Hausdorff dimension and uniform exponents in dimension two

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Abstract

In this paper we prove the Hausdorff dimension of the set of (nondegenerate) singular two-dimensional vectors with uniform exponent $\mu \in (1/2, 1)$ is $2(1-\mu)$ when $\mu \geq \sqrt{2}/2$, whereas for $\mu < \sqrt{2}/2$ it is greater than $2(1-\mu)$ and at most $(3-2\mu)(1-\mu)/(1+\mu+\mu^2)$. We also establish that this dimension tends to 4/3 (which is the dimension of the set of singular two-dimensional vectors) when μ tends to 1/2. These results improve upon previous estimates of R. Baker, joint work of the first author with M. Laurent, and unpublished work of M. Laurent. We also prove a lower bound on the packing dimension that is strictly greater than the Hausdorff dimension for $\mu \geq 0.565\ldots$.

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1 Introduction and results

1.1 Overview of known results

Let $\underline{\theta}$ be a (column) vector in $\mathbb{R}^n$. We denote by $|\underline{\theta}|_\infty$ the maximum of the absolute values of its coordinates and by

$$\|\underline{\theta}\| = \min_{\underline{x} \in \mathbb{Z}^n} |\underline{\theta} - \underline{x}|_\infty$$

the maximum of the distances of its coordinates to the rational integers.

Let m, n be positive integers and A a real $n \times m$ matrix. Dirichlet's Theorem implies that, for any $X > 1$, the system of inequalities

$$\|A\underline{x}\| \leq X^{-m/n}, \quad 0 < |\underline{x}|_\infty \leq X$$

have a solution $\underline{x}$ in $\mathbb{Z}^m$. This leads to the following definitions. The second one was introduced by Davenport and Schmidt [13].

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Definition 1. Let m, n be positive integers and A a real $n \times m$ matrix. The matrix A is badly approximable if there exists a positive constant c such that the system of inequalities

$$\|A\underline{x}\| \leq c\, X^{-m/n}, \qquad 0 < |\underline{x}|_\infty \leq X \tag{1.1}$$

has no solution $\underline{x}$ in $\mathbb{Z}^m$ for any $X \geq 1$.

Definition 2. Let m, n be positive integers and A a real $n \times m$ matrix. We say that Dirichlet's Theorem can be improved for the matrix A if there exists a positive constant $c < 1$ such that the system of inequalities (1.1) has a solution $\underline{x}$ in $\mathbb{Z}^m$ for any sufficiently large X.

If the subgroup $G = A\mathbb{Z}^m + \mathbb{Z}^n$ of $\mathbb{R}^n$ generated by the m rows of the matrix ${}^tA$ (here and below, ${}^tM$ denