. By assumption $|u-v|_{u,v} \geq 1$, therefore, by the distance formula (Lemma 14) with $a = 1$ and $b = 1$, we obtain $\mathrm{d}(w_2, 1) \geq 1$. Since $w_1 \in \overline{\mathcal{C}} \setminus \{0, 1\} \subset \mathbb{D}(\frac{1+i}{2}, \frac{1}{\sqrt{2}})$, $\mathrm{d}(w_2, 1-i) \geq 1$ again by Lemma 14 with $a = 1$ and $b = 1-i$.

Finally z' and w_2 satisfy the inequalities

$$z' \notin \mathbb{D}(\tfrac{3}{2}i, \tfrac{3}{2}),\ \Re z' > \frac{1}{2}, \text{ and } \mathrm{d}(w_2, 1) \geq 1,\ \mathrm{d}(w_2, 1-i) \geq 1$$

contradicting $\mathrm{d}(z', w_2) \leq \frac{1}{3}$ because $\frac{\sqrt{3}}{2} - \frac{1}{2} > \frac{1}{3}$ (see Figure 1).

Case 2: $\mathrm{d}(w_1, i) < 1$. We already know that $|z'| \geq \frac{3}{4}$ and that z' is neither in the open disk $\mathbb{D}(-\frac{3}{2}, \frac{3}{2})$ nor in the open disk $\mathbb{D}(\frac{3}{2}i, \frac{3}{2})$

As in case (1), making use of lemma 14 with $a = 1$ and $b = 1$ we see that $\mathrm{d}(w_2, 1) \geq 1$. Since $\mathrm{d}(w_1, i) < 1$ (case(2)), again with $a = i$ and $b = 1$ we see that $\mathrm{d}(w_2, -i) \geq 1$.

It follows that $\mathrm{d}(w_2, z') > \frac{1}{3}$ (see Figure 2), a contradiction. □

6) It remains to study the case $|g|$ and $|h| \leq 6$. Indeed, if g and h are nonzero complex numbers such that $|g|$ or $|h| > 6$, say $|g| > 6$, and taking into account that g and h are both Gaussian integers or both in J, we have

- either, $|h| \geq 1$ and
  - either, $\frac{|g|}{|h|} > 2$, and item 4. with $a = 1$ implies $|gu - hv|_{u,v} > 1$,
  - or, $\frac{|g|}{|h|} \leq 2$ and $|h| \geq 3$, and item 5. implies $|gu - hv|_{u,v} > 1$.
- or, $|h| = \frac{1}{\sqrt{2}}$ and $\frac{|g|}{|h|} \geq \frac{6}{\sqrt{2}} > 1 + \sqrt{2}$. Now item 4. with $a = \sqrt{2}$ implies $|gu - hv|_{u,v} > 1$.

Observe that up to now we have only used the hypothesis: $w_1 \in \overline{\mathcal{C}} \setminus \{0, 1\}$, $w_2 \in D(0, 1)$ and $|gu - hv|_\infty \geq 1$ for $(g, h) \in \{(1, 1), (i, 1), (1, 1-i)\}$.

We now use a computer to prove two lemmas.

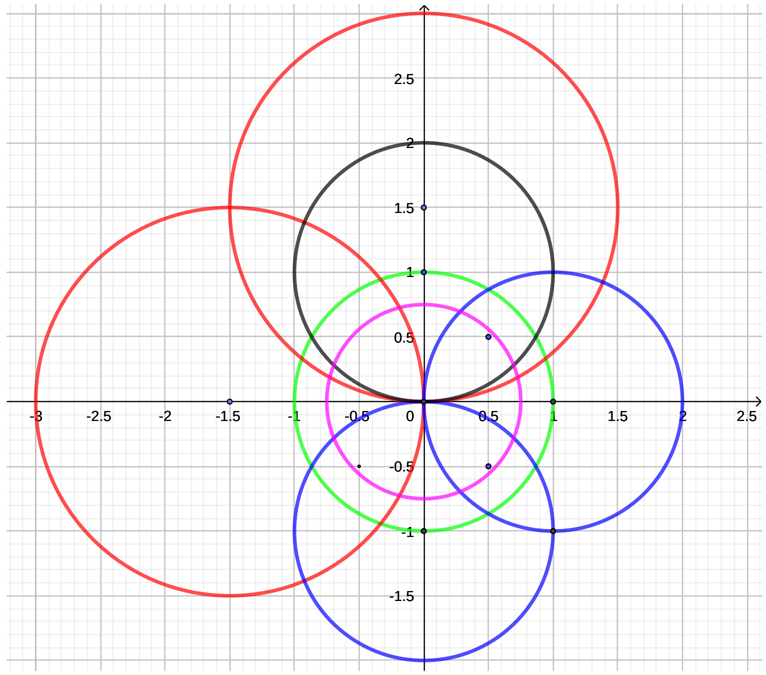

FIGURE 2. Proof of Lemma 15, case 2

Lemma 17 (First set of critical pairs)**.** *For all nonzero Gaussian integers g and h with $|g|$ and $|h| \leq 6$, if the pair (g,h) is not in $\mathbb{Z}[i]G_1$ where*

$$\begin{aligned}G_1 = \{&(1,-i),(1,1),(1,i),(1,-1),(1,1+i),(1,1-i),(1,-1+i),(1,-1-i),\\&(1,2i),(1,-2i),(1,-2),(1+i,1),(1+i,i),(1+i,2-i),(2,1),(2,1-2i),\\&(2-i,-2i),(2+i,2-2i)\},\end{aligned}$$

then

$$|h|\,\mathrm{d}(\tfrac{g}{h},\mathcal{C}) \text{ or } |g|\,\mathrm{d}(\tfrac{h}{g},\mathcal{D}) > 1.$$

The set G_1 is called the first set of critical pairs.

Proof using a computer.

- Let L_1 be the set of pairs of nonzero Gaussian integers with moduli ≤ 6. The set L_1 is finite with less than $(6+1+6)^4 = 28501$ elements and can be generated using a simple computer code (we use a Python code).
- One can write two functions that calculate the two distances $\mathrm{d}(z,\mathcal{C})$ and $\mathrm{d}(z,\mathcal{D})$ for any complex number z. See Appendix Section 14 where it is explained how to calculate $\mathrm{d}(z,\mathcal{D})$. This calculation can be performed with standard floating point arithmetic. The distance to $\mathcal{C}$ can be calculated the same way.
- Using these two functions one can obtain the set L_1' of pairs $(g,h) \in L_1$ such that
$$|h|\,\mathrm{d}(\mathcal{C},\tfrac{g}{h}) \leq 1+\varepsilon \text{ and } |g|\,\mathrm{d}(\mathcal{D},\tfrac{h}{g}) \leq 1+\varepsilon,$$
with $\varepsilon = 0.001$, a numerical safety margin. The set L_1' certainly contains all the pairs such that $|h|\,\mathrm{d}(\mathcal{C},\frac{g}{h}) \leq 1$ and $|g|\,\mathrm{d}(\mathcal{D},\frac{h}{g}) \leq 1$.
- Finally extract from L_1', a minimal subset G_1' such that for each pair $(a,b) \in L_1'$ there exist $z \in \mathbb{Z}[i]$ and $(g,h) \in G_1'$ such that $(a,b) = z(g,h)$. For this step observe that if $(a,b) \in L_1'$, then there exists a primitive pair $(g,h) \in \mathbb{Z}[i]^2$ which is in the line $\mathbb{C}(a,b)$ and which is also in L_1' because $|au - bv|_{u,v} \geq |gu - bv|_{u,v}$.
- The pairs added in G_1' due to the numerical margin are validated using calculation by hand. This lead to the set G_1 (actually, with the margin $\varepsilon = 0.001$, $G_1 = G_1'$).

Suppose now that (a,b) is a pair of nonzero Gaussian integers such that $|h|,|g|\leq 6$ and $|au-bv|_{u,v}\leq 1$. There exists a primitive pair $(g,h)\in\mathbb{Z}[i]^2$ such that $(a,b)=z(g,h)$ with $|z|\geq 1$. Since $|z|\geq 1$, $1\geq |au-bv|_{u,v}\geq |gu-bv|_{u,v}$. Therefore $(g,h)\in L_1'$. Since (g,h) is primitive, on the one hand, one of the pairs $\alpha(g,h)$, $\alpha\in\mathbb{U}_4$ must be in G_1', and on the other hand, $z\in\mathbb{Z}[i]$. It follows that $(a,b)\in\mathbb{Z}[i]G_1$. □

Remark 3. Without the safety margin ε in the above proof, some pairs may be missing from the set G_1 as the referee pointed out.

Lemma 18 (Second set of critical pairs)**.** *For all $(g,h)\in J^2$ with $|g|$ and $|h|\leq 6$, and $(g,h)\notin G_2=\{(\frac{a}{1+i},\frac{b}{1+i}):a,b\in\mathbb{U}_4\}$, we have*

$$|h|\,\mathrm{d}(\tfrac{g}{h},\overline{\mathcal{C}}\setminus\mathbb{D}(-i,\sqrt{2}))\ \textit{or}\ |g|\,\mathrm{d}(\tfrac{h}{g},\mathcal{T})>1.$$

Proof using a computer.

- Let L_2 be the set of pairs of nonzero elements in J with moduli ≤ 6. The set L_2 is finite with less than $(8+1+8)^4=83521$ and can be generated a using simple computer code (we use a Python code).
- One can write a function that calculates the distance $\mathrm{d}(z,\mathcal{T})$ from z to $\mathcal{T}$ for any complex number z. See Appendix Section 14 where it is explained how to calculate $\mathrm{d}(z,\mathcal{D})$. The distance to $\mathcal{T}$ can be calculated the same way.
- Since $\overline{\mathcal{C}}\setminus\mathbb{D}(-i,\sqrt{2}))\subset(-i)\overline{\mathcal{T}}$, for each nonzero g, h,
$$\mathrm{d}(\tfrac{g}{h},\overline{\mathcal{C}}\setminus\mathbb{D}(-i,\sqrt{2})))\leq\mathrm{d}(i\tfrac{g}{h},\mathcal{T}).$$
- Using the function $\mathrm{d}(z,\mathcal{T})$, one can obtain the set L_2' of pairs $(g,h)\in L_2$ such that
$$|h|\,\mathrm{d}(i\tfrac{g}{h},\mathcal{T})\leq 1+\varepsilon\text{ and }|g|\,\mathrm{d}(\tfrac{h}{g},\mathcal{T})\leq 1+\varepsilon,$$
with $\varepsilon=0.001$, a numerical safety margin. We obtain
$$L_2'=\{(\tfrac{a}{1+i},\tfrac{b}{1+i}):a,b\in\mathbb{U}_4\}.$$

□

End of proof of Part 1 in Proposition 13. Recall that we suppose $w_1\neq 0,1$ and $w_2\neq -i,-1$. By 2), we already know that $w_2\in\overline{\mathcal{D}}$. It remains to prove that, if $|gu-hv|_{u,v}\geq 1$ (resp. >1) for all (g,h) in

$$F=\{(1,1),(1,-i),(1,1-i),(1,1+i),(1+i,1)\},$$

then $|gu-hv|_{u,v}\geq 1$ (resp. >1) for all nonzero g,h in $\mathbb{Z}[i]$.

By Lemma 17, if $|gu-hv|_{u,v}\geq 1$ (resp. >1) for all $(g,h)\in G_1$, then $|gu-hv|_{u,v}\geq 1$ (resp. >1) for all pairs (g,h) of nonzero Gaussian integers. We prove that we can remove the pairs in $G_1\setminus F$ and get the same conclusion.

When $(g,h)\in\{(1,i),(1,-1),(1,-1+i),(1,-1-i),(1,2i),(1,-2)\}$, we have

$$\begin{aligned}&|h|\,\mathrm{d}(\tfrac{g}{h},\mathcal{C})=1,\\&\forall w_1\in\overline{\mathcal{C}}\setminus\{0\},\ |h|\,\mathrm{d}(\tfrac{g}{h},w_1)>1.\end{aligned}$$

So by the distance formula, these six pairs can be removed from G_1 when dealing with the large inequality or the strict inequality.

When $(g,h)\in\{(1+i,i),(2,1)\}$, we have

$$\begin{aligned}&|h|\,\mathrm{d}(\tfrac{g}{h},\mathcal{C})=1,\\&\forall w_1\in\overline{\mathcal{C}}\setminus\{1\},\ |h|\,\mathrm{d}(\tfrac{g}{h},w_1)>1.\end{aligned}$$

So these two pairs can be removed.

When $(g,h) \in \{(1,-2i),(1+i,2-i),(2,1-2i),(2+i,2-2i)\}$, we have

$$|g|\,\mathrm{d}(\tfrac{h}{g},\mathcal{D}) = 1,$$
$$\forall w_2 \in \overline{\mathcal{D}} \setminus \{-i\},\ |g|\,\mathrm{d}(\tfrac{h}{g},w_2) > 1.$$

So these four pairs can be removed from G_1.

Finally, consider the pair $(g,h) = (-2+i,2i)$. Since the disk $D(\frac{g}{h},\frac{1}{|h|})$ is included in the disk $D(\frac{g'}{h'},\frac{1}{|h'|})$ where $(g',h') = (1,1+i)$ and since the portion of the disk $D(\frac{h}{g},\frac{1}{|g|})$ lying in the unit disk is included in the disk $D(\frac{h'}{g'},\frac{1}{|g'|})$, the inequality $|g'u-h'v|_{u,v} \geq 1$ (resp. >1) implies the inequality $|gu-hv|_{u,v} \geq 1$ (resp. >1) which means that we can remove the pair $(-2+i,2i)$.

It follows that for $w_1 \in \overline{\mathcal{C}} \setminus \{0,1\}$ and $w_2 \in D(0,1) \setminus \{-i\}$, if for all $(g,h) \in F$ $|gu-hv|_{u,v} \geq 1$, (resp. >1) then $|gu-hv|_{u,v} \geq 1$ (resp. >1) for all nonzero Gaussian integers g and h.

End of proof of Part 2 in Proposition 13. By 3), we already know that $w_2 \in \overline{\mathcal{T}}$. It remains to prove that if $|gu-hv|_{u,v} \geq 1$ (resp. >1) for all $(g,h) \in \{(\frac{a}{1+i},\frac{b}{1+i}) : a,b \in \mathbb{U}_4\}$, then $w_1 \in \overline{\mathcal{C}} \setminus \mathbb{D}(-i,\sqrt{2})$ and $|gu-hv|_{u,v} \geq 1$ (resp. >1) for all nonzero g,h both in $\mathbb{Z}[i]$ or both in J

Since $w_2 \in \overline{\mathcal{T}}$ and since $w_2 \neq -1$, $\mathrm{d}(w_2,i) < \sqrt{2}$. Now by assumption, $|\frac{1}{1+i}u - \frac{i}{1+i}v|_{u,v} \geq 1$, hence $\mathrm{d}(w_1,-i) \geq \sqrt{2}$ by the distance formula. This means that $w_1 \in \mathcal{C} \setminus \mathbb{D}(-i,\sqrt{2})$.

Since $w_1 \in \mathcal{C} \setminus \mathbb{D}(-i,\sqrt{2})$ and $w_2 \in \overline{\mathcal{T}}$, we can use Lemma 18, we obtain that $|gu-hv|_{u,v} \geq 1$ (resp. >1) for all g,h both in J.

It remains to see what is happening when g and h are Gaussian integers. Since $\mathcal{T} \subset \mathcal{D}$, using the first set of critical pairs and the Part 1 of the proposition, we see that only the pairs (g,h) in F must be examined. For each of these pairs, we have $|g|\,\mathrm{d}(\frac{h}{g},\mathcal{T}) > 1$ except for $(g,h) = (1,1+i)$, and for this latter pair $\mathrm{d}(w_2,\frac{h}{g}) > 1$ for all $w_2 \in \mathcal{T} \setminus \{i\}$, so we are done. □

4.3. **Constraints on the pairs (w_1,w_2) when $w_1 \in \mathcal{C}$.** Let's remember the sets we need

$$\mathcal{C} = \{z \in \mathbb{C} : |z| < 1,\ \arg z \in [0,\tfrac{\pi}{4}]\},$$
$$\mathcal{D} = \{z \in \mathbb{C} : |z| < 1,\ \mathrm{d}(z,1) > 1,\ \mathrm{d}(z,1-i) > 1\},$$
$$\mathcal{T} = \{z \in \mathbb{C} : |z| < 1,\ \mathrm{d}(z,1) > \sqrt{2},\ \mathrm{d}(z,-i) > \sqrt{2}\},$$
$$F = \{(1,1),(1,-i),(1,1-i),(1,1+i),(1+i,1)\}.$$

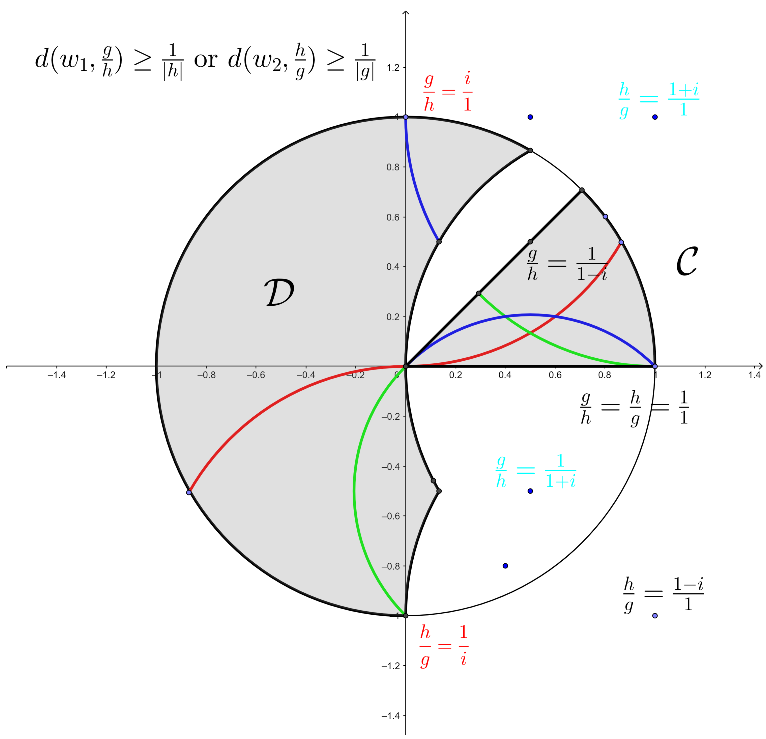

FIGURE 3. The constraints on consecutive minimal vectors.

Consider the following pairs of open disks in $\mathbb{C}$

$$Blue_1 = \mathbb{D}(\tfrac{1-i}{2}, \tfrac{1}{\sqrt{2}}),\ Blue_2 = \mathbb{D}(1+i, 1)$$
$$Red_1 = \mathbb{D}(i, 1),\ \ Red_2 = \mathbb{D}(-i, 1)$$
$$Green_1 = \mathbb{D}(1+i, 1),\ Green_2 = \mathbb{D}(\tfrac{1-i}{2}, \tfrac{1}{\sqrt{2}}).$$

Corollary 19. *Let $u = (u_1, v_2 w_2)$ and $v = (u_1 w_1, v_2)$ be two vectors in $\mathbb{C}^2$ with $|u_1|, |v_2| > 0$, $|w_1|, |w_2| < 1$, and $w_1 \in \mathcal{C} \setminus \{0\}$.*

- *No nonzero vector in $\mathbb{Z}[i]u + \mathbb{Z}[i]v$ are in $\overset{o}{C}(u, v)$ iff $w_2 \in \overline{\mathcal{D}}$ and one of the four conditions*
  (1) *$w_2 \in Green_2$ and $w_1 \notin Red_1 \cup Green_1$,*
  (2) *$w_2 \in Red_2 \setminus Green_2$ and $w_1 \notin Red_1$,*
  (3) *$w_2 \notin Red_2 \cup Blue_2$,*
  (4) *$w_2 \in Blue_2$ and $w_1 \notin Blue_1$,*
  holds.
- *No nonzero vector in $\langle u, v\rangle_J$ are in $\overset{o}{C}(u, v)$ iff $(w_1, w_2) \in (\mathcal{C} \setminus \mathbb{D}(-i, \sqrt{2})) \times \overline{\mathcal{T}}$.*

Remark 4. We do not consider the particular cases $w_1 = 0$ or $|w_i| = 1$ because we will not use them in the sequel. But clearly, it is not difficult to find the constraints on (w_1, w_2) in these cases using Proposition 13. In fact, when $w_1 = 0$ then $(\mathbb{Z}[i]u+\mathbb{Z}[i]v)\cap\overset{o}{C}(u, v) = \{0\}$ whatever the value of $w_2 \in \mathcal{D}$, and $\langle u, v\rangle_J \cap \overset{o}{C}(u, v) \neq \{0\}$ whatever the value of $w_2 \in \mathcal{D}$.

Proof. By Proposition 13, no nonzero vector in $\mathbb{Z}[i]u+\mathbb{Z}[i]v$ are in $\overset{o}{C}(u, v)$ iff for all (g, h) in

$$F = \{(1, 1), (1, -i), (1, 1-i), (1, 1+i), (1+i, 1)\},$$

$|gu - hv|_{u,v} \geq 1$. By the distance formula (Lemma 14), it means that for all $(g,h) \in F$,

$$\mathrm{d}(w_1, \tfrac{g}{h}) \geq \tfrac{1}{|h|} \text{ or } \mathrm{d}(w_2, \tfrac{h}{g}) \geq \tfrac{1}{|g|}.$$

Taking $(g,h) = (1,1)$ and $(g,h) = (1, 1-i)$, we obtain that $w_2 \notin \mathbb{D}(1,1)$ and $w_2 \notin \mathbb{D}(1-i,1)$ which implies $w_2 \in \overline{\mathcal{D}}$.

Taking $(g,h) = (1,-i)$, we obtain $w_1 \notin \mathbb{D}(i,1) = Red_1$ or $w_2 \notin \mathbb{D}(-i,1) = Red_2$.

Taking $(g,h) = (1, 1+i)$, we obtain $w_1 \notin \mathbb{D}(\frac{(1-i)}{2}, \frac{1}{\sqrt{2}}) = Blue_1$ or $w_2 \notin \mathbb{D}(i+i,1) = Blue_2$.

Taking $(g,h) = (1+i,1)$, we obtain $w_1 \notin \mathbb{D}(1+i,1) = Green_1$ or $w_2 \notin \mathbb{D}(\frac{1-i}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = Green_2$.

Taking into account the positions of the disks Red_2, $Blue_2$ and $Green_2$ in $\overline{\mathcal{D}}$, we obtain that $|gu - hv|_{u,v} \geq 1$ for all $(g,h) \in F$ iff one of the conditions (1) or (2) or (3) or (4) holds.

Again by Proposition 13, no nonzero vector in $\langle u, v\rangle_J$ are in $\overset{o}{C}(u,v)$ iff it is true for all $(g,h) \in \{(\frac{1}{1+i}, \frac{1}{1+i}), (\frac{1}{1+i}, \frac{i}{1+i})\}$. The distance formula shows that it is equivalent to $(w_1, w_2) \in (\mathcal{C} \setminus D(-i, \sqrt{2})) \times \overline{\mathcal{T}}$. □

4.4. **An example of a lattice with linearly independent equivalent minimal vectors.** We give an example of two vectors u and v such that

- u and v are consecutive minimal vectors of $\Lambda = \mathbb{Z}[i]u + \mathbb{Z}[i]v$,
- $u - (1-i)v$ is a minimal vector equivalent to v,
- $(1+i)u - v$ is a minimal vector equivalent to u.

Let $s \in (\frac{4}{3}\pi, \frac{3}{2}\pi)$, $t \in (\frac{5}{6}\pi, \pi)$, $w_1 = 1 + i + e^{is}$, $w_2 = 1 - i + e^{it}$, $u = r(1, e^{i\alpha} w_2)$ and $v = r(w_1, e^{i\alpha})$ where $r > 0$ and $\alpha \in \mathbb{R}$ are such that

$$\det_{\mathbb{C}}(u,v) = r^2 e^{i\alpha}(1 - w_1 w_2) = 1.$$

Consider the lattice $\Lambda = \mathbb{Z}[i]u + \mathbb{Z}[i]v$. We have $w_1 \in \mathcal{C} \cap \partial Green_1 \setminus \overline{Red_1}$ and $w_2 \in \overline{\mathcal{D}} \cap \mathbf{C}(1-i,1)$, so that thanks to Corollary 19,

$$\overset{o}{C}(u,v) \cap \Lambda = \{0\}.$$

However, since $w_1 \in \partial Green_1$ and $w_2 \in \mathbf{C}(1-i,1)$, $\partial C(u,v)$ contains not only the subsets $\mathbb{U}_4 u$ and $\mathbb{U}_4 v$ of Λ, but also the subsets of Λ

$$\mathbb{U}_4((1+i)u - v) \text{ and } \mathbb{U}_4(u - (1-i)v).$$

By Lemma 17 about the first set of critical pairs, the only others (g,h) such that $|gu - hv|_{u,v} = 1$ are in $\mathbb{U}_4 G_1$. Furthermore, since $w_1 \neq 0, 1$ and $w_2 \neq -1, -i$, we see as in the end of the proof of Proposition 13 part 1, that if $(g,h) \in \mathbb{U}_4(G_1 \setminus F)$, then $|gu - hv|_{u,v} > 1$. By checking the values of $|gu - hv|_{u,v}$ for the pairs $(g,h) \in F$, we see that

$$\partial C(u,v) \cap \Lambda = \mathbb{U}_4 u \cup \mathbb{U}_4 v \cup \mathbb{U}_4((1+i)u - v) \cup \mathbb{U}_4(u - (1-i)v).$$

Now

$$\begin{aligned}
u-(1-i)v &= r(1-(1-i)w_1, e^{i\alpha}(w_2-(1-i))) \\
&= r(1-(1-i)(1+i+e^{is}), e^{i\alpha}(1-i+e^{it}-(1-i))) \\
&= r(-1-(1-i)e^{is}, e^{i\alpha}e^{it}) \\
&= r(x_1, x_2) \\
(1+i)u-v &= r(1+i-w_1, e^{i\alpha}((1+i)w_2-1)) \\
&= r(1+i-(1+i+e^{is}), e^{i\alpha}((1+i)(1-i+e^{it})-1)) \\
&= r(-e^{is}, e^{i\alpha}((1+i)e^{it}+1) \\
&= r(y_1, y_2)
\end{aligned}$$

and one check that

$$\begin{aligned}
|x_1|^2 = 3+2(\cos s+\sin s) = |w_1|^2 \text{ and } |x_2| = 1 \\
|y_1|^2 = 1 \text{ and } |y_2| = 3+2(\cos t-\sin t) = |w_2|
\end{aligned}$$

Hence v and $u-(1-i)v$ are two equivalent minimal vectors and u and $(1+i)u-v$ as well.

5. No consecutive pairs of index 2

In Proposition 10 we have seen that two consecutive minimal vectors have index one or two. We show now that the case of index two cannot occur twice consecutively.

Proposition 20. *Suppose that $u=(u_1, v_2w_2)$ and $v=(u_1w_1, v_2)$ are two consecutive minimal vectors of index 2 in a lattice Λ in $\mathbb{C}^2$. Let w be a minimal vector such that v and w are two consecutive minimal vectors. Then $\mathbb{Z}[i]v+\mathbb{Z}[i]w=\Lambda$.*

Remark 5. In fact, with the assumptions of the proposition, it is possible to prove that $\mathbb{Z}[i]u+\mathbb{Z}[i]w=\Lambda$ also holds. The proof of this latter fact goes as the proof of the proposition but is slightly more difficult. We shall not do it.

The proof of the proposition uses two lemmas. The first lemma is well known and its proof is a straightforward calculation we omit.

Lemma 21. *Let w and z be two complex numbers and let $k<1$ be a nonnegative real number. Then*

$$\begin{aligned}
|z-w| < k|w| \Leftrightarrow \mathrm{d}(w, \frac{z}{1-k^2}) < \frac{k|z|}{1-k^2}, \\
|z-w| = k|w| \Leftrightarrow \mathrm{d}(w, \frac{z}{1-k^2}) = \frac{k|z|}{1-k^2}.
\end{aligned}$$

Lemma 22. *Suppose that $u=(u_1, v_2w_2)$ and $v=(u_1w_1, v_2)$ are two consecutive minimal vectors of index 2 in a lattice Λ. Suppose that w is a minimal vector such that v and w are two consecutive minimal vectors and such that v and w has index 2 in Λ. Then $w=a(1+i)v+\alpha u$ where a and α are Gaussian integers such that $0<|a|\le 2$ and $|\alpha|=1$.*

Proof. By Proposition 10, $(U = \frac{1}{1+i}(u+v), V = v)$ is a basis of Λ and $(V = v, W = \frac{1}{1+i}(v+w))$ as well. Therefore,

$$\begin{cases} u = -V + (1+i)U \\ v = V \\ w = -V + (1+i)W \\ W = bV - \alpha U \end{cases}$$

where $b \in \mathbb{Z}[i]$ and $\alpha \in \mathbb{U}_4$ because the determinant of the matrix of the coordinates of the vectors V and W in the basis (U, V) is a unit of $\mathbb{Z}[i]$. It follows that

$$\begin{aligned} w &= -v + (1+i)(bv - \alpha \tfrac{1}{1+i}(u+v)) \\ &= (-1 - \alpha + (1+i)b)v - \alpha u \\ &= a(1+i)v - \alpha u \end{aligned}$$

where $a \in \mathbb{Z}[i]$ because $-1-\alpha \in (1+i)\mathbb{Z}[i]$. In coordinates this gives

$$w = (u_1(a(1+i)w_1 - \alpha), v_2(a(1+i) - \alpha w_2))$$

and since w follows v,

$$|a(1+i)w_1 - \alpha| < |w_1|.$$

Therefore, $|w_1| > |a(1+i)||w_1| - |\alpha|$. Making use of Corollary 19 and of Proposition 11, we see that w_1 is in $\varphi(\mathcal{C} \setminus D(-i, \sqrt{2}))$ for some $\varphi \in \mathbb{D}_8$, hence $|w_1| > \frac{\sqrt{3}-1}{\sqrt{2}}$. The last two inequalities imply that $|a| < \frac{1}{|1+i|}(1 + \frac{1}{|w_1|}) < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}-1} < \sqrt{5}$. It follows that $|a| \leq 2$. Finally, a cannot be 0 because w and u are not proportional. $\square$

Proof of the proposition. We proceed by contradiction and suppose that $\mathbb{Z}[i]v + \mathbb{Z}[i]w$ has index two. By the above lemma, we have $w = (1+i)av - \alpha u$ where a and α are Gaussian integers with $|\alpha| = 1$ and $0 < |a| \leq 2$. We have

$$w = (u_1((1+i)aw_1 - \alpha), v_2((1+i)a - \alpha w_2)).$$

Since the minimal vector w follows v, we have $|(1+i)aw_1 - \alpha| < |w_1|$ which is equivalent to

$$|w_1 - z|^2 < |z|^2 |w_1|^2$$

where $z = c + id = \frac{\alpha}{(1+i)a}$. With the above lemma, we see that the latter inequality is equivalent to

$$\mathrm{d}(w_1, \frac{z}{1-|z|^2}) < \frac{|z|^2}{1-|z|^2}.$$

Given $\beta \in \mathbb{U}_4$, consider the complex numbers $x = -\frac{1}{1+i} + (1+i)a$ and $y = -\frac{\beta}{1+i} + \alpha$; they are both in J. So that by Proposition 10, the vector

$$w' = w - xv + yu = \tfrac{1}{1+i}(v - \beta u) = (w'_1, w'_2)$$

is in Λ. If we can choose β so that

$$\begin{cases} |w'_1| \leq |u_1 w_1| \\ |w'_2| \leq |v_2((1+i)a - \alpha w_2)| \end{cases}$$

with one strict inequality at least, it contradicts that v and w are two consecutive minimal vectors. The strategy is now to prove that either we can choose β or that the inequality $\mathrm{d}(w_1, \frac{z}{1-|z|^2}) < \frac{|z|^2}{(1-|z|^2)}$ does not hold.

Using the symmetries and Proposition 11 as in section 4, we can suppose that $w_1 \in \mathcal{C}$. By Corollary 19,

$$w_1 \in \mathcal{C} \setminus \mathbb{D}(-i, \sqrt{2}) \text{ and } w_2 \in \mathbb{D} \setminus (\mathbb{D}(1, \sqrt{2}) \cup \mathbb{D}(-i, \sqrt{2})).$$

With $t = \frac{1}{z} = \frac{(1+i)a}{\alpha}$, the above inequalities about w_1' and w_2' are equivalent to

$$\begin{cases} \frac{1}{\sqrt{2}}|w_1 - \beta| \leq |w_1| \\ \frac{1}{\sqrt{2}}|w_2 - \bar{\beta}| \leq |t - w_2| \end{cases}.$$

A short calculation shows that the latter inequalities are equivalent to

$$\begin{cases} |w_1 + \beta|^2 \geq 2 \\ |w_2 - (2t - \bar{\beta})|^2 \geq 2|t - \bar{\beta}|^2 \end{cases}.$$

Since $w_1 \in \mathcal{C} \setminus \mathbb{D}(-i, \sqrt{2})$, the first inequality holds when $\beta = 1$ or i.

Suppose first that $|a| = 1$. We have $|t|^2 = 2$ hence $t = \pm 1 \pm i$.

If $t = 1 + i$, choose $\beta = 1$. We have $t - \bar{\beta} = i$ and $2t - \bar{\beta} = 1 + 2i$, hence the second inequality is equivalent to $|w_2 - (1 + 2i)|^2 > 2$ which holds because $\Re w_2 < 0$ and $\Im w_2 < 1$.

If $t = 1 - i$, choose $\beta = 1$. We have $t - \bar{\beta} = -i$ and $2t - \bar{\beta} = 1 - 2i$, hence the second inequality is equivalent to $|w_2 - (1 - 2i)|^2 > 2$ which holds because $\Im w_2 > 0$.

If $t = -1 + i$, then $z = -\frac{1+i}{2}$. Therefore, $\mathrm{d}(w_1, \frac{z}{1-|z|^2})^2 > \frac{|z|^4}{(1-|z|^2)^2} = 1$ a contradiction.

If $t = -1 - i$, then $z = \frac{-1+i}{2}$. Therefore, $\mathrm{d}(w_1, \frac{z}{1-|z|^2})^2 > \frac{|z|^4}{(1-|z|^2)^2} = 1$ a contradiction.

Suppose that $|a| = \sqrt{2}$. We have $|t| = 2$, hence $t = \pm 2$ or $\pm 2i$. Therefore, $\frac{z}{1-|z|^2} = \frac{4}{3}z$, and the information $w_1 \in \mathcal{C} \setminus D(-i, \sqrt{2})$ implies that if $t = 2i$ or -2 or $-2i$ then $\mathrm{d}(w_1, \frac{4}{3}z)^2 > \frac{|z|^4}{(1-|z|^2)^2} = \frac{1}{9}$.

If $t = 2$, choose $\beta = 1$. We have $t - \bar{\beta} = 1$ and $2t - \bar{\beta} = 3$, hence the second inequality becomes $|w_2 - 3|^2 > 2$ which holds because $\Re w_2 < 0$.

Suppose that $|a| = 2$. We have $|t| = 2\sqrt{2}$ hence $t = \pm 2(1 + i)$ or $\pm 2i(1 + i)$. Therefore, $\frac{z}{1-|z|^2} = \frac{8}{7}z$, and the information $w_1 \in \mathcal{C} \setminus D(-i, \sqrt{2})$ implies that if $t = 2(1+i)$ or $-2(1+i)$ or $-2i(1 + i)$ then $\mathrm{d}(w_1, \frac{8}{7}z)^2 > \frac{|z|^4}{(1-|z|^2)^2} = \frac{1}{49}$.

If $t = 2(1-i)$, choose $\beta = 1$. We have $t - \bar{\beta} = 1 - 2i$ and $2t - \bar{\beta} = 3 - 4i$, hence the second inequality becomes $|w_2 - (3 - 4i)|^2 > 2 \times 5$ which holds because $|\Re(w_2 - (3 - 4i))| \geq 2$ and $|\Im(w_2 - (3 - 4i))| \geq 3$. $\square$

6. Definitions and parametrization of the transversals

6.1. **The open transversal.** Let $\mathbb{U}_4$ be the group of units in $\mathbb{Z}[i]$. The *open transversal* T is the set of Gauss lattices Λ in $\mathbb{C}^2$ such that $\det_{\mathbb{C}} \Lambda \in \mathbb{U}_4$ and such that there exist two vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in Λ such that

(1) $|u_2|, |v_1| < |u_1| = |v_2| = r$,
(2) the only nonzero vectors of Λ in the ball $B_\infty(0, r)$ are in $\mathbb{U}_4 u \cup \mathbb{U}_4 v$.

Observe that the two vectors u and v are minimal vectors in Λ and that by Lemma 4, they are consecutive. The vectors u and v are the vectors *associated* with Λ. They are unique up to a multiplicative factor in $\mathbb{U}_4$:

Lemma 23. *Let Λ be a lattice in the open transversal T and let u, v be two vectors in Λ satisfying (1) and (2) in the above definition. If u' and v' are two vectors in Λ such that (1) and (2) hold then $u' \in \mathbb{U}_4 u$ and $v' \in \mathbb{U}_4 v$.*

Proof. Set $r = |u|_\infty$ and $r' = |u'|_\infty$. The balls $B_\infty(0,r)$ and $B_\infty(0,r')$ are nested, therefore by (2) they are equal. So that by (1), $|u'_1| = |u_1| = |v'_2| = |v_2|$. Hence u' and $v' \in B_\infty(0,r)$. Again by (2), this imply that $u' \in \mathbb{U}_4 u$ and $v' \in \mathbb{U}_4 v$. □

6.2. **The full transversal.** The *full transversal* T' is the set of Gauss lattices Λ in $\mathbb{C}^2$ such that $\det_{\mathbb{C}} \Lambda \in \mathbb{U}_4$ and such that there exist two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in Λ such that

(1') $|u_2|, |v_1| < |u_1| = |v_2| = r$,

(2') the only nonzero vector of Λ in the open ball $\overset{o}{B}_\infty(0,r)$ is 0.

Clearly

$$T \subset T' \subset \{\Lambda \in \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}(2,\mathbb{Z}[i]) : \lambda_1(\Lambda, |.|_\infty, \mathbb{C}) = \lambda_2(\Lambda, |.|_\infty, \mathbb{C})\}\,.$$

The vectors u and v are the vectors *associated* with Λ. They are no longer unique up to a multiplicative factor in $\mathbb{U}_4$, see the example subsection 4.4. By Lemma 4, they are consecutive.

By Proposition 10, the lattice $L = \mathbb{Z}[i]u + \mathbb{Z}[i]v$ has index 1 or 2 in Λ. Therefore, the transversal T (resp. T') is the union of two pieces T_1 and T_2 (resp. T'_1 and T'_2) according to the index of L. The above lemma implies that T_1 and T_2 are disjoint but as the example in the Subsection 4.4 shows,

$$T'_1 \cap T'_2 \neq \emptyset.$$

However, $T'_1 \cap T'_2$ is a small set. It is a consequence of the following Lemma.

Let $\mathcal{N}$ be the set of unimodular lattices $\Lambda \subset \mathbb{C}^2$ such that either there exists a nonzero vector $(u_1, u_2) \in \Lambda$ with $u_1 u_2 = 0$ or there exist two linearly independent vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in Λ such that $|u_1| = |v_1|$ or $|u_2| = |v_2|$.

Lemma 24. *The following properties hold*

(1) $\mathcal{N}$ *contains the set*

$$\{\Lambda \in \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}(2,\mathbb{Z}[i]) : \lambda_1(\Lambda, |.|_\infty, \mathbb{C}) = \lambda_2(\Lambda, |.|_\infty, \mathbb{C})\} \setminus T.$$

(2) $\mathcal{N}$ *is stable under the action of the flow* g_t, $t \in \mathbb{R}$.

(3) $\mathcal{N}$ *has zero Haar measure.*

Remark 6. $T' \setminus T \subset \mathcal{N}$.

Proof. 1. Let Λ be a unimodular lattice not in the open transersal T such that $\lambda_1(\Lambda, |.|_\infty, \mathbb{C}) = \lambda_2(\Lambda, |.|_\infty, \mathbb{C}) = r$. The equality of the two minima implies that there exist two linearly independent vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ such that $|u|_\infty = |v|_\infty = r$ and $|u_1| \geq |v_1|$. If Λ were not in $\mathcal{N}$, we would have $r \geq |u_1| > |v_1|$ and therefore, $r = |v_2|$. Since $|v_2| \neq |u_2|$, we would have $|v_2| > |u_2|$. Since Λ is not in T, this implies that there exists a nonzero vector $w = (w_1, w_2) \in \Lambda \cap B_\infty(0,r)$ and not in $\mathbb{U}_4 u \cup \mathbb{U}_4 v$. Then we have either $|w_1| = r = |u_1|$ or $|w_2| = r = |v_2|$, a contradiction.

2. Clear.

3. Let $\mathcal{M}$ be the set of matrices $M \in M_2(\mathbb{C})$ such that either there exists $X \in \mathbb{Z}[i]^*$ such that the product of the coordinates of MX is zero, or there are two linearly independent vectors X and Y in $\mathbb{Z}[i]$ such that either the moduli of the first coordinates of MX and MY are equal, or the moduli of the second coordinates of MX and MY are equal. It suffices to prove that $\mathcal{M}$ has zero Lebesgue measure. By definition, the set $\mathcal{M}$ is the union of two sets $\mathcal{M}_1$ and $\mathcal{M}_2$. The first set is a countable union of hyperplanes in $M_2(\mathbb{C})$ and thus is of zero Lebesgue measure. Let us deal now with $\mathcal{M}_2$.

Since $\mathbb{Z}[i]$ is countable, considering each row of the matrix M, we are reduced to prove that given two linearly independent vectors $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{C}^2$, the set of $(a, b) \in \mathbb{C}^2$ such that $P(a, b) = |ax_1 + bx_2|^2 - |ay_1 + by_2|^2 = 0$, is of zero Lebesgue measure in $\mathbb{C}^2$.

Now P can be considered as a real polynomial of four variable. Since $P(y_2, -y_1) = |\det_{\mathbb{C}}(x, y)|^2 \neq 0$, the polynomial P is not zero and the set of (a, b) such that $P(a, b) = 0$ has measure zero. $\square$

6.3. **Properties of the open transversal.**

Lemma 25. *The open transversal T is a real submanifold of* $\mathrm{SL}(2, \mathbb{C}) / \mathrm{SL}(2, \mathbb{Z}[i])$. *Furthermore, The flow $(g_t)_{t\in\mathbb{R}}$ is transverse to T.*

Proof. Let Λ_0 be in T and let u_0 and v_0 be the two vectors associated with Λ_0 by the definition of T. By Proposition 10, either (u_0, v_0) form a basis of Λ_0 and we can suppose $\det(u_0, v_0) = 1$ w.l.o.g. (case of index 1) or $U_0 = u_0$ and $V_0 = \frac{1}{1+i}(u_0 + v_0)$ form a basis of Λ_0 and we can suppose $\det(u_0, v_0) = (1 + i)$ w.l.o.g. (case of index 2). We can find a small enough positive real number ε such that for any (u, v) in the open set

$$W = B_{\mathbb{C}^2}(u_0, \varepsilon) \times B_{\mathbb{C}^2}(v_0, \varepsilon),$$

- the matrix $M = M(u, v)$ the columns of which are $u = (u_1, u_2)$ and $v = (v_1, v_2)$, is in $\mathrm{GL}(2, \mathbb{C})$ and the sets WP, $P \in \mathrm{SL}(2, \mathbb{Z}[i])$ are disjoint,
- the vectors in $\mathbb{U}_4 u$ and $\mathbb{U}_4 v$ are the only nonzero vectors of the lattice $\Lambda = M\mathbb{Z}[i]^2$ in the cylinder $C(u, v)$ in the index 1 case, or of the lattice $\mathbb{Z}[i]u + \mathbb{Z}[i]\frac{1}{1+i}(u + v)$ in the index 2 case,
- $|u_1| > |u_2|$ and $|v_1| < |v_2|$,
- for all $M \in W$, $|\det M - \det(u_0, v_0)| \leq \frac{1}{10}$.

Consider the map

$$\begin{aligned} \phi &: W \to \mathbb{C} \times \mathbb{R} \\ &: M = (u, v) \to (\phi_1(M) = \frac{\det M}{\det(u_0, v_0)}, \phi_2(M) = |u_1|^2 - |v_2|^2). \end{aligned}$$

In the index 1 case, a lattice $\Lambda = M\mathbb{Z}[i]^2$ with $M \in W$, is in T iff $\phi(M) = (1, 0)$. In index 2 case, a lattice $\Lambda = \mathbb{Z}[i]u + \mathbb{Z}[i]\frac{1}{1+i}(u + v)$ is in T iff $\phi(M) = (1, 0)$. Hence, to prove that T is a submanifold, it is enough to show that the differential $D\phi(M)$ is onto at every point M in W. The differential of ϕ_1 is $\mathbb{C}$-linear and is given by

$$D\phi_1(M).(x, y) = \frac{1}{\det(u_0, v_0)}(v_2 x_1 + u_1 y_2 - v_1 x_2 - u_2 y_1)$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The differential of ϕ_2 is given by

$$D\phi_2(M).(x, y) = u_1\bar{x}_1 + \bar{u}_1 x_1 - v_2\bar{y}_2 - \bar{v}_2 y_2.$$

Call γ_M the $\mathbb{C}$-linear map defined by $\gamma_M(x, y) = \bar{u}_1 x_1 - \bar{v}_2 y_2$. On the one hand, $D\phi_2(M) = \gamma_M + \bar{\gamma}_M$. On the other hand, γ_M and $D\phi_2(M)$ are $\mathbb{C}$-linearly independent because $|v_2|^2 + |u_1|^2 \neq 0$. Therefore, the three $\mathbb{R}$-linear maps $\Re\gamma_M$, $\Re D\phi_1(M)$ and $\Im D\phi_1(M)$ are $\mathbb{R}$-linearly independent. It follows that $D\phi(M)$ is onto which implies that T is a submanifold of Ω_2.

To show that the flow is transverse to T, we have to check that for any matrix $M = M(u, v)$ in W such that $\phi(M) = (1, 0)$, we have
$D\phi(M).((u_1, -u_2), (v_1, -v_2)) \neq 0$. Now, $D\phi_2(M).((u_1, -u_2), (v_1, -v_2)) = 2|u_1|^2 + 2|v_2|^2$, hence $D\phi(M).((u_1, -u_2), (v_1, -v_2))$ is not zero. $\square$

7. First return map, proof of Theorem 3

Lemma 26. *Let Λ be a unimodular lattice that is in the full transversal T' and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two consecutive minimal vectors associated with Λ. Let $t = \inf\{s > 0 : g_s(\Lambda) \in T'\}$. Then $t < +\infty$ if and only if $v_1 \neq 0$. Moreover, in the latter case, there exists a minimal vector $z = (z_1, z_2)$ such that v and z are two consecutive minimal vectors and such that*

$$t = \frac{1}{2} \ln \frac{|z_2|}{|v_1|}.$$

Consequently the first return map applied to Λ is $g_t\Lambda$.

Proof. If $v_1 \neq 0$, then by Lemma 6, a minimal element $z = (z_1, z_2) \in \Lambda$ for the lexicographic preoder $\prec$ in the infinite vertical cylinder $\overset{o}{C}_1(|v_1|)$ is a minimal vector. By definition v and z are consecutive. So, by Lemma 4, $\overset{o}{C}(v, z) \cap \Lambda \setminus \{0\} = \emptyset$, hence $\overset{o}{C}(g_t v, g_t z) \cap g_t(\Lambda \setminus \{0\}) = \emptyset$, thus $g_t\Lambda$ is in T' when $t = \frac{1}{2} \ln \frac{|z_2|}{|v_1|}$. Let $0 < s < t$. We want to show that $g_s\Lambda \notin T'$.

- If $r_2 = e^{-s}|v_2| \leq r_1 = e^s|v_1|$, since $r_1 = e^s|v_1| < e^t|v_1| = e^{-t}|z_2| < e^{-s}|z_2|$, we have $B_\infty(0, r_1 = |g_s v|_\infty) \subset C(g_s v, g_s z)$. Now, there is no vector in $g_s(\Lambda \setminus \{0\})$ of the shape (x_1, x_2) with $|x_1| < r_1$ and $|x_2| = r_1 < e^{-s}|z_2|$ because such vector would be in $\overset{o}{C}(g_s v, g_s z) = g_{s-t}\overset{o}{C}(g_t v, g_t z)$. Therefore, $g_s\Lambda$ is not in the full transversal T'.
- If $r_2 = e^{-s}|v_2| > r_1 = e^s|v_1|$, since $r_2 = e^{-s}|v_2| = e^{-s}|u_1| < e^s|u_1|$, we have $B_\infty(0, r_2 = |g_s v|_\infty) \subset C(g_s u, g_s v)$ and there is no vector in $g_s(\Lambda \setminus \{0\})$ of the shape (x_1, x_2) with $|x_1| = r_2 < e^s|u_1|$ and $|x_2| < r_2$. Therefore, $g_s\Lambda$ is not in the full transversal T'.

□

Let $u = (u_1, v_2 w_2)$ and $v = (u_1 w_1, v_2)$ be the two consecutive minimal vectors associated with a lattice Λ that is in the full transversal T'. By the above lemma, the computation of the first return map is reduced to the computation of the minimal vector $v' \in \Lambda$ such that v and v' are two consecutive minimal vectors. This is the purpose of Theorem 3 that we recall below. To perform this calculation, we must take into account the component of the transversal which contains Λ.

Recall that the lexicographic preoder on $\mathbb{C}^2$ is defined by

$$(x_1, x_2) \prec (y_1, y_2)$$

iff $|x_2| < |y_2|$ or $|x_2| = |y_2|$ and $|x_1| \leq |y_1|$. We recall the statement of Theorem 3 for the convenience of the reader.

Theorem 3. *Let $u = (u_1, v_2 w_2)$ and $v = (u_1 w_1, v_2)$ be the two minimal consecutive vectors associated with a lattice Λ that is in the full transversal T'. If $w_1 \neq 0$ then there exists $v' \in \Lambda$ a minimal vector such that v and v' are two consecutive minimal vectors and*

- *if $\det_{\mathbb{C}}(u, v) = 1$, then v' is any vectors that is minimal for the preoder $\prec$ in the set*

$$\left\{ z = -au + gv : a \in \{1, 1+i\},\, g \in \mathbb{Z}[i],\, |\tfrac{a}{w_1} - g| < 1 \right\}.$$

Moreover with $u' = v = (u'_1, v'_2 w'_2)$ *and* $v' = -au + gv = (u'_1 w'_1, v'_2)$ *we have*

$$w'_1 = g - \frac{a}{w_1}, \qquad w'_2 = \frac{1}{g - aw_2}. \tag{3}$$

- *if* $\det_{\mathbb{C}}(u, v) = 1 + i$, *then* v' *is any vectors that is minimal in the set*

$$\left\{ z = -\tfrac{1}{1+i}(u + v) + gv : g \in \mathbb{Z}[i],\, |\tfrac{1}{(1+i)w_1} + \tfrac{1}{(1+i)} - g| < 1 \right\}.$$

Moreover with $u' = v = (u'_1, v'_2 w'_2)$ *and* $v' = -au + gv = (u'_1 w'_1, v'_2)$ *we have*

$$w'_1 = g - \tfrac{1}{(1+i)w_1} - \tfrac{1}{(1+i)}, \qquad w'_2 = \frac{1}{g - \tfrac{1}{(1+i)} w_2 - \tfrac{1}{(1+i)}}. \tag{4}$$

Proof. If $w_1 \neq 0$ then $v_1 = u_1 w_1 \neq 0$ and by Minkowski convex body theorem, the cylinder $\overset{o}{C}_1(|v_1|) = \{(x, y) \in \mathbb{C}^2 : |x| < |v_1|\}$ contains at least one nonzero vector of Λ. By Lemma 6, a vector of Λ in this cylinder which is minimal for the preorder $\prec$ is a minimal vector v' that follows v.

Case 1: $\det_{\mathbb{C}}(u, v) = 1$. Let $L = \mathbb{Z}[i]v + \mathbb{Z}[i]v'$ be the lattice generated by v and v'. Since L has index 1 or 2 in Λ, the determinant of v and v' in the basis u, v is a unit of $\mathbb{Z}[i]$ or $(1 + i)$ times a unit. This implies that $v' = -au + gv$ with $g \in \mathbb{Z}[i]$ and $a \in \mathbb{U}_4$ or $a \in (1 + i)\mathbb{U}_4$. We can suppose that $a \in \{1, 1 + i\}$ w.l.o.g.. The condition $v' \in \overset{o}{C}_1(|v_1|)$ is equivalent to,

$$|au_1 - gu_1 w_1| < |u_1 w_1|$$

which in turn is equivalent to

$$|\frac{a}{w_1} - g| < 1.$$

By definition v' is minimal for the preorder $\prec$ among the vectors $-au + gv$ such that the latter inequality holds. An easy calculation leads to the formula for w'_1 and w'_2.

Case 2: $\det_{\mathbb{C}}(u, v) = 1 + i$. By Proposition 20, $\mathbb{Z}[i]v + \mathbb{Z}[i]v'$ has index one in Λ and by Proposition 10, $\frac{1}{1+i}(u + v)$, v is a basis of Λ, therefore, z' is of the shape

$$v' = a\tfrac{1}{1+i}(u + v) + gv$$

with $g \in \mathbb{Z}[i]$ and $a \in \mathbb{U}_4$. We can suppose that $a = -1$ w.l.o.g. As before, we have

$$|\tfrac{1}{1+i}(u_1 + u_1 w_1) - gu_1 w_1| < |u_1 w_1|,$$

which is equivalent to

$$|\tfrac{1}{(1+i)w_1} + \tfrac{1}{(1+i)} - g| < 1.$$

We conclude as in the first case. □

8. Parametrization of the open transversal and the first return map

We first give a parametrization of the open transversal with coordinates $(\theta, w_1, w_2) \in \mathbb{R} \times \mathbb{D}^2$. Then, we want to describe the open transversal with the (θ, w_1, w_2) coordinates. To do this, we first write the symmetries of the transversal with these coordinates. Finally, we give some explicit formulas for the first return map in terms of the coordinates (θ, w_1, w_2).

8.1. **Parametrization of the open transversal T.**

Proposition 27. *Let $\Psi_k : \mathbb{R} \times \mathbb{D}^2 \to \Omega_2$, $k = 1, 2$ be the maps defined by*

$$\Psi_1(\theta, w_1, w_2) = \mathbb{Z}[i]u + \mathbb{Z}[i]v,$$
$$\Psi_2(\theta, w_1, w_2) = \mathbb{Z}[i]u + \tfrac{1}{1+i}\mathbb{Z}[i](u + v)$$

where

$$\begin{aligned} u &= u(\theta, w_1, w_2) = r(u_1, v_2 w_2), \\ v &= v(\theta, w_1, w_2) = r(u_1 w_1, v_2), \\ r &= \frac{k^{1/4}}{\sqrt{|1 - w_1 w_2|}}, \\ u_1 &= \exp i\theta, \\ v_2 &= \exp i\theta' = \exp i((k-1)\tfrac{\pi}{4} - \theta - \arg(1 - w_1 w_2)). \end{aligned}$$

For $k = 1, 2$, let $C_k(\theta, w_1, w_2) = C(u(\theta, w_1, w_2), v(\theta, w_1, w_2))$. Then for all Λ in T_k there exists exactly one element $(\theta, w_1, w_2) \in [0, \frac{\pi}{2}) \times \mathbb{D}^2$ such that $\Lambda = \Psi_k(\theta, w_1, w_2)$ and $\Lambda \cap \overset{o}{C}_k(\theta, w_1, w_2) = \{0\}$.

Proof. Existence. Let Λ be a unimodular lattice in $\mathbb{C}^2$ that belongs to T_k and call u and v the two minimal vectors associated with Λ. Denoting $r = |u|_\infty$, u and v can be written $u = r(u_1, v_2 w_2)$ and $v = r(u_1 w_1, v_2)$ with

$$|w_1|, |w_2| < 1 = |u_1| = |v_2|.$$

The unimodularity implies that

$$\det_{\mathbb{C}}(u, v) = r^2 u_1 v_2 (1 - w_1 w_2) \in \mathbb{U}_4 \text{ or } \in (1 + i)\mathbb{U}_4$$

according to $\Lambda \in T_1$ or T_2. On the one hand, this implies that $r = \frac{k^{1/4}}{\sqrt{|1-w_1 w_2|}}$. On the other hand, since u and v can be changed in ωu and $\omega' v$ with $\omega, \omega' \in \mathbb{U}_4$, we can impose $u_1 = \exp i\theta$ with $\theta \in [0, \frac{\pi}{2}[$ and $v_2 = \exp i\theta'$ where

$$\theta' = (k - 1)\tfrac{\pi}{4} - \theta - \arg(1 - w_1 w_2).$$

With our choices, the triple (θ, w_1, w_2) belongs to $U = [0, \frac{\pi}{2}) \times \mathbb{D}^2$.

If $k = 1$, we have $\Lambda = \Psi_1(\theta, w_1, w_2) \in \Psi_1(U)$. If $k = 2$, thanks to Proposition 10 Part 3, we have $\Lambda = \Psi_2(\theta, w_1, w_2) \in \Psi_2(U)$.

Uniqueness. Let Λ be in T_k. Suppose that $\Lambda = \Psi_k(\alpha, w_1, w_2) = \Psi_k(\alpha', w_1', w_2')$ with (α, w_1, w_2) and $(\alpha', w_1', w_2') \in U$. Let $u = u(\alpha, w_1, w_2)$, $v = v(\alpha, w_1, w_2)$, $u' = u(\alpha', w_1', w_2')$ and $v' = v(\alpha', w_1', w_2')$. The two cylinders $\overset{o}{C}_k(\alpha, w_1, w_2)$ and $\overset{o}{C}_k(\alpha, w_1, w_2)$ must be equal, otherwise one of them would contains nonzero elements of Λ. Hence $r = r'$ and by Lemma 23, $u' \in \mathbb{U}_4 u$. Since α and α' are both in $[0, \frac{\pi}{2}[$, it follows that $u = u'$ and $\alpha = \alpha'$. Hence $v_2 w_2 = v_2' w_2'$. Again $v' = \omega v$ with $\omega \in \mathbb{U}_4$, therefore $w_1' u_1 = \omega w_1 u_1$ and $v_2' = \omega v_2$. It follows that $w_1' = \omega w_1$ and $w_2' \omega = w_2$ which in turn imply $w_1' w_2' = w_1 w_2$. Now by definition of Ψ_k, $\det_{\mathbb{C}}(u, v) = \det_{\mathbb{C}}(u', v') = 1$ or $1 + i$, hence $u_1 v_2 (1 - w_1 w_2) = u_1' v_2' (1 - w_1' w_2')$ and taking into account the relations $u_1' = u_1$ and $w_1' w_2' = w_1 w_2$, we obtain $v_2 = v_2'$. Finally, this implies, $\omega = 1$ and $(\alpha', w_1', w_2') = (\alpha, w_1, w_2)$. $\square$

8.2. **Symmetries of the transversal.** Given $(\theta, w_1, w_2) \in \mathbb{R}\times\mathbb{D}^2$, we would like to know what are the conditions on (θ, w_1, w_2) in order that $\Psi_1(\theta, w_1, w_2) \in T_1$ and $\Psi_2(\theta, w_1, w_2) \in T_2$. Thanks to Theorem 7 and to the distance formula (Lemma 14), these conditions are given by a finite set of inequalities on w_1 and w_2 and do not depend on θ. As for Theorem 7, the symmetries of the transversal simplify the statement. Actually, it can be reduced to the case

$$w_1 \in \mathcal{C} = \{z \in \mathbb{D} : \arg z \in [0, \tfrac{\pi}{4}]\}$$

and the other cases, $w_1 \in \{z \in \mathbb{D} : \arg z \in [\frac{k\pi}{4}, \frac{(k+1)\pi}{4}]\}$, $k = 1, \ldots, 7$, will be obtained by simple transformations.

Let T_1^0 and T_2^0 be the subset of T_1 and T_2 defined by

$$\begin{aligned} T_1^0 =& T_1 \cap \{\Lambda = \Psi_1(\theta, w_1, w_2) : \\ &(\theta, w_1, w_2) \in [0, \tfrac{\pi}{2}) \times \mathcal{C} \times \mathcal{D}),\ \Lambda \cap \overset{o}{C}_1(\theta, w_1, w_2) = \{0\}\}, \\ T_2^0 =& T_2 \cap \{\Lambda = \Psi_2(\theta, w_1, w_2) : \\ &(\theta, w_1, w_2) \in [0, \tfrac{\pi}{2}) \times \mathcal{C} \times \mathcal{D}),\ \Lambda \cap \overset{o}{C}_1(\theta, w_1, w_2) = \{0\}\}. \end{aligned}$$

Recall that $\mathbb{D}_8$ is the group of isometries acting on $\mathbb{C}$ generated by the multiplications by elements in $\mathbb{U}_4$ and by the conjugation. For $\varphi \in \mathbb{D}_8$, consider the map $F_{k,\varphi} : T_k \to \Omega_2$ defined by

$$F_{k,\varphi}(\Psi_k(\theta, w_1, w_2)) = \Psi_k(\theta, \varphi(w_1), \tfrac{1}{\varphi(1)^2}\varphi(w_2)).$$

This map is well defined because by Proposition 27, for each $\Lambda \in T_k$, there exists (θ, w_1, w_2) unique in $[0, \frac{\pi}{2}[\times\mathbb{D}^2$ such that $\psi_k(\theta, w_1, w_2) = \Lambda$ and $\Lambda \cap \overset{o}{C}_k(\theta, w_1, w_2) = \{0\}$.

Our aim is to prove:

Proposition 28. *For $k = 1, 2$,*

$$T_k = \bigcup_{\varphi\in\mathbb{D}_8} F_{k,\varphi}(T_k^0).$$

Since

$$\mathbb{D} = \cup_{\varphi\in\mathbb{D}_8}\varphi(\mathcal{C}),$$

the proposition is an obvious consequence of the following lemma.

Lemma 29. *For $k = 1, 2$ and $\varphi \in \mathbb{D}_8$*

$$F_{k,\varphi}(T_k) = T_k.$$

Proof of Lemma 29. 1. It is enough to prove that $F_{k,\varphi}(T_k) \subset T_k$ for all $\varphi \in \mathbb{D}_8$. Indeed, if so, we have $F_{k,\varphi^{-1}}(F_{k,\varphi}(T_k)) \subset F_{k,\varphi^{-1}}(T_k) \subset T_k$ and since the elements in $\mathbb{D}_8$ are $\mathbb{R}$-linear, for $\Lambda = \Psi_k(\theta, w_1, w_2) \in T_k$, we have

$$\begin{aligned} F_{k,\varphi^{-1}}(F_{k,\varphi}(\Psi_k(\theta, w_1, w_2))) &= F_{k,\varphi^{-1}}(\Psi_k(\theta, \varphi(w_1), \tfrac{1}{\varphi(1)^2}\varphi(w_2)) \\ &= \Psi_k(\theta, \varphi^{-1}(\varphi(w_1)), \tfrac{1}{\varphi^{-1}(1)^2}\varphi^{-1}(\tfrac{1}{\varphi(1)^2}\varphi(w_2))) \\ &= \Psi_k(\theta, w_1, \tfrac{1}{\varphi^{-1}(1)^2}\tfrac{1}{\varphi(1)^2}w_2) \\ &= \Psi_k(\theta, w_1, w_2), \end{aligned}$$

which implies that $F_{k,\varphi^{-1}}(F_{k,\varphi}(T_k)) = T_k$.

2. Call $E_1 = (\mathbb{Z}[i] \setminus \{0\})^2$ and $E_2 = (\mathbb{Z}[i] \setminus \{0\})^2 \cup J^2$. For each $\varphi, \psi \in \mathbb{D}_8$, the maps $f(a, b) = (\psi(a), \varphi(b))$ induce a bijection on E_k. It is an immediate consequence of $\varphi(\mathbb{Z}[i]) = \psi(\mathbb{Z}[i]) = \mathbb{Z}[i]$ and $\varphi(J) = \psi(J) = J$.

3. Let $\theta \in [0, \frac{\pi}{2}[$, $w_1, w_2 \in \mathbb{D}$, $w_1' = \varphi(w_1)$, and $w_2' = \frac{1}{\varphi(1)^2}\varphi(w_2)$

$$u = u(\theta, w_1, w_2) = (u_1, v_2 w_2),\ \ v = v(\theta, w_1, w_2) = (u_1 w_2, v_2)$$
$$u' = u(\theta, w_1', w_2') = (u_1', v_2' w_2'),\ \ v' = v(\theta, w_1', w_2') = (u_1' w_2', v_2').$$

Suppose that $\Lambda = \mathbb{Z}[i]u + \mathbb{Z}[i]v \in T_k$ and $\Lambda \cap C_k(\theta, w_1, w_2) = \{0\}$. Consider $\Lambda' = \mathbb{Z}[i]u' + \mathbb{Z}[i]v'$. By definition $\Lambda' = F_{k,\varphi}(\Lambda)$. We want to show that $\Lambda' \in T_k$. By Proposition 11 about the symmetries, for all nonzero complex numbers a, b,

$$|au - bv|_{u,v} = |\varphi(1)\varphi(a)u' - \varphi(b)v'|_{u',v'}.$$

By 2, it follows that $|a'u' - b'v'|_{u',v'} > 1$ for all $(a', b') \in E_k$ iff $|au - bv|_{u,v} > 1$ for all $(a, b) \in E_k$. Since Λ is in T_k, $|au - bv|_{u',v'} > 1$ for all $(a, b) \in E_k$ which implies that $|a'u' - b'v'|_{u',v'} > 1$ for all $(a', b') \in E_k$ and we are done. $\square$

8.3. **Determination of the open transversals T_1 and T_2 with the (θ, w_1, w_2)-coordinates.** Recall that

$$\mathcal{C} = \{w \in \mathbb{C} : |w| < 1,\ \arg w \in [0, \tfrac{\pi}{4}]\},$$
$$\mathcal{D} = \{w \in \mathbb{C} : |w| < 1,\ \mathrm{d}(w, 1) > 1,\ \mathrm{d}(w, 1-i) > 1\},$$
$$\mathcal{T} = \{w \in \mathbb{C} : |w| < 1,\ \mathrm{d}(w, 1) > \sqrt{2},\ \mathrm{d}(w, -i) > \sqrt{2}\}$$

and that the parametrizations Ψ_1 and Ψ_2 have been defined in Proposition 27.

Consider the following pairs of closed disks in $\mathbb{C}$:

$$\bar{Red}_1 = D(i, 1),\ \bar{Red}_2 = D(-i, 1)$$
$$\bar{Blue}_1 = D(\tfrac{1-i}{2}, \tfrac{1}{\sqrt{2}}),\ \bar{Blue}_2 = D(1+i, 1)$$
$$\bar{Green}_1 = D(1+i, 1),\ \bar{Green}_2 = D(\tfrac{1-i}{2}, \tfrac{1}{\sqrt{2}}).$$

see the Figure 3 in Subsection 4.3.

Let W_1^0 be the set of (w_1, w_2) in $\mathcal{C} \times \mathcal{D}$ such that one of the four conditions

(1) $w_2 \in \bar{Green}_2$ and $w_1 \notin \bar{Red}_1 \cup \bar{Green}_1$,
(2) $w_2 \in \bar{Red}_2 \setminus \bar{Green}_2$ and $w_1 \notin \bar{Red}_1$,
(3) $w_2 \notin \bar{Red}_2 \cup \bar{Blue}_2$ and $w_1 \neq 0$,
(4) $w_2 \in \bar{Blue}_2$ and $w_1 \notin \bar{Blue}_1$,

holds, and let

$$W_1 = \{(\varphi(w_1), \tfrac{1}{\varphi(1)^2}\varphi(w_2)) : \varphi \in \mathbb{D}_8,\ (w_1, w_2) \in W_1^0\}.$$

Let

$$W_2^0 = \mathcal{C} \setminus D(-i, \sqrt{2}) \times \mathcal{T}$$

and let

$$W_2 = \{(\varphi(w_1), \tfrac{1}{\varphi(1)^2}\varphi(w_2)) : \varphi \in \mathbb{D}_8,\ (w_1, w_2) \in W_2^0\}.$$

Theorem 8. *Let (θ, w_1, w_2) be in $[0, \frac{\pi}{2}[\times \mathbb{D}^2$. Then*

- $\Psi_1(\theta, w_1, w_2) \in T_1$ *iff* $(w_1, w_2) \in W_1$,
- $\Psi_2(\theta, w_1, w_2) \in T_2$ *iff* $(w_1, w_2) \in W_2$.

Abridge proof of Theorem 8. With $(w_1', w_2') = (\varphi(w_1), \frac{1}{\varphi(1)^2}\varphi(w_2))$, we have $\Psi_k(\theta, w_1, w_2) \in T_k$ iff $\Psi_k(\theta, w_1', w_2') \in T_k$ according to Proposition 11 about the symmetries of the transversal. Then, we follow the proof of Corollary 19 using Proposition 13 with strict inequalities. $\square$

Remark 7. The conditions for $\Psi_k(\theta, w_1, w_2)$ to be in the full transversal are similar, just replace the closed disks by open disks and take care of the particular case $w_1 = 0$. In this latter case $\Psi_1(\theta, w_1, w_2)$ is in the transversal whatever w_2 is.

8.4. **The first return map in the** (θ, w_1, w_2)**-coordinates.** Let us denote by $R : T \to T'$ the first return map in the full transversal T'. We want to find the formula

$$R(\theta, w_1, w_2) = (\theta', w_1', w_2')$$

in the coordinates (θ, w_1, w_2). As the example in Subsection 4.4, shows the minimal vectors following one minimal vector are not necessarily unique up to a multiplicative factor in $\mathbb{U}_4$. This makes the map T_G multi-valued. In order to avoid this drawback we restrict the first return map to $T' \setminus \mathcal{N} = T \setminus \mathcal{N}$ (see the definition 6.2 where $\mathcal{N}$ is defined).

Recall that the set $\mathcal{N}$ is negligible, is invariant by the flow and contains the lattices with nonzero vectors on the axes. Therefore, the restriction of the first return map

$$R : T \setminus \mathcal{N} \to T \setminus \mathcal{N}$$

is a bijection.

For $k = 1, 2$, let W_k' be the set of $(\theta, w_1, w_2) \in W_k$ such that $\Psi_k(\theta, w_1, w_2) \notin \mathcal{N}$. Let T_G be the map defined on the disjoint union of W_1' and W_2' according to Theorem 3:

- If $(w_1, w_2) \in W_1$, let $a \in \{1, 1+i\}$ and $g \in \mathbb{Z}[i]$ be such that $-a(1, w_2) + g(w_1, 1)$ is minimal for the preorder $\prec$ in the set
$$\left\{-a(1, w_2) + g(w_1, 1) : a \in \{1, 1+i\},\ g \in \mathbb{Z}[i],\ |\tfrac{a}{w_1} - g| < 1\right\}.$$
We then have
$$T_G(w_1, w_2) = \left(g - \frac{a}{w_1}, \frac{1}{g - aw_2}\right)$$
and $T_G(w_1, w_2) \in W_1$ or W_2 according to $a = a_1(w_1, w_2) = 1$ or $a = a_1(w_1, w_2) = 1 + i$.
- If $(w_1, w_2) \in W_2$, let $g \in \mathbb{Z}[i]$ be such that $-\frac{1}{1+i}(1+w_1, w_2+1)+g(w_1, 1)$ is minimal for the preorder $\prec$ in the set
$$\left\{-\tfrac{1}{1+i}(1 + w_1, w_2 + 1) + g(w_1, 1) : g \in \mathbb{Z}[i],\ |\tfrac{1}{(1+i)w_1} + \tfrac{1}{(1+i)} - g| < 1\right\}.$$
We then have
$$T_G(w_1, w_2) = \left(g - \frac{1}{(1+i)w_1} - \frac{1}{(1+i)}, \frac{1}{g - \frac{1}{(1+i)}w_2 - \frac{1}{(1+i)}}\right)$$
and $T_G(w_1, w_2)$ is always in W_1. In that case, we set $a = a_2(w_1, w_2) = \frac{1}{1+i}$.

Formally the map T_G should be defined on $(\{1\} \times W_1') \cup (\{2\} \times W_2')$ with values in the same set.

Now we are able to compute the first return map in (θ, w_1, w_2) coordinates.

Theorem 9. *Let $k = 1$ or 2. Let (θ, w_1, w_2) be in $[0, \frac{\pi}{2}[\times W_k'$. Then*

$$R \circ \Psi_k(\theta, w_1, w_2) = \Psi_j(\theta', \alpha^2 T_G(w_1, w_2))$$

where

- *$j = 2$ when $a_k(w_1, w_2) = 1 + i$ and $j = 1$ otherwise,*
- *$\alpha = \alpha(\theta, w_1)$ is the only element in $\mathbb{U}_4$ such that $\theta' = \theta + \arg w_1 + \arg \alpha \in [0, \frac{\pi}{2}[$.*

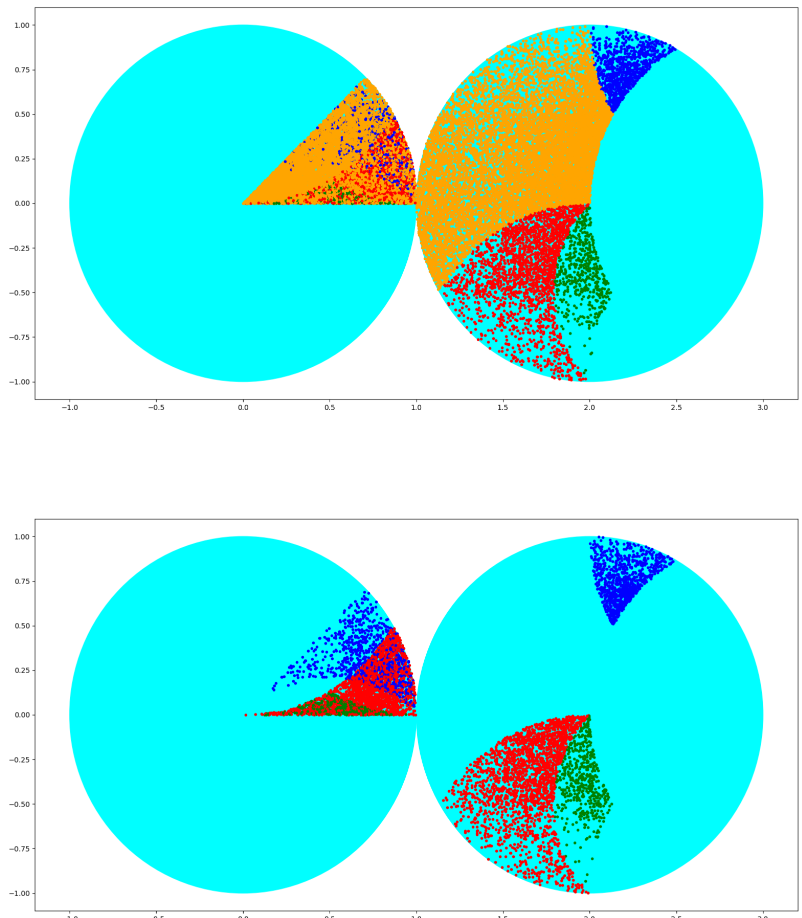

FIGURE 4. Iterates of the map T_G with one initial point. Only the couple (w_1, w_2) in T_1 with $w_1 \in \mathcal{C}$ are plotted, w_1 in the left disk and w_2 in the right disk. The color is chosen according to the regions in Corollary 19. In the second rectangle, the orange points have been suppressed.

Proof. Set $(w_1', w_2') = T_G(w_1, w_2)$. By Theorem 3 and the definitions of the parametrizations Ψ_1 and Ψ_2 and of the map T_G,

$$R \circ \Psi_k(\theta, w_1, w_2) = \Psi_j(\theta + \arg w_1, w_1', w_2')$$

where $j = 2$ iff $a_k(w_1, w_2) = 1+i$. The only thing we have to worry about is that $\theta + \arg w_1$ might be outside the interval $[0, \frac{\pi}{2}[$. In any cases, there exists $\alpha \in \mathbb{U}_4$ unique such that $\theta' = \theta + \arg w_1 + \arg \alpha \in [0, \frac{\pi}{2}[$. So, if we change the vectors $u(\theta + \arg w_1, w_1', w_2')$ and $v(\theta + \arg w_1, w_1', w_2')$ in $\alpha u(\theta + \arg w_1, w_1', w_2')$ and $\frac{1}{\alpha} v(\theta + \arg w_1, w_1', w_2')$, we obtain the same lattice and we do not change the determinant (see Proposition 27 the definition of $u(.)$ and $v(.)$). Now,

$$\begin{aligned} \alpha u(\theta + \arg w_1, w_1', w_2') &= u(\theta + \arg w_1 + \arg \alpha, \tfrac{1}{\alpha^2} w_1', \alpha^2 w_2') \\ \tfrac{1}{\alpha} v(\theta + \arg w_1, w_1', w_2') &= v(\theta + \arg w_1 + \arg \alpha, \tfrac{1}{\alpha^2} w_1', \alpha^2 w_2') \end{aligned}$$

and since $\frac{1}{\alpha^2} = \alpha^2$, we are done. $\square$

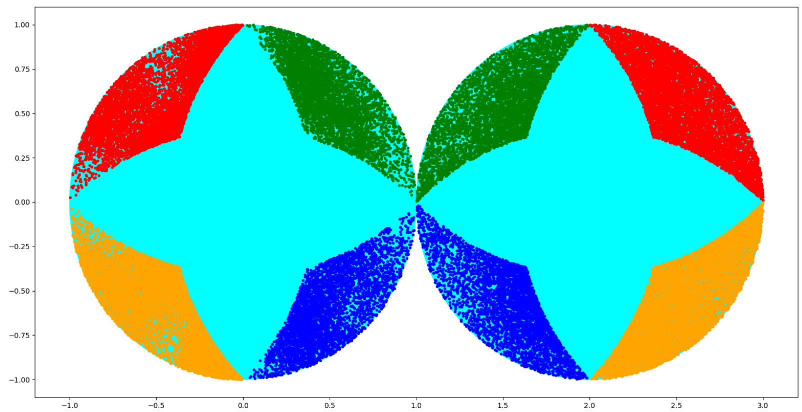

FIGURE 5. Iterates of the map T_G: the points in T_2.

9. INVARIANT MEASURES, PROOF OF THEOREM 4

We want to find the measure induced by the Haar measure and the flow $(g_t)_t$ on the transversal T in the coordinates systems $\Psi_1(\theta, w_1, w_2)$ and $\Psi_2(\theta, w_1, w_2)$ (see Proposition 27 the definitions of the parametrizations Ψ_k). The Haar measure is defined up to a multiplicative constant and can be defined with an invariant volume form α on $\mathrm{SL}(2, \mathbb{C})$. To take advantage of the $\mathbb{C}$-linearity of the differential of Ψ_k with respect to w_1, w_2, we use the following volume form. Let ω be the differential form of degree 3 defined on $M_2(\mathbb{C})$ by

$$\omega_M(M_1, M_2, M_3) = \det(M, M_1, M_2, M_3)$$

where $M_2(\mathbb{C})$ is identified with $\mathbb{C}^4$. Since for every matrix $A \in M_2(\mathbb{C})$,

$$\begin{aligned}\det(AM, AM_1, AM_2, AM_3) &= \det(MA, M_1A, M_2A, M_3A) \\ &= (\det A)^2 \det(M, M_1, M_2, M_3),\end{aligned}$$

the form ω is $\mathrm{SL}(2, \mathbb{C})$-invariant and

$$\alpha = -i\omega \wedge \overline{\omega}$$

is a volume form on $\mathrm{SL}(2, \mathbb{C})$ and defines a Haar measure μ_1 on $\mathrm{SL}(2, \mathbb{C})$ (see below the definition of $\overline{\omega}$). We restate Theorem 4.

Theorem 10. *Using the parametrization Ψ_k, $k = 1, 2$ of the transversal T_k, the Haar measure μ_1 associated with the volume form α and the flow g_t induce a measure ν with density*

$$h(\theta, w_1, w_2) = \frac{32}{|1 - w_1 w_2|^4}$$

with respect of the Lebesgue measure of $[0, \pi/2] \times \mathbb{D}^2$.

Before we prove the above theorem, we wish to compute the volume of the space of lattices $\mathrm{SL}(2, \mathbb{C})/\mathrm{SL}(2, \mathbb{Z}[i])$. This volume can be deduced from the volume of $\mathrm{SL}(2, \mathbb{Z}[i]))\backslash\mathbb{H}_3$ where $\mathbb{H}_3 = \mathbb{C} + j\mathbb{R}_{>0}$ is the three-dimensional hyperbolic space. Indeed, consider the left action of $\mathrm{SL}(2, \mathbb{C})$ on the three-dimensional hyperbolic space $\mathbb{H}_3$ defined by

$$M.(z + rj) = \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2 + jr}{|cz + d|^2 + |c|^2 r^2}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ and $z + rj \in \mathbb{H}_3$. The stabilizer of j is $K = \mathrm{SU}(2,\mathbb{C})$. Choosing an appropriate Haar measure μ_2 on $\mathrm{SL}(2,\mathbb{C})$, we have

$$\mathrm{Vol}(\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i])) = \mathrm{Vol}(\mathrm{SL}(2,\mathbb{Z}[i]))\backslash\mathbb{H}_3) \times \mathrm{Vol}(\mathrm{SU}(2,\mathbb{C})).$$

The measure μ_2 can be chosen in order that in the above formula,

$$\mathrm{Vol}(\mathrm{SU}(2,\mathbb{C})) = \mathrm{Vol}(S_3) = 2\pi^2$$

is the volume of the unit sphere S_3 with respect to the standard Euclidean distance and $\mathrm{Vol}(\mathrm{SL}(2,\mathbb{Z}[i]))\backslash\mathbb{H}_3)$ is computed with respect to the hyperbolic metric on $\mathbb{H}_3$. In that case

$$\mathrm{Vol}(\mathrm{SL}(2,\mathbb{Z}[i]))\backslash\mathbb{H}_3) = \frac{|d|^{\frac{3}{2}}}{4\pi^2}\zeta_K(2)$$

where $d = -4$ is the discriminant of the quadratic field $K = \mathbb{Q} + i\mathbb{Q} = \mathbb{Q}[\sqrt{-1}]$ and $\zeta_K(s) = \sum \frac{1}{(a^2+b^2)^s}$ where the sum is computed over all the nonzero ideals $(a+ib)\mathbb{Z}[i]$ in $\mathbb{Z}[i]$ (see [8] p. 311-312). Actually, $\zeta_K(2) = \frac{\pi^2}{6}C$ where $C = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^2}$ is the Catalan number.

So we have two Haar measures on $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i])$, one obtained with the volume form α and a second obtained from $\mathbb{H}_3$ and $\mathrm{SU}(2,\mathbb{C})$. It is possible to compute the normalization factor between the two Haar measures μ_1 and μ_2, in fact

$$\mu_1 = 16\mu_2,$$

which leads to

$$\begin{aligned}\mathrm{Vol}_\alpha(\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i])) &= 16 \times (2\pi^2) \times (\frac{|-4|^{\frac{3}{2}}}{4\pi^2}\zeta_K(2)) \\ &= 64\zeta_K(2) = \frac{32\pi^2}{3}\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2}.\end{aligned}$$

Proof of Theorem 10. The parametrizations Ψ_k, $k = 1,2$ can be factorized, $\Psi_k = \Phi_k \circ F_k$. Indeed, let $\Phi_k : \mathbb{C}^* \times \mathbb{D}^2 \to \mathrm{SL}(2,\mathbb{C})$, $k = 1,2$, be the maps defined by

$$\Phi_1(u_1, w_1, w_2) = \begin{pmatrix} u_1 & u_1 w_1 \\ v_2 w_2 & v_2 \end{pmatrix}$$

where $v_2 = v_2(u_1, w_1, w_2) = \frac{1}{u_1(1-w_1w_2)}$ and

$$\Phi_2(u_1, w_1, w_2) = \begin{pmatrix} u_1 & u_1 w_1 \\ v_2' w_2 & v_2' \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{1+i} \\ 0 & \frac{1}{1+i} \end{pmatrix}$$

where $v_2' = v_2'(u_1, w_1, w_2) = (1+i)v_2(u_1, w_1, w_2)$. Let $F_k : \mathbb{R} \times \mathbb{D}^2 \to \mathbb{C}^* \times \mathbb{D}^2$, $k = 1,2$, be the maps defined by $F_k(\theta, w_1, w_2) = (u_1 = re^{i\theta}, w_1, w_2)$ where $r = \frac{k^{1/4}}{\sqrt{|1-w_1w_2|}}$. By definition, $\Psi_k = \Phi_k \circ F_k$.

The first step is to compute the pull back $\Phi_{k*}\omega$. Let $p = (u_1, w_1, w_2)$. Straightforward calculations lead to

$$
\begin{aligned}
(\Phi_{1*}\omega)_p(\tfrac{\partial}{\partial u_1}, \tfrac{\partial}{\partial w_1}, \tfrac{\partial}{\partial w_2}) &= \det \begin{pmatrix} u_1 & 1 & 0 & 0 \\ u_1 w_1 & w_1 & u_1 & 0 \\ v_2 w_2 & w_2 \frac{\partial v_2}{\partial u_1} & w_2 \frac{\partial v_2}{\partial w_1} & v_2 + w_2 \frac{\partial v_2}{\partial w_2} \\ v_2 & \frac{\partial v_2}{\partial u_1} & \frac{\partial v_2}{\partial w_1} & \frac{\partial v_2}{\partial w_2} \end{pmatrix} \\
&= \frac{-2}{u_1(1 - w_1 w_2)^2},
\end{aligned}
$$

hence

$$(\Phi_{1*}\omega)_p = \frac{-2}{u_1(1 - w_1 w_2)^2} du_1 \wedge dw_1 \wedge dw_2.$$

Using that $v_2' = (1 + i)v_2$, we obtain

$$
\begin{aligned}
(\Phi_{2*}\omega)_p &= \det \begin{pmatrix} 1 & \frac{1}{1+i} \\ 0 & \frac{1}{1+i} \end{pmatrix}^2 \frac{-2(1+i)^2}{u_1(1 - w_1 w_2)^2} du_1 \wedge dw_1 \wedge dw_2 \\
&= \frac{-2}{u_1(1 - w_1 w_2)^2} du_1 \wedge dw_1 \wedge dw_2.
\end{aligned}
$$

Let us now use the conjugation. Consider the maps $c : \mathbb{C}^* \times \mathbb{D}^2 \to \mathbb{C}^* \times \mathbb{D}^2$ and $C : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SL}(2, \mathbb{C})$ defined by

$$
\begin{aligned}
c(u_1, w_1, w_2) &= (\overline{u}_1, \overline{w}_1, \overline{w}_2), \\
C(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) &= \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}.
\end{aligned}
$$

The form $\overline{\omega}$ is defined by $\overline{\omega} = C_*\omega$. Since $C \circ \Phi_k = \Phi_k \circ c$, we have

$$
\begin{aligned}
\Phi_{k*}\overline{\omega} &= (C \circ \Phi_k)_*\omega \\
&= (\Phi_k \circ c)_*\omega \\
&= c_*\Phi_{k*}\omega.
\end{aligned}
$$

With $u_1 = u_{11} + iu_{12}$, $w_1 = w_{11} + iw_{12}$ and $w_2 = w_{21} + iw_{22}$, we have

$$du_1 = du_{11} + idu_{12} \text{ and } \overline{du_1} = c_* du_1 = du_{11} - idu_{12}.$$

Hence,

$$
\begin{aligned}
&c_*(\frac{-2}{u_1(1 - w_1 w_2)^2} du_1 \wedge dw_1 \wedge dw_2) \\
&= \frac{-2}{c(u_1)(1 - c(w_1)c(w_2))^2} c_* du_1 \wedge c_* dw_1 \wedge c_* dw_2 \\
&= \overline{\frac{-2}{u_1(1 - w_1 w_2)^2}} \overline{du_1} \wedge \overline{dw_1} \wedge \overline{dw_2}
\end{aligned}
$$

Therefore, in the coordinates system (u_1, w_1, w_2), the Haar measure is associated with the differential form

$$
\begin{aligned}
(\Phi_{k*}\alpha)_p &= -i \frac{4}{|u_1|^2 |1 - w_1 w_2|^4} du_1 \wedge dw_1 \wedge dw_2 \wedge \overline{du_1} \wedge \overline{dw_1} \wedge \overline{dw_2} \\
&= \frac{32}{|u_1|^2 |1 - w_1 w_2|^4} du_{11} \wedge du_{12} \wedge dw_{11} \wedge dw_{12} \wedge dw_{21} \wedge dw_{21}.
\end{aligned}
$$

In coordinates (u_1, w_1, w_2), the diagonal flow g_t writes

$$g_t(u_1, w_1, w_2) = (e^t u_1, w_1, w_2)$$

and is associated with the vector field $X(p) = (u_1, 0, 0)$.

In order to compute the measure induced by the Haar measure and the flow g_t, it is enough to compute the Jacobian of the map

$$(t, \theta, w_1, w_2) \to g_t \circ F_k(\theta, w_1, w_2) = (re^{t+i\theta}, w_1, w_2)$$

at the point $(0, \theta, w_1, w_2)$. It is the 6×6 determinant

$$\det \begin{pmatrix} r\cos\theta & -r\sin\theta & . & . & . & . \\ r\sin\theta & r\cos\theta & . & . & . & . \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = r^2.$$

Finally, we obtain the density

$$h(\theta, w_1, w_2) = \frac{32r^2}{|u_1|^2|1 - w_1w_2|^4} = \frac{32}{|1 - w_1w_2|^4}.$$

$\square$

10. Dirichlet best constant, proof of Theorem 5

Let θ be complex number. The Dirichlet constant associated with θ is the infimum $C(\theta)$ of constants C such that for any real number $Q \geq 1$ there exist $p, q \in \mathbb{Z}[i]$ such that

$$\begin{cases} 0 < |q| < Q \\ |q\theta - p| \leq \frac{C}{Q} \end{cases}.$$

The best constant in Theorem 5 is then $C_D = \sup\{C(\theta) : \theta \in \mathbb{C}\}$.

Let $(p_n, q_n) \in \mathbb{Z}[i]^2$, $n \in I_\theta \subset \mathbb{N}$, be a sequence of best approximations vectors of θ such that

$$1 = |q_0| < |q_1| < \cdots < |q_n| < \ldots,$$

and including all the the best approximation denominators: if (p, q) is a best approximation vector then there is an $n \in I_\theta$ such that $|q| = |q_n|$. The sequence is infinite, i.e. $I_\theta = \mathbb{N}$, iff $\theta \notin \mathbb{Q}[i]$.

Then, it is clear that

$$C(\theta) = \sup\{|q_{n+1}||q_n\theta - p_n| : n, n+1 \in I_\theta\}.$$

If we want to study the best Dirichlet constant for all large enough Q when $\theta \notin \mathbb{Q}[i]$, we have to use the constant

$$C'(\theta) = \limsup_{n\to\infty} |q_{n+1}||q_n\theta - p_n|.$$

instead of the constant $C(\theta)$. By Proposition 9, the sequence of best approximation vectors of θ is the sequence of minimal vectors of the lattice

$$\Lambda_\theta = \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \mathbb{Z}[i]^2 = M_\theta \mathbb{Z}[i]^2.$$

More precisely, the sequence $M_\theta \begin{pmatrix} p_n \\ q_n \end{pmatrix}$, $n \in I_\theta$, contains exactly one element equivalent to any minimal vector of Λ_θ with nonzero second coordinate. It follows that the best Dirichlet constant C_D is bounded above by

$$C_S = \sup |u_1||v_2|$$

where the supremum is taken over all Gauss unimodular lattices Λ in $\mathbb{C}^2$ and all the pairs $u = (u_1, u_2), v = (v_1, v_2)$ of consecutive minimal vectors in Λ with $0 < |u_2| < |v_2|$. The proof is now done in two steps:

(1) We prove that $C_S = \frac{1}{\sqrt{6-3\sqrt{3}}} = \frac{\sqrt{2}}{3-\sqrt{3}}$,
(2) We prove that for almost all $\theta \in \mathbb{C}$, $C'(\theta) = C_S$.

10.1. **Step 1.**

10.1.1. *A first reduction to compute C_S.* Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two consecutive minimal vectors of a unimodular Gauss lattice Λ in $\mathbb{C}^2$. Then $|u_1| > |v_1|$ and $|v_2| > |u_2|$ and by Theorem 1, the index of the sublattice $\mathbb{Z}[i]u + \mathbb{Z}[i]v$ in Λ, is one or two. Thus, we can write

$$\begin{cases} u = (u_1, v_2 w_2), & |w_2| < 1 \\ v = (u_1 w_1, v_2), & |w_1| < 1 \end{cases}$$

and $|\det_{\mathbb{C}}(\mathbb{Z}[i]u + \mathbb{Z}[i]v)| = |u_1 v_2 (1 - w_1 w_2)| = 1$ or $\sqrt{2}$ depending on the index 1 or 2. Set

$$C_1 = \sup u_1 v_2 = \sup \frac{1}{|1 - w_1 w_2|}$$

where the supremum is taken over all unimodular lattices Λ and all pairs of consecutive minimal vectors u, v of index 1 in Λ and set

$$C_2 = \sup u_1 v_2 = \sup \frac{\sqrt{2}}{|1 - w_1 w_2|}$$

where the supremum is taken over all unimodular lattices Λ and all pairs of consecutive minimal vectors u, v of index 2 in Λ. Then

$$C_S = \max(C_1, C_2).$$

Thanks to Proposition 11, using the symmetries associated with $\varphi \in \mathbb{D}_8$, we can suppose that $w_1 \in \mathcal{C}$. We can now evaluate C_1 and C_2 using Corollary 19 that gives necessary and sufficient conditions on w_1 and w_2 so that u and v are consecutive minimal vectors.

10.1.2. *We show that $C_1 = \frac{1}{\sqrt{6-3\sqrt{3}}}$.* We want to bound above the function $f(w_1, w_2) = \frac{1}{|1-w_1 w_2|}$. In the particular case $w_1 = 0$, $f(w_1, w_2) = 1$ so that $f(w_1, w_2) \leq \frac{1}{\sqrt{6-3\sqrt{3}}}$. From now on, we suppose $w_1 \neq 0$.

Recall the notations

$$\begin{aligned} &Red_1 = \mathbb{D}(i, 1),\ Red_2 = \mathbb{D}(-i, 1) \\ &Blue_1 = \mathbb{D}(\tfrac{1-i}{2}, \tfrac{1}{\sqrt{2}}),\ Blue_2 = \mathbb{D}(1+i, 1) \\ &Green_1 = \mathbb{D}(1+i, 1),\ Green_2 = \mathbb{D}(\tfrac{1-i}{2}, \tfrac{1}{\sqrt{2}}). \end{aligned}$$

Let $u = (u_1, v_2 w_2)$ and $v = (u_1 w_1, v_2)$ be two vectors in $\mathbb{C}^2$ with $|u_1|, |v_2| > 0$, $|w_1|, |w_2| < 1$, and $w_1 \in \mathcal{C} \setminus \{0\}$.

By Corollary 19, zero is the only vector of $\mathbb{Z}[i]u+\mathbb{Z}[i]v$ that is in $\overset{o}{C}(u,v)$ iff $w_2 \in \overline{\mathcal{D}}$ and one of the four conditions

(1) $w_2 \in Green_2$ and $w_1 \notin Red_1 \cup Green_1$,
(2) $w_2 \in Red_2 \setminus Green_2$ and $w_1 \notin Red_1$,
(3) $w_2 \notin Red_2 \cup Blue_2$,
(4) $w_2 \in Blue_2$ and $w_1 \notin Blue_1$,

holds.

So we have to compute the supremum of the function $f(w_1,w_2)=\frac{1}{|1-w_1w_2|}$ over the four regions defined by (1), (2), (3) and (4).

In the following we assume the arguments of complex numbers are in $[0,2\pi[$.

Region 1: $(w_1,w_2)\in \mathcal{C}\setminus\{0\}\times\overline{\mathcal{D}}$, $w_2\in Green_2$ and $w_1\notin Red_1\cup Green_1$.

We have to minimize the distance from w_1w_2 to 1 when (w_1,w_2) is in region 1. If w_1 is on the circle of radius r_1 centered at 0 and w_2 on the circle of radius r_2 centered at 0, the point w_1w_2 is on the circle of radius r_1r_2 and will be closest to 1 when the arguments of w_1 and w_2 are maximal. It follows that the infimum of the distances $|1-w_1w_2|$ is reached when w_1 and w_2 are in the following arcs of circle (see Figure 3 in Subsection 4.3)

(a) w_1 is in the arc $\mathbf{C}_a$ of the circle $\mathbf{C}(i,1)$ from $z_0=0$ to $z_1=1/2+(1-\sqrt{3}/2)i$ (positive orientation),
(b) w_1 is in the arc $\mathbf{C}_b$ of the circle $\mathbf{C}(1+i,1)$ from z_1 to $z_2=1$

and

(c) w_2 is in the arc $\mathbf{C}_c$ of the circle $\mathbf{C}(1,1)$ from z_0 to $z_3=-iz_1$,
(d) w_2 is in the arc $\mathbf{C}_d$ of the circle $\mathbf{C}(1-i,1)$ from z_3 to $z_4=-i$.

We are going to show that the infimum of $|1-w_1w_2|$ is

$$r=|1-z_1z_3|=\sqrt{6-3\sqrt{3}}.$$

Case $w_1\in\mathbf{C}_a$ and $w_2\in\mathbf{C}_c$. Clearly $|w_1w_2|\le|z_1z_3|$ and $3\pi/2\le\arg w_1w_2\le\arg z_1z_3$. Now $z_1z_3=1-\sqrt{3}/2-i(\sqrt{3}-3/2)$, $|z_1z_3|=2-\sqrt{3}$ and $\arg z_1z_3=2\pi-\pi/3$, hence w_1w_2 is in the sector

$$S=\{z\in\mathbb{C}:3\pi/2\le\arg w_1w_2\le 2\pi-\pi/3,\ |z|\le 2-\sqrt{3}\}.$$

Since $z_1z_3\in\mathbf{C}(1,r)$ and $|z_1z_3|=2-\sqrt{3}<1/2=\cos\pi/3$, this sector doesn't intersect the open disk $\mathbb{D}(1,r)$, hence $|1-w_1w_2|\le r$.

Case $w_1\in\mathbf{C}_a$ and $w_2\in\mathbf{C}_d$. Suppose first that $w_1\notin D(1,r)$. It is enough to prove that $\Re(w_1w_2)\le 1-r$. We have $\Re(w_1w_2)=|w_1w_2|\cos(\arg w_1w_2)$. As before $\arg w_1w_2\in[3\pi/2,\arg z_1z_3]$, so that $\cos(\arg w_1w_2)\le\cos(\arg z_1z_3)=1/2$. Let $w_0=x+iy$ be the unique point in $\mathbf{C}_a\cap\mathbf{C}(1,r)$. We have that $|w_1|\le|w_0|$ so that $\Re(w_1w_2)=|w_1w_2|\cos(\arg w_1w_2)\le|w_0|/2$. The complex number $w_0=x+iy$ can easily be calculated because its real part is solution of an equation of degree 2. We find that $x\le 0.103...$ and $y\le 0.0054...$ so that $|w_0|\le 0.11$. Since $1-r\ge 0.10>|w_0|/2$, we are done.

When $w_1\in D(1,r)$ and in the remaining cases below, we shall use the following simple lemma. It is an easy consequence of the fact that two circles meet in two points at most.

Lemma 30. *Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two circles in the plane and let p_1,p_2 and p_3 be three distinct points in $\mathbf{C}_1$. If p_1 and p_2 are not in the interior of $\mathbf{C}_2$ while p_3 is in the closed disk associated with $\mathbf{C}_2$, then the closed arc of the circle $\mathbf{C}_1$ between p_1 and p_2 that doesn't contain p_3, doesn't intersect the interior of $\mathbf{C}_2$.*

We now use the lemma with $\mathbf{C}_1 = w_1\mathbf{C}(1-i,1)$ and $\mathbf{C}_2 = \mathbf{C}(1,r)$. Since $w_1 \in \mathbf{C}_a$ and $z_3 \in \mathbf{C}_c$, the products $w_1 z_3$ is not in the interior of $\mathbf{C}_2$. Next $w_1 z_4 = -iw_1 \in \mathbf{C}(1,1)$ is not in the interior of $\mathbf{C}_2$ while the product $w_1 \times 1$ is in the closed disk associated with $\mathbf{C}_2$. Therefore, using Lemma 30 with $p_1 = w_1 z_3$, $p_2 = w_1 z_4$ and $p_3 = w_1 \times 1$, we obtain that $w_1\mathbf{C}_d$ doesn't meet the interior of $\mathbf{C}_2$.

Case $w_1 \in \mathbf{C}_b$ and $w_2 \in \mathbf{C}_c$. We have $w_1 w_2 = w_1' w_2'$ with $w_1' = iw_2 \in \mathbf{C}_a$ and $w_2' = -iw_1 \in \mathbf{C}_d$ so we are done thanks to the above case.

Case $w_1 \in \mathbf{C}_b$ and $w_2 \in \mathbf{C}_d$. We fix $w_2 \in \mathbf{C}_d$. The points $w_2 z_1$ and $w_2 z_2 = w_2$ are not in the interior of $\mathbf{C}_2$ while $w_2 \times i$ is in the interior of $\mathbf{C}_2$, therefore, by Lemma 30, $w_2\mathbf{C}_b$ doesn't meet the interior of $\mathbf{C}_2$.

Region 2: $(w_1, w_2) \in \mathcal{C} \setminus \{0\} \times \overline{\mathcal{D}}$, $w_2 \in \mathit{Red}_2 \setminus \mathit{Green}_2$ and $w_1 \notin \mathit{Red}_1$.

As before, if w_1 is on the circle of radius r_1 and w_2 on the circle of radius r_2, the point $w_1 w_2$ is on the circle of radius $r_1 r_2$ and will be closest to 1 when the arguments of w_1 and w_2 are maximal. It follows that the infimum is reached when w_1 and w_2 are in the following arcs of circle (see Figure 3)

(a) w_1 is in the arc $\in \mathbf{C}_a$ of the circle $\mathbf{C}(i,1)$ from the point $z_0 = 0$ to the point $z_1 = z_1 = i + e^{i\pi/3} = e^{i\pi/6}$ (positive orientation).
(b) w_2 is in the arc $\mathbf{C}_b$ of the circle $\mathbf{C}(\frac{1-i}{2}, \frac{1}{\sqrt{2}})$ from the point $z_0 = 0$ to the point $z_2 = -i$ (positive orientation).

The points three points $z_0 = 0$, $z_2 = -i$ et $z_3 = 1-i$ are on the circle $\mathbf{C}_1 = \mathbf{C}(\frac{1-i}{2}, \frac{1}{\sqrt{2}})$. The products $z_1 z_0$ and $z_1 z_2 = e^{-i\pi/3}$ are on the circle $\mathbf{C}_2 = \mathbf{C}(1,1)$ and the product $z_1 z_3 = \sqrt{2}e^{-i\pi/12}$ is in the interior of $\mathbf{C}_2$. Therefore, by Lemma 30, the arc $z_1\mathbf{C}_b$ doesn't meet the interior of $\mathbf{C}_2$ which means that

$$|1 - z_1 w_2| \geq 1$$

for all $w_2 \in \mathbf{C}_b$.

Next fix $w_2 \in \mathbf{C}_b$. The three points $z_1' = w_2 z_0$, $z_2' = w_2 z_1$ and $z_3' = w_2(1+i)$ are on the circle $\mathbf{C}_3 = w_2\mathbf{C}(i,1)$. The point z_1' is not in the interior of the circle $\mathbf{C}_2$ and we just proved that z_2' is not in the interior of the circle $\mathbf{C}_2$ either, while z_3' is on the circle $(1+i)\mathbf{C}(\frac{1-i}{2}, \frac{1}{\sqrt{2}}) = \mathbf{C}_2$. Therefore, by Lemma 30, the arc $w_2\mathbf{C}_a$ doesn't meet the interior of the circle $\mathbf{C}_2$ which means that

$$|1 - w_2 w_1| \geq 1$$

for all $w_1 \in \mathbf{C}_a$.

Region 3: $(w_1, w_2) \in \mathcal{C} \setminus \{0\} \times \overline{\mathcal{D}}$, $w_2 \notin \mathit{Red}_2 \cup \mathit{Blue}_2$.

If $w_2 \notin \mathit{Red}_2 \cup \mathit{Blue}_2$, then either $\arg w_2 \leq \pi/2$ or

$$\pi/2 \leq \arg w_2 \leq 7\pi/6.$$

In the latter case, since $w_1 \in \mathcal{C}$, we have

$$\begin{aligned} w_1 w_2 \in &\{z : \pi/2 \leq \arg z \leq 7\pi/6 + \pi/4\} \\ &\subset \{z : \Re z \leq 0\}. \end{aligned}$$

Therefore, $|1 - w_1 w_2| \geq 1$. In the former case, $\arg w_1 w_2 \geq \arg((1 - \sqrt{3}/2) + i/2) = \pi/2 - \pi/12$. Therefore, $|1 - w_1 w_2| \geq \cos(\pi/12) > r = \sqrt{6 - 3\sqrt{3}}$.

Region 4: $(w_1, w_2) \in \mathcal{C} \setminus \{0\} \times \overline{\mathcal{D}}$, $w_2 \in \mathit{Blue}_1$ and $w_1 \notin \mathit{Blue}_2$.

Again, if w_1 is on the circle of radius r_1 and w_2 on the circle of radius r_2, the point w_1w_2 is on the circle of radius r_1r_2 and will be closest to 1 when the arguments of w_1 and w_2 are minimal. It follows that the infimum is reached when w_1 and w_2 are in the following arcs of circle (see Figure 3)

(a) w_1 in the arc $\mathbf{C}_a$ of the circle $\mathbf{C}(\frac{1-i}{2}, \frac{1}{\sqrt{2}})$ from the point $z_1 = 1$ to the point $z_0 = 0$.
(b) w_2 in the arc $\mathbf{C}_b$ of the circle $\mathbf{C}(1,1)$ from the point $z_2 = 1/2 + i\sqrt{3}/2$ to the point $z_3 = (1-\sqrt{3}/2) + i/2$.

Fix w_2 in $\mathbf{C}_b$. The extremities of $w_2\mathbf{C}_a$ are 0 and w_2 and they are not inside the circle $\mathbf{C}(1,1)$. While the point $w_2(-i) \in w_2\mathbf{C}(\frac{1-i}{2}, \frac{1}{\sqrt{2}})$ is inside the circle $\mathbf{C}(1,1)$. Therefore, by Lemma 30, the arc $w_2\mathbf{C}_a$ is outside the circle $\mathbf{C}(1,1)$ which means that $|1 - w_1w_2| \geq 1$ for all $w_1 \in \mathbf{C}_a$

10.1.3. *We show that* $C_2 = \frac{1}{\sqrt{6-3\sqrt{3}}}$. We suppose that u and v are of index 2. Thanks to Proposition 11, we can suppose $w_1 \in \mathcal{C}$ and thanks to Corollary 19, we know that, zero is the only vector of $\langle u, v\rangle_J$ in $\overset{o}{C}(u,v)$, iff $(w_1, w_2) \in (\mathcal{C} \setminus \mathbb{D}(-i, \sqrt{2})) \times \overline{\mathcal{T}}$. We want to show that $\inf |1 - w_1w_2| = 3 - \sqrt{3}$.

Once again, if w_1 is on the circle of radius r_1 and w_2 on the circle of radius r_2, the point w_1w_2 is on the circle of radius r_1r_2 and will be closest to 1 when the arguments of w_1 and w_2 are minimal. It follows that the infimum is reached when w_1 and w_2 are in the following arcs of circle (see Figures 5 and 6)

(a) w_1 is in the arc $\mathbf{C}_a$ of the circle $\mathbf{C}(-i, \sqrt{2})$ with extremities $z_0 = 1$ and $z_1 = \frac{\sqrt{3}-1}{2}(1+i)$,
(b) w_2 is in the arc $\mathbf{C}_b$ of the circle $\mathbf{C}(1, \sqrt{2})$ with extremities and $z_2 = i$ and $z_3 = \frac{\sqrt{3}-1}{2}(-1+i)$.

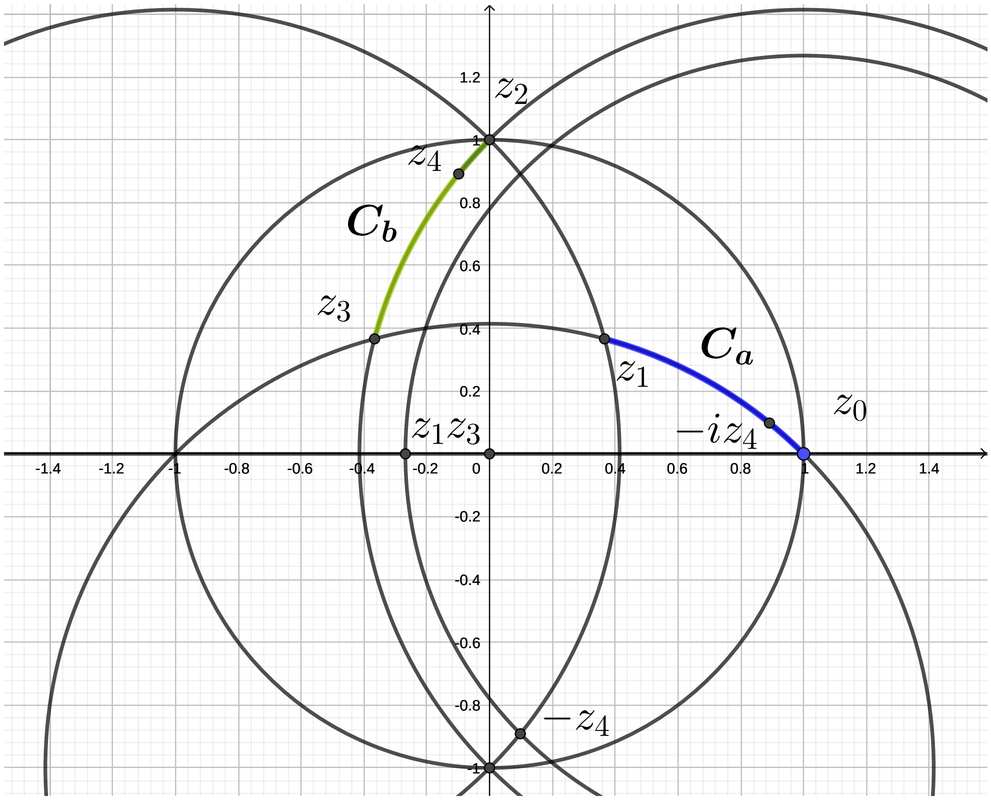

FIGURE 6. The infimum of $|1 - w_1w_2|$ on T_2.

A short computation shows that $\rho = |1 - z_1z_3| = 3 - \sqrt{3} = \sqrt{2}\sqrt{6 - 3\sqrt{3}}$.

Our objective is to show that if $w_1 \in \mathbf{C}_a$ and $w_2 \in \mathbf{C}_b$ then

$$|1 - w_1w_2| \geq 3 - \sqrt{3}.$$

It will implies that $C_2 = \frac{\sqrt{2}}{3-\sqrt{3}}$.

When $w_1 = z_1$, the points w_1z_2 and w_1z_3 are not inside the circle $\mathbf{C} = \mathbf{C}(1, \rho)$ and the point $w_1(-i)$ is inside $\mathbf{C}$. Therefore, by Lemma 30, the arc $w_1\mathbf{C}_b$ doesn't meet the interior of $\mathbf{C}$.

Fix $w_2 \in \mathbf{C}_b$. The points $w_2z_0 = w_2$ and w_2z_1 are not inside the circle $\mathbf{C}$. If the point $w_2(-1)$ were inside $\mathbf{C}$, by Lemma 30, $w_2\mathbf{C}_a$ would be outside $\mathbf{C}$ which means that

$$|1 - w_1w_2| \geq 3 - \sqrt{3}.$$

However, $w_2(-1)$ could be outside the circle $\mathbf{C}$. Let us determine the points $w_2 = x+iy \in \mathbf{C}_b$ such that $w_2(-1)$ is inside the circle $\mathbf{C}$: $w_2(-1)$ is inside $\mathbf{C}$ iff

$$\begin{aligned}
&|1 - w_2(-1)|^2 < \rho^2 \\
\Leftrightarrow\ &(1 + x)^2 + y^2 < (3 - \sqrt{3})^2 \\
\Leftrightarrow\ &x^2 + y^2 + 2x + 1 < 12 - 6\sqrt{3}.
\end{aligned}$$

Since $w_2 \in \mathbf{C}(1, \sqrt{2})$, $x^2 + y^2 = 1 + 2x$. Thus, $w_2(-1)$ is inside $\mathbf{C}$ iff $x < \frac{5-3\sqrt{3}}{2}$. So by Lemma 30, if $x \leq \frac{5-3\sqrt{3}}{2}$ then $|1 - w_1w_2| \geq 3 - \sqrt{3}$.

Call $x_0 = \frac{5-3\sqrt{3}}{2}$, $y_0 = \sqrt{2 - (x_0 - 1)^2}$ and $z_4 = x_0+iy_0$. Let $\mathbf{C}'_b$ and $\mathbf{C}''_b$ be the portions of the arc $\mathbf{C}_b$ from $z_2 = i$ to z_4 and from z_4 to z_3. Let $\mathbf{C}'_a$ and $\mathbf{C}''_a$ be the portions of the arc $\mathbf{C}_a$ from $z_0 = 1$ to $(-i)z_4$ and from $(-i)z_4$ to $z_1 = (-i)z_3$.

We already know that if $w_1 \in \mathbf{C}_a$ and $w_2 \in \mathbf{C}_b''$ then $|1-w_1w_2| \geq 3-\sqrt{3}$. Since $w_2 \in \mathbf{C}_b$ and $w_1 \in \mathbf{C}_a''$ imply $(-i)w_2 \in \mathbf{C}_a$ and $iw_1 \in \mathbf{C}_b''$, it follows that if $w_2 \in \mathbf{C}_b$ and $w_1 \in \mathbf{C}_b''$ then

$$|1-w_1w_2| = |1-iw_1(-1)w_2| \geq 3-\sqrt{3}.$$

So we are left with the case $w_1 \in \mathbf{C}_a'$ and $w_2 \in \mathbf{C}_b'$. Since $|1-iw_1| = \sqrt{2}$,

$$\begin{aligned}|1-w_1w_2|^2 &= |1-w_1(w_2-i+i)|^2 = |(1-iw_1)-w_1(w_2-i)|^2\\ &= 2+|w_1(w_2-i)|^2 - 2\Re((1-iw_1)\overline{w_1(w_2-i)}).\end{aligned}$$

Furthermore, when $w_1 \in \mathbf{C}_a'$ and $w_2 \in \mathbf{C}_b'$,

$$7\pi/4 \leq \arg(1-iw_1) \leq \arg(1-z_4)$$
$$2\pi - \arg(-iz_4) \leq \arg \overline{w}_1 \leq 2\pi$$
$$2\pi - \arg(z_4-i) \leq \arg(\overline{w_2-i}) \leq 2\pi - 5\pi/4,$$

and $\arg(z_4-i) = 5\pi/4+\varepsilon_1$, $\arg(-iz_4) = \varepsilon_2$ and $\arg(1-z_4) = 7\pi/4+\varepsilon_3$ where $\varepsilon_1, \varepsilon_2$ and ε_3 are positive and small. It follows that, modulo 2π, we have

$$\begin{aligned}7\pi/4 - \arg(-iz_4) - \arg(z_4-i) \leq \arg((1-iw_1)\overline{w_1(w_2-i)}) &\leq \arg(1-z_4) - 5\pi/4\\ \pi/2-\varepsilon_1-\varepsilon_2 \leq \arg((1-iw_1)\overline{w_1(w_2-i)}) &\leq \pi/2+\varepsilon_3.\end{aligned}$$

Now $\varepsilon_1 = \arg(z_4-i) - 5\pi/4 = 0.0.0518\cdots \leq 0.06$, and $\varepsilon_2 = \arg(-iz_4) = 0.109\cdots \leq 0.11$ therefore,

$$\begin{aligned}2\Re((1-iw_1)\overline{w_1(w_2-i)}) &\leq 2\sqrt{2}|w_2-i|\cos(\pi/2-\varepsilon_1-\varepsilon_2)\\ &\leq 2\sqrt{2}\times 0.15\times 0.17\\ &\leq 0.08.\end{aligned}$$

Finally, we obtain

$$|1-w_1w_2| \geq \sqrt{2-0.08} \geq 1.38 \geq 3-\sqrt{3}$$

and we are done.

10.2. **Step 2, $C'(\theta) = C_S$ for almost all $\theta \in \mathbb{C}$.** Consider the lattice $\Lambda_0 = \mathbb{Z}[i]u + \mathbb{Z}[i]v$ defined by the vectors $u = (u_1, u_2) = r_0(1, e^{i\alpha}w_2)$ and $v = (v_1, v_1) = r_0(w_1, e^{i\alpha})$ where

$$\begin{aligned}w_1 &= \tfrac{1}{2} + (1-\tfrac{\sqrt{3}}{2})i,\\ w_2 &= (-i)w_1,\\ r_0 &= \frac{1}{\sqrt{|1-w_1w_2|}},\\ \alpha &= -\arg(1-w_1w_2).\end{aligned}$$

The lattice Λ_0 is unimodular and by Theorem 2 (more precisely by Corollary 19), there is no nonzero vector of Λ_0 in the open cylinder $\overset{o}{C}(u,v) = \overset{o}{B}_\infty(0,r_0)$. The vectors u and v has been chosen so that

$$|u_1v_2| = r_0^2 = C_S = \frac{\sqrt{2}}{3-\sqrt{3}}.$$

Lemma 31. *Let Λ be a lattice in $\mathbb{C}^2$ and let r be a positive real number. Suppose that for some real number t, $g_t\Lambda \cap B_\infty(0,r) = \{0\}$. Then there exist two consecutive minimal vectors $u = (u_1, u_2)$ and $v = (u_2, v_2)$ in Λ with $|v_2| > |u_2|$ such that $|u_1v_2| \geq r^2$ and $|v_2| \geq re^t$.*

Proof. Let $u = (u_1, u_2)$ be a minimal vector in $g_t\Lambda$ with $|u_1| \geq r$ and $|u_1|$ minimal. Such a minimal vector exists because by Lemma 6, there exist minimal vectors in $\overset{o}{C}_2(r)$ and such minimal vectors have a first coordinate with modulus $\geq r$ because $g_t\Lambda \cap B_\infty(0, r) = \{0\}$. Let $v = (v_1, v_2)$ be a minimal element for the lexicographic preorder $\prec$ in $g_t\Lambda \cap \overset{o}{C}_1(r)$. By Lemma 6, v is a minimal vector and again $|v_2| > r$ because $g_t\Lambda \cap B_\infty(0, r) = \{0\}$. If $w = (w_1, w_2)$ is a minimal vector in $g_t\Lambda$ with $|w_2| > |u_2|$ then $|w_1| < |u_1|$. By definition of u, this implies $|w_1| < r$ which implies successively that $w \in \overset{o}{C}_1(r)$ then that $v \prec w$ and finally that $|w_2| \geq |v_2|$. It follows that u and v are consecutive minimal vectors in $g_t\Lambda$ and that $B_\infty(0, r) \subset C(u, v)$. It follows that $g_{-t}u$ and $g_{-t}v$ are consecutive minimal vectors in Λ and since $|v_2| > r$, we have $|e^t v_2| > re^t$. Moreover $|e^{-t}u_1 e^t v_2| = |u_1||v_2| \geq r \times r$. □

Lemma 32. *Let r be a positive real number. The set F of unimodular lattices Λ in $\mathbb{C}^2$ such that $\Lambda \cap B_\infty(0, r) \neq \{0\}$, is closed in* $\mathrm{SL}(2, \mathbb{C})/\mathrm{SL}(2, \mathbb{Z}[i])$.

Proof. Let $(\Lambda_n = M_n\mathbb{Z}[i]^2)_n$ be a sequence of lattices in F. Suppose that the sequence converges to a lattice $\Lambda = M\mathbb{Z}[i]^2$. We want to show that $\Lambda \in F$. We can suppose that the sequence of matrices $(M_n)_n$ converges to M w.l.o.g.. For each n, there exists a nonzero vector $X_n \in \mathbb{Z}[i]^2$ such that $Y_n = M_nX_n \in B_\infty(0, r)$. Changing X_n into 2^kX_n for some non-negative integer k, we can suppose that $r/2 \leq |Y_n|_\infty \leq r$. Since the matrices M_n are all invertible and since the sequence (M_n) is convergent, there exists $\delta > 0$ such that $\|M_n\| \geq \delta > 0$ for all n, where $\|A\|$ is the operator norm of the matrix A associated with the sup norm on $\mathbb{C}^2$. Therefore, $|X_n|_\infty \leq \frac{r}{\delta}$ for n large enough. Thus there exist a vector $X \in \mathbb{Z}[i]^2$ and an increasing sequence of integers n_k such that $X_{n_k} = X$ for all k. Since $|M_{n_k}X_{n_k}|_\infty = |Y_{n_k}|_\infty \geq r/2$, $MX = \lim_{k\to\infty} M_{n_k}X = \lim_{k\to\infty} Y_{n_k}$ is a nonzero vector of $\Lambda = M\mathbb{Z}[i]^2$ in the ball $B_\infty(0, r)$, which means that $\Lambda \in F$. □

End of proof of Theorem 5 and Theorem 5 bis. By Lemma 31, to prove that $C'(\theta) \geq r_0$ for almost all θ, it suffices to prove that the set

$$\{\theta \in [0, 1] + i[0, 1] : \forall T \geq 0, \forall\varepsilon > 0, \exists t \geq T, g_t\Lambda_\theta \cap B_\infty(0, r_0 - \varepsilon) = \{0\}\}$$

has full Lebesgue measure in $[0, 1] + i[0, 1]$. Suppose on the contrary that the set

$$\{\theta \in [0, 1] + i[0, 1] : \exists T \geq 0, \exists\varepsilon > 0, \forall t \geq T, g_t\Lambda_\theta \cap B_\infty(0, r_0 - \varepsilon) \neq \{0\}\}$$

has positive Lebesgue measure. Then there exist $T \geq 0$ and $\varepsilon > 0$ such that the set

$$N = \{\Lambda_\theta : \theta \in [0, 1] + i[0, 1] \text{ and } \forall t \geq T, g_t\Lambda_\theta \cap B_\infty(0, r_0 - \varepsilon) \neq \{0\}\}$$

has positive measure. By definition of N, for all $\Lambda_\theta \in N$ and all $t \geq T$, there exists a nonzero vector $X(\theta, t) \in \mathbb{Z}[i]^2$ such that

$$Y(\theta, t) = g_tM_\theta X(\theta, t) \in B_\infty(0, r_0 - \varepsilon).$$

Let $\mathcal{H}_\leq$ be the subgroup of $\mathrm{SL}(d + 1, \mathbb{C})$ defined by

$$\mathcal{H}_\leq = \left\{h = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) : a \in \mathbb{C}^*,\ b \in \mathbb{C}\right\}.$$

There exists $\delta > 0$ such that for all $A \in \mathcal{H}_\leq$

$$|A - Id|_\infty \leq \delta \Rightarrow \forall t \geq 0, \|g_tAg_t^{-1} - Id\| \leq \frac{\varepsilon}{2r_0}$$

where $|M|_\infty$ is the sup norm of the matrix M and $\|M\|$ is its operator norm associated with the sup norm. For all $\Lambda_\theta \in N$, all $t \geq T$ and all $A \in \mathcal{H}_\leq$ with $|A - Id|_\infty \leq \delta$, we have

$$\begin{aligned}|g_tAM_\theta X(\theta,t)|_\infty &= |g_tAg_t^{-1}g_tM_\theta X(\theta,t)|_\infty\\ &= |(g_tAg_t^{-1}-Id)g_tM_\theta X(\theta,t)+g_tM_\theta X(\theta,t)|_\infty\\ &\le |(g_tAg_t^{-1}-Id)g_tM_\theta X(\theta,t)|_\infty+|g_tM_\theta X(\theta,t)|_\infty\\ &\le (\|g_tAg_t^{-1}-Id\|+1)|g_tM_\theta X(\theta,t)|_\infty\\ &\le (\frac{\varepsilon}{2r_0}+1)(r_0-\varepsilon)\\ &\le r_0-\frac{\varepsilon}{2}.\end{aligned}$$

Therefore,

$$g_t\Lambda\cap B_\infty(0,r_0-\tfrac{\varepsilon}{2})\neq\{0\}$$

for all $\Lambda\in B_{\mathcal{H}_\le}(Id,\delta)N$ where $B_{\mathcal{H}_\le}(Id,\delta)$ is the set of matrices M in the subgroup $\mathcal{H}_\le$ such that $|M-Id|_\infty\le\delta$ and $B_{\mathcal{H}_\le}(Id,\delta)N$ is the set of lattices of the shape $A\Lambda$ with $A\in B_{\mathcal{H}_\le}(Id,\delta)$ and $\Lambda\in N$.

Let U be the set of unimodular lattices Λ such that $\Lambda\cap B_\infty(0,r_0-\frac{\varepsilon}{2})=\{0\}$. By the choice of r_0, the lattice Λ_0 is in U. By Lemma 32, U is open. So that U is a nonempty open set and has a positive Haar measure in $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i])$. The action of the flow g_t, $t\in\mathbb{R}$, on $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i])$ is ergodic, see [1] page 90 (it is also a direct consequence of Mautner's lemma and of the fact that $\mathrm{SL}(2,\mathbb{C})$ is generated by the matrices of the shape $\begin{pmatrix}1&*\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\ *&1\end{pmatrix}$). It follows, by Birkhoff ergodic theorem applied to the flow g_t and to the function $f=1_U$, that for almost all lattices Λ, there exist arbitrarily large t such that $g_t\Lambda\in U$. Now, the set $B_{\mathcal{H}}(Id,\delta)N$ has positive Haar measure and by construction for all lattice Λ in this set and all $t\ge T$

$$g_t\Lambda\cap B(r_0-\tfrac{\varepsilon}{2})\neq\{0\},$$

and therefore $g_t\Lambda\notin U$ for all $t\ge T$, a contradiction. □

11. Search of minimal vectors in a Gauss lattice in $\mathbb{C}^2$

In this subsection we address the problem of finding two consecutive minimal vectors in a Gauss lattice in $\mathbb{C}^2$. The first step is to find one minimal vector and the second step the next one.

Thanks to the Gauss reduction algorithm it can be done very efficiently.

11.1. **The Gauss reduction algorithm.**

Given Gauss lattice in $\mathbb{C}^2$ we want to find a minimal vector in this lattice. This can be done with a Gauss reduction algorithm and the following observation:

If Λ is Gauss lattice in $\mathbb{C}^2$ and if u is a shortest vector of Λ for the standard Hermitian norm then u is a minimal vector in Λ.

Indeed, if $u=(u_1,u_2)$ is a shortest vector for the standard Hermitian norm then any vector $v=(v_1,v_2)$ in the cylinder $C(u)$ such that $|v_1|<|u_1|$ or $|v_2|<|u_2|$ has a strictly smaller Hermitian norm.

Given a basis of a lattice in a two-dimensional (real) Euclidean vector space, the Gauss reduction algorithm provides a reduced basis of the lattice. This algorithm can be adapted to the case of Gauss lattices in two-dimensional $\mathbb{C}$-vector spaces equipped with an Hermitian norm. In [26] it is proved that the Gauss reduction algorithm works for lattices in $\mathbb{C}^2$ on Euclidean integer rings of imaginary quadratic fields. We state their result for lattices on $\mathbb{Z}[i]$ without proof.

Definition 33. Let E be a two-dimensional $\mathbb{C}$-vector space equipped with a norm $\|.\|$. A basis (u, v) of a Gauss lattice $\Lambda = \mathbb{Z}[i]u + \mathbb{Z}[i]v$ is reduced with respect to the norm $\|.\|$ if $\|u\| = \lambda_1(\Lambda, \|.\|, \mathbb{C})$ and $\|v\| = \lambda_2(\Lambda, \|.\|, \mathbb{C})$.

Let E be a two-dimensional $\mathbb{C}$-vector space equipped with an Hermitian norm $|.|_E$.
The Gauss reduction algorithm proceed as follows.

Input: A basis (u, v) of a Gauss lattice Λ in E.

(1) If $|v|_E < |u|_E$, exchange $u \leftrightarrow v$.
(2) $A :=$ False
(3) Main loop: while $A =$ False
   (a) Compute $w = (a + ib)u$ the orthogonal projection of v on the line $\mathbb{C}u$.
   (b) Find the Gaussian integer p closest to $a + ib$ and replace v with $v - pu$.
   (c) If $|u|_E \le |v|_E$, $A :=$ True, else exchange $u \leftrightarrow v$.

Output A reduced basis of Λ.

Proposition 34. *The above algorithm find a reduced basis of $\Lambda = \mathbb{Z}[i]u + \mathbb{Z}[i]v$ for the norm $|.|_E$ in finitely many steps.*

11.2. **An algorithm finding consecutive minimal vectors.** Let Λ be a Gauss lattice in $\mathbb{C}^2$ and let u be a minimal vector in Λ. How to find a minimal vector v in Λ such that u and v are consecutive ?

This problem can be solved with the following proposition.

Notation. For a positive real number t denote $|.|_t$ the Hermitian norm on $\mathbb{C}^2$ defined by

$$|(z_1, z_2)|_t^2 = |tz_1|^2 + |\tfrac{1}{t} z_2|^2.$$

Proposition 35. *Let Λ be a Gauss lattice in $\mathbb{C}^2$ and let $u = (u_1, u_2)$ be a minimal vector in Λ. Set*

$$s = \sqrt{\frac{4}{\pi} |\det_{\mathbb{C}}(\Lambda)|} \quad \text{and } t = \frac{s}{|u_1|}.$$

Let (w, w') be a reduced basis of Λ with respect to the norm $|.|_t$. Then the minimal vector v such that u and v are consecutive minimal vectors, belong to the set of vectors $zw + z'w'$ with $z, z' \in \mathbb{Z}[i]$ and $(|z|^2 + |z'^2|) < 23$.

Proof. Let $v = (v_1, v_2)$ be a minimal vector in Λ such that u and v are consecutive minimal vectors, let $s = \sqrt{a|\det_{\mathbb{C}}(\Lambda)|}$ where a is a positive constant and let $t = \frac{s}{|u_1|}$. We will make the choice $a = \frac{4}{\pi}$ only at the end of the proof. It is enough to prove that a can be chosen so that for all $z, z' \in \mathbb{C}$, $zw + z'w' \in C(u, v)$ implies $|z|^2 + |z'|^2 < 23$.

The sup norm defined by

$$\|(z_1, z_2)\|_t = \max(|tz_1|, |\tfrac{1}{t} z_2|)$$

is bounded below by $\frac{1}{\sqrt{2}}|(z_1,z_2)|_t$. By Lemma 7, $\frac{1}{2}|\det_{\mathbb{C}}(\Lambda)| \le |u_1||v_2| \le \frac{4}{\pi}|\det_{\mathbb{C}}(\Lambda)| = C|\det_{\mathbb{C}}(\Lambda)|$. Since,

$$\begin{aligned}\lambda_2(\Lambda, |.|_t, \mathbb{C}) &\le \sqrt{2}\,\lambda_2(\Lambda, \|.\|_t, \mathbb{C})\\ &\le \sqrt{2}\max(\|u\|_t, \|v\|_t)\\ &= \sqrt{2}\max(s, \frac{1}{s}|u_1||u_2|, s\tfrac{|v_1|}{|u_1|}, \frac{1}{s}|u_1||v_2|)\\ &= \sqrt{2}\max(s, \frac{1}{s}|u_1||v_2|),\end{aligned}$$

with $s = \sqrt{a|\det_{\mathbb{C}}(\Lambda)|}$, we obtain

$$\begin{aligned}|w'|_t = \lambda_2(\Lambda, |.|_t, \mathbb{C}) &\le \sqrt{2}\max(s, \frac{1}{s}|u_1||v_2|)\\ &\le \sqrt{2}\max(s, \frac{C}{s}|\det_{\mathbb{C}}(\Lambda)|)\\ &= \sqrt{2}\max(1, \frac{C}{a})s = bs.\end{aligned}$$

Now by Hadamard inequality,

$$|\det_{\mathbb{C}}(\Lambda)| = |\det(w,w')| = |\det_{|.|_t}(w,w')| \le |w|_t|w'|_t$$

where $\det_{|.|_t}(w,w')$ is the determinant computed in a $|.|_t$-orthonormal basis, hence

$$|w|_t \ge \frac{|\det_{\mathbb{C}}(\Lambda)|}{|w'|_t} \ge \frac{s}{ab}.$$

Again, since $|u_1||v_2| \le C|\det_{\mathbb{C}}(\Lambda)|$ and $s = \sqrt{a|\det_{\mathbb{C}}(\Lambda)|}$, the cylinder $C(|u_1|,|v_2|)$ is included in the closed ball of radius $\max(1, \frac{C}{a})s$ associated with this sup norm $\|.\|_t$. Therefore, it is enough to find a constant A such that $|z|^2+|z'^2| > A$ implies $|zw+zw'|_t > \sqrt{2}\max(1, \frac{C}{a})s = bs$. Now, since the basis (w,w') is reduced, $|w \pm w'|_t^2 \ge |w'|_t^2$, which implies $|\Re\langle w,w'\rangle_t| \le \frac{1}{2}|w|_t^2$, and $|w \pm iw'|_t^2 \ge |w'|_t^2$ implies $|\Im\langle w,w'\rangle_t| \le \frac{1}{2}|w|_t^2$ as well. Hence $|\langle w,w'\rangle_t| \le \frac{1}{\sqrt{2}}|w|_t^2$, and

$$\begin{aligned}|zw+z'w'|_t^2 &\ge |z|^2|w|_t^2 + |z'|^2|w'|_t^2 - \sqrt{2}|z||z'||w|_t^2\\ &\ge (1-\tfrac{1}{\sqrt{2}})(|z|^2+|z'|^2)|w|_t^2\\ &\ge (\tfrac{\sqrt{2}-1}{\sqrt{2}})(|z|^2+|z'|^2)(\tfrac{s}{ab})^2.\end{aligned}$$

Therefore, if $|z|^2+|z'^2| > A$, then $|zw+z'w'|_t^2 \ge (\frac{\sqrt{2}-1}{\sqrt{2}})(\frac{1}{ab})^2 s^2 A$. So we are done if $A \ge \frac{\sqrt{2}}{\sqrt{2}-1}a^2b^4$. The value $a = C$ minimizes a^2b^4 and gives (recall that $C = \frac{4}{\pi}$),

$$\tfrac{\sqrt{2}}{\sqrt{2}-1}a^2b^4 = \tfrac{4\sqrt{2}C^2}{\sqrt{2}-1} = \tfrac{64\sqrt{2}}{(\sqrt{2}-1)\pi^2} = 22.139...$$

$\square$

Thanks to the proposition, the algorithm that finds two consecutive minimal vectors in a Gauss lattice $\Lambda \subset \mathbb{C}^2$ goes as follows:

- Use the Gauss reduction algorithm to find a reduced base (u,u') in Λ with respect to the standard Hermitian norm. u is the first minimal vector of the pair.
- Once again, use the Gauss reduction algorithm to find a reduced base (w,w') with respect to the Hermitian norm $|.|_t$ where t is the parameter associated with u defined in the proposition.

- Find a minimal element for the lexicographic preorder among the vectors $zw+z'w'$ with $z, z' \in \mathbb{Z}[i]$ and $(|z|^2+|z'^2|) < 23$ that are in the infinite cylinder $C_1(u)$.

12. Miscellaneous questions and comments

12.1. **The Hurwitz algorithm in the space of bases of $\mathbb{C}^2$.** Consider a basis $u = (u_1, u_2)$, $v = (v_1, v_2)$ of the vector space $\mathbb{C}^2$. We define a map H that associates a new basis $u' = (u'_1, u'_2)$, $v' = (v'_1, v'_2)$ to each basis u, v. This map is defined only when $u_1 \neq 0$ and $v_1 \neq 0$. The first vector u' is defined by $u' = v$ and v' is defined as follows. Set $w_1 = \frac{v_1}{u_1}$. We define $v' = (v'_1, v'_2) = u - gv$ where g is the Gaussian integer such that

$$\frac{v'_1}{v_1} = \frac{u_1 - gw_1u_1}{w_1u_1} = \frac{1}{w_1} - a \in S = [-\tfrac{1}{2}, \tfrac{1}{2}) + [-\tfrac{1}{2}, \tfrac{1}{2})i.$$

For the new basis u', v', we have

$$\frac{v'_1}{u'_1} = \frac{v'_1}{v_1} = \frac{1}{w_1} - g = w'_1 \in S$$

and therefore $v'_1 = w'_1u'_1$ with $w'_1 \in S$. We recognize the Hurwitz continued fraction algorithm applied to w_1. The map H is defined on the set of pairs of independent vectors (u, v) such that the first coordinates of u and v are nonzero. Observe that $\mathbb{Z}[i]u' + \mathbb{Z}[i]v' = \mathbb{Z}[i]u + \mathbb{Z}[i]v$ and that $\det_{\mathbb{C}} H(u, v) = -\det_{\mathbb{C}}(u, v)$.

Remark 8. In Theorem 3, the map T_G was defined by a good choice of a Gaussian integer g and of $a \in \{1, 1+i\}$ such that $|\frac{a}{w_1} - g| < 1$, while in the Hurwitz algorithm there is a unique Gaussian integer g such $\frac{1}{w_1} - g = w'_1 \in S$.

There are two simple questions:

- If (u, v) is a pair of consecutive minimal vectors in a Gauss lattice $\Lambda \subset \mathbb{C}^2$ and $(u', v') = H(u, v)$ is defined, is it true that v' is a minimal vector in Λ ?
- Is it possible to continue the process : if v' is still a minimal vector and $H(u', v') = (u'', v'')$, is v'' minimal ?

12.2. **Ergodic theory and the first return map.** Let θ be in $\mathbb{C}$ and let $X_n(\Lambda) = (x_n(\theta), y_n(\theta))$, $n \in \mathbb{N}$, be the sequence of minimal vectors of the lattice Λ_θ. We can ask several questions about the quantities $x_n(\theta)$ and $y_n(\theta)$.

- (Levy-Khintchin theorem) Show that for almost all $\theta \in \mathbb{C}$
$$\lim_{n\to\infty} \frac{1}{n} \ln |y_n(\theta)| = C$$
where C is a constant that can be computed with the Haar measure of $\mathrm{SL}(2, \mathbb{C})/\mathrm{SL}(2, \mathbb{Z}[i])$ and the induced measure ν (see Theorem 10). What can be said about the sequence $(\frac{y_n(\theta)}{|y_n(\theta)|})_n$? Is this sequence almost surely equidistributed in the unit circle?
- (Bosma-Jager-Wiedijk theorem) Show that for almost all $\theta \in \mathbb{C}$, the sequence of probabilities
$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{x_n(\theta)y_n(\theta)}$$
converges in measure to a probability λ in $\mathbb{C}$. Show that λ has a density with respect to the Lebesgue measure and compute this density. The question can be studied with the product $y_{n+1}(\theta)x_n(\theta)$ instead of $x_n(\theta)y_n(\theta)$.

For these two questions the method in [5] should lead to the existence of the limit almost everywhere. The explicit computations of the limits C and λ could be more difficult.

As we have seen the core of the first return map on the transversal T is the map T_G (see subsection 8.4). In the definition of the map T_G two coefficients appear: $a \in \{1, 1+i\}$ and $g \in \mathbb{Z}[i]$. By Proposition 20, if $a = 1+i$ for some iterate of T_G, the next iterate should be with $a = 1$.

- More generally, find the succession laws for the coefficients.
- What is the almost-sure frequency of $a = 1$ when computing the sequence of iterates of T_G?
- Is there a Borel-Bernstein theorem for the coefficient g?

13. Appendix 1, Gauss lattices

Definition 36. Let E be a finite dimensional $\mathbb{C}$-vector space. A subset Λ in E is a Gauss lattice if it is a $\mathbb{Z}[i]$-submodule of E, if it is a discrete subset of E and if it generates the vector space E.

Lemma 37. *Let E be a $\mathbb{C}$-vector space of dimension n and let Λ be a Gauss lattice in E. Then there exists a basis $u_1, \dots, u_n$ of E such that*

$$\Lambda = \oplus_{j=1}^{n} \mathbb{Z}[i] u_j.$$

Proof. Denote $\|.\|$ a Hermitian norm in E (an Hermitian structure is used only for convenience). We proceed by induction. If $n = 1$, $E = \mathbb{C}u$ and $\Lambda = \mathbb{Z}[i]\lambda u$ where λu is a shortest vector in Λ. Indeed for all $zu \in \Lambda$, there exits $p \in \mathbb{Z}[i]$ such that $|\frac{z}{\lambda} - p| < 1$, hence $\|zu - p\lambda u\| = |z - p\lambda| \|u\| < |\lambda| \|u\|$ and therefore $z = p\lambda$.

Suppose the result holds for all $n-1$-dimensional vector spaces. Let E be a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} E = n$ and let Λ be a Gauss lattice in E. Since Λ generates the vector space E, there is a basis $u_1, \dots, u_n$ of E with $u_1, \dots, u_n \in \Lambda$. Let F be the vector space spanned by $u_1, \dots, u_{n-1}$. By induction hypothesis there exists a basis $v_1, \dots, v_{n-1}$ of F such that $F \cap \Lambda = \oplus_{j=1}^{n-1} \mathbb{Z}[i] v_j$. The orthogonal projection Λ' of Λ on the line D orthogonal to F is discrete. Indeed, suppose there is a sequence $w_n \in \Lambda'$ of nonzero vectors which converges to zero. We can suppose that the vectors w_n, $n \in \mathbb{N}$, are distinct. For each n, let $w'_n \in \Lambda$ be a vector whose projection is w_n. The vectors w'_n can be chosen in order that their orthogonal projections on F are in the bounded set $\{\sum_{j=1}^{n-1} z_j v_j \in F : (\Re z_j, \Im z_j) \in [0,1]^2\}$. It follows that the w_n are in a bounded set which is not possible because they are distinct. Therefore Λ' is discrete. Let v'_n be a shortest nonzero vector in Λ'. Since Λ' is a Gauss lattice, the step $n = 1$ of the induction implies that $\Lambda' = \mathbb{Z}[i] v'_n$. Finally choose any vector $v_n \in \Lambda$ whose projection on Λ' is v'_n. If $v \in \Lambda$ then its projection v' on the line D is in $\mathbb{Z}[i] v'_n$. It follows that $v' = g v'_n$ for some $g \in \mathbb{Z}[i]$. Therefore, the projection of $v - g v_n$ is 0 which implies that $v - g v_n \in F \cap \Lambda$. We conclude that $v_1, \dots, v_n$ generate Λ. □

A direct adaptation of Theorem I page 11 of Cassels' book, [4], An introduction to the geometry of numbers, shows that

Theorem 11. *Let E be a n-dimensional $\mathbb{C}$-vector space, let Λ be a Gauss lattice in E and let $L \subset \Lambda$ be a lattice in E.*

A. To every basis $b_1, \dots, b_n$ of Λ, there can be found a basis $a_1, \dots, a_n$ of L of the shape

$$\begin{cases} a_1 = z_{11}b_1 \\ a_2 = z_{21}b_1 + z_{22}b_2 \\ \quad \vdots \\ a_n = z_{n1}b_1 + \cdots + z_{nn}b_n \end{cases}$$

where the z_{ij} are in $\mathbb{Z}[i]$ and z_{ii} are nonzero for all i.
B. Conversely, to every basis $a_1, \dots, a_n$ of L, there exists a basis $b_1, \dots, b_n$ of Λ such the above system holds.

Proof of A. Pick one basis of Λ and one basis of L and call D the determinant of the second in the first. The determinant D is a Gaussian integer and Cramer formula shows that $D\Lambda \subset L$.

Let $(b_1, \dots, b_n)$ be a basis of Λ. For each $i \in \{1, \dots, n\}$ there exist points a_i in L of the shape

$$a_i = z_{i1}b_1 + \cdots + z_{ii}b_i$$

where the z_{ij} are Gaussian integers and $z_{ii} \neq 0$ for $Db_i \in L$. We choose for a_i such an element in L for which $|z_{ii}|$ is as small as possible but zero. We are going to show that $a_1, \dots, a_n$ is a basis of L. Since $a_1, \dots, a_n$ are in L, so is every vector $w = w_1a_1 + \cdots + w_na_n$ where $w_1, \dots, w_n$ are Gaussian integers. Suppose by contradiction that there exists a vector c of L not of the latter shape. Since c is Λ, $c = t_1b_1 + \cdots + t_kb_k$ where $1 \leq k \leq n$, $t_k \neq 0$ and $t_1, \dots, t_k$ are Gaussian integers. If there are several such c, then we choose one for which k is minimal. Now since $z_{kk} \neq 0$, we may choose a Gaussian integer s such that

$$|t_k - sz_{kk}| < |z_{kk}|.$$

The vector

$$c - sa_k = (t_1 - sz_{k1})b_1 + \cdots + (t_k - sz_{kk})b_k$$

is in L since a_k and c are; but it is not of the shape $w_1a_1 + \cdots + w_na_n$ since c is not. Hence $t_k - sz_{kk}$ cannot be zero by assumption that k was minimal. But then $|t_k - sz_{kk}| < |z_{kk}|$ contradicts the assumption that the nonzero modulus $|z_{kk}|$ was minimal. □

Proof of B. Let $a_1, \dots, a_n$ be some basis of L. Since $D\Lambda \subset L$, by part A, there exists a basis $Db_1, \dots, Db_n$ of $D\Lambda$ such that

$$\begin{cases} Db_1 = z_{11}a_1 \\ Db_2 = z_{21}a_1 + z_{22}a_2 \\ \quad \vdots \\ Db_n = z_{n1}a_1 + \cdots + z_{nn}b_n \end{cases}$$

with $z_{i,j}$ Gaussian integers and $z_{ii} \neq 0$. Solving the above system we can express $a_1, \dots, a_n$ in the basis $b_1, \dots, b_n$, we obtain a triangular system with coefficients in the fields $\mathbb{Q}(i)$. But $b_1, \dots, b_n$ is basis of Λ and the a_i are in Λ so the coefficients must be Gaussian integers. □

The norm $\|.\|$ we consider on $\mathbb{C}$-vector spaces are suppose to verify

$$\|\lambda u\| = |\lambda| \|u\|$$

for all vector u of the vector space and all complex number λ.

Definition 38. Let E be a finite dimensional $\mathbb{C}$-vector space equipped with a norm $\|.\|$ and let Λ be a discrete subset in E. For $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and for $1\leq j\leq \dim_{\mathbb{K}} E$, the j-th minimum of Λ with respect to the norm $\|.\|$ and to the field $\mathbb{K}$ is the infimum of all the real numbers λ such that there exist j $\mathbb{K}$-linearly independent vectors in Λ with norms $\leq \lambda$. It is denoted by $\lambda_j(\Lambda,\|.\|,\mathbb{K})$ or simply by $\lambda_j(\Lambda,\mathbb{K})$ or even by λ_j when there is no ambiguity.

Lemma 39. *Let E be a $\mathbb{C}$-vector space of dimension n equipped with a norm $\|.\|$ and let Λ a Gauss lattice in E. Then for $j=1,\dots,n$*

$$\lambda_j(\Lambda,\|.\|,\mathbb{C})=\lambda_{2j-1}(\Lambda,\|.\|,\mathbb{R})=\lambda_{2j}(\Lambda,\|.\|,\mathbb{R}).$$

Proof. If $u_1,\dots,u_j$ are j $\mathbb{C}$-linearly independent vectors with norms $\leq\lambda$ then $u_1,iu_1,\dots,u_j,iu_j$ are $2j$ $\mathbb{R}$-linearly independent with norms $\leq\lambda$, therefore $\lambda_j(\Lambda,\|.\|,\mathbb{C})\geq\lambda_{2j}(\Lambda,\|.\|,\mathbb{R})$. Since a $\mathbb{C}$-vector space of $\mathbb{C}$-dimension $\leq j-1$ has a real dimension $\leq 2j-2$, $2j-1$ $\mathbb{R}$-linearly independent vectors $u_1,\dots,u_{2j-1}$ in Λ generate a $\mathbb{C}$-vector space of dimension $>j-1$. Therefore, $\lambda_j(\Lambda,\|.\|,\mathbb{C})\leq\lambda_{2j-1}(\Lambda,\|.\|,\mathbb{R})$. □

14. Appendix 2, computing the distance to $\mathcal{D}$

There is a simple algorithm that calculate the distances from a complex number z to the regions $\mathcal{D}$, $\mathcal{C}$ and $\mathcal{T}$. We explain it for the distance to the region $\mathcal{D}=\{z\in\mathbb{C}:|z|<1,\ \mathrm{d}(z,1)>1,\ \mathrm{d}(z,1-i)>1\}$. The distances to $\mathcal{C}$ and to $\mathcal{T}$ can be calculated the same way. The complex plane is the union of seven regions $\mathcal{D}_0,\dots,\mathcal{D}_6$, see Figure 7. For each of these regions, there is a simple formula giving the distance $\mathrm{d}(z,\mathcal{D})$:

(1) If $z\in\mathcal{D}_0=\overline{\mathcal{D}}$ then $\mathrm{d}(z,\mathcal{D})=0$.
(2) If $z\in\mathcal{D}_1=\{z\in D(1,1):\arg(z-1)\in[\frac{2\pi}{3},\frac{7\pi}{6}]\}$ then $\mathrm{d}(z,\mathcal{D})=1-|z-1|$.
(3) If $z\in\mathcal{D}_2=\{z\in\mathbb{C}:\arg z\leq\frac{\pi}{3},\arg(z-1)\in[-\frac{\pi}{12},\frac{2\pi}{3}]\}$ then $\mathrm{d}(z,\mathcal{D})=\mathrm{d}(z,z_2)$ where $z_2=\frac{1}{2}+\frac{\sqrt{3}}{2}i$.
(4) If $z\in\mathcal{D}_3=\{z\in\mathbb{C}:|z|\geq 1,\arg z\in[\frac{\pi}{3},\frac{3\pi}{2}]\}$ then $\mathrm{d}(z,\mathcal{D})=|z|-1$.
(5) If $z\in\mathcal{D}_4=\{z\in\mathbb{C}:\Re z\geq 0,\arg(z-1+i)\in[\pi,\frac{23}{12}\pi]\}$ then $\mathrm{d}(z,\mathcal{D})=\mathrm{d}(z,-i)$.
(6) If $z\in\mathcal{D}_5=\{z\in D(1-i,1):\arg(z-1+i)\in[\frac{5}{6}\pi,\pi]\}$ then $\mathrm{d}(z,\mathcal{D})=1-|z-1+i|$.
(7) If $z\in\mathcal{D}_6=\{z\in\mathbb{C}:\arg(z-1)\in[\frac{7\pi}{6},2\pi],\arg(z-1+i)\in[-\frac{\pi}{12},\frac{5\pi}{6}]\}$ then $\mathrm{d}(z,\mathcal{D})=\mathrm{d}(z,z_1)$ where $z_1=1-\frac{\sqrt{3}}{2}-\frac{1}{2}i$.

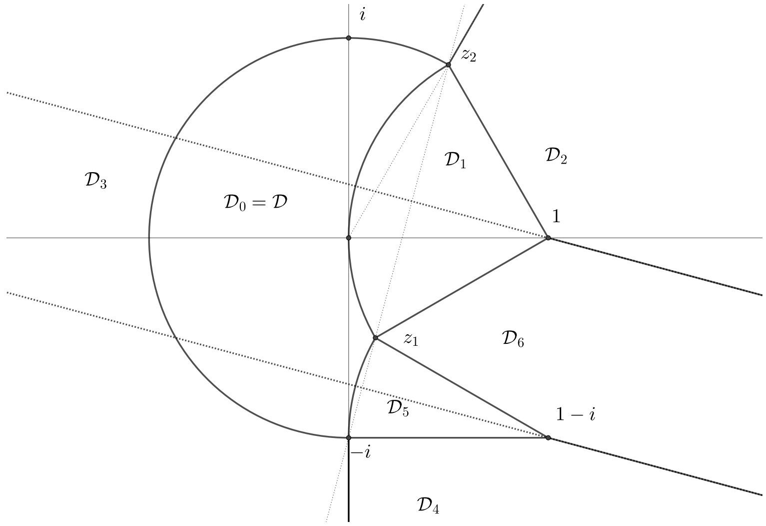

FIGURE 7. Distance to $\mathcal{D}$.

Furthermore, it is easy to check whether a point z belongs a region $\mathcal{D}_j$. For instance $z \in \mathcal{D}_1$ if and only if

$$|z-1| \leq 1 \text{ and } \Im(\frac{z-1}{z_2-1}) \geq 0 \text{ and } \Im(\frac{z-1}{z_1-1}) \leq 0.$$

REFERENCES

[1] M. B. Bekka, M. Mayer, *Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces*, London Mathematical Society, Lecture Note Series 269 (2000).

[2] J. Buchmann, A Generalization of Voronoi's Unit Algorithm I, *Journal of Number Theory* **20** (1985), 177-191.

[3] J. Buchmann, A Generalization of Voronoi's Unit Algorithm II, *Journal of Number Theory* **20** (1985), 192-209.

[4] W. S. Cassels, *An Introduction to the Geometry of Numbers*, Classics in Mathematics, Springer (1997).

[5] Y. Cheung, Hausdorff dimension of singular couples, *Ann. of Math.* **173**, no. 1 (2011), 127–167.

[6] Y. Cheung, N. Chevallier, Lévy-Khintchin Theorem for best simultaneous Diophantine approximations, *arXiv* (2019).

[7] S. G. Dani, A. Nogueira, Continued fractions for complex numbers and values of binary quadratic forms, *Transaction of the A.M.S.* **366** (2014), 3553–3583.

[8] J. Elstrodt, F. Grunewald, J. Mennicke, *Groups Acting on Hyperbolic Space*, Springer Monographs in Mathematics (1998).

[9] G. Fejes Tóth, Thinnest Covering a Circle by Eight, Nine, or Ten Congruent Circles. In J.E. Goodman, J. Pach, and E. Welzl, editors, *Combinatorial and Computational Geometry*, vol. 52 of MSRI Publ., Cambridge University Press (2005), 359–374.

[10] H. Ei, S. Ito, H. Nakada, R. Natsui, On the construction of the natural extension of the Hurwitz complex continued fraction map, *Monatshefte Mathematik* **188** (2019), 37–86.

[11] M. Einsiedler, T. Ward, *Ergodic theory with a view towards number theory.* Graduate Texts in Mathematics, 259. Springer-Verlag London, Ltd., London, (2011) xviii+481 pp.

[12] L. R. Ford, On the closeness of approach of complex rational fractions to a complex irrational number. *Trans. Amer. Math. Soc.* **27** (1925), 146–154.

[13] D. J. Grabiner, J. C. Lagarias, Cutting Sequences for Geodesic flow on the Modular Surface and Continued Fractions *Monatshefte für Mathematik* **133** (2001), 295–339.

[14] D. Hensley, *Continued Fractions*, Word Scientific Publishing (2006).

[15] C. Hermite, Sur différents objets de la théorie des nombres (French), *J. Reine Angew. Math.* 40 (1850), 261–315.

[16] N. Hofreiter, Über die Kettenbruchentwicklung komplexer Zahlen und Anwendungen auf diophantische Approximationen. *Monatshefte f. Math. u. Physik*, **46** (1938), 379–383.

[17] A. Hurwitz, Ueber die Entwicklung complexer Grössen in Kettenbrüche. *Acta Math.*, **XI** (1888), 187–200.

[18] J. Hurwitz, Ueber eine besondere Art der Kettenbruch-Entwicklung complexer Grössen, Dissertation, University of Halle (1895). Printed by Ehrhardt Karras in Halle.

[19] J. C. Lagarias, Geodesic multidimensional continued fractions, *Proc. London Math. Soc.* **63** (1994) no. 3, 464–488.

[20] J. C. Lagarias, Best simultaneous Diophantine approximations. I. Growth rates of best approximation denominators, *Trans. A.M.S.* 272 (1982), no 2, 545–554.

[21] J. C. Lagarias, Best simultaneous Diophantine approximations. II. Behavior of consecutive best approximations, *Pacific J. Math.* 102 (1982), no. 1, 61–88.

[22] J. C. Lagarias, *Some new results in simultaneous Diophantine approximation,* Proc. Queens's University Number Theory Conference 1979, (P. Ribenboim, Ed.), Queen's Papers in Pure and Applied Math. No. 54, Queen's University, 1980, 453–474.

[23] R. B. Lakein, Approximation Properties of Some Complex Continued Fractions, *Monatshefte Mathematik* **77** (1973), 396–403.

[24] R. B. Lakein, A continued fraction proof of Ford's theorem on complex rational approximations. *J. Reine Angew. Math.*, **272** (1975), 1–13.

[25] P. Lunz, Kettenbrüche, deren Teilnenner dem Ring der Zahlen 1 und $\sqrt{-2}$ angehören, Dissertation in Munich. printed by A. Ebner - München 1937.

[26] S. Lyu, C. Porter, C. Ling, Lattice Reduction over Imaginary Quadratic Fields with an Application to Compute-and-Forward, *arXiv:* 1806.03113v5 [cs.IT], 6 May 2019.

[27] N. Michelangeli, On some properties of continued fractions with complex partial quotients (in Italian). Napoli. A. Bellisario e C, (1887).

[28] N.Oswald, Hurwitz's Complex Continued Fractions - A Historical Approach and Modern Perspectives, Phd Thesis, Würzburg University (2014).

[29] N. Oswald, J. Steuding, Complex continued fractions: early work of the brothers Adolf and Julius Hurwitz, *Arch. Hist. Exact. Sci.* **68** (2014), 499–528.

[30] G. G. Robert, Complex Continued Fractions Theoretical Aspects of Hurwitz's Algorithm, Phd Thesis, Aarhus University Denmark September 2018.

[31] A. L. Schmidt, Diophantine approximation of complex numbers. *Acta Math.*, **134** (1975), 1–85.

[32] A. Tanaka, A Complex Continued Fraction Transformation and Its Ergodic Properties. *Tokyo J. Math.*, **8** (1985), 191–214.

[33] G. Voronoï, *On a generalization of the algorithm of continued fractions*, Dissertation, Warsaw (1896) [Russian]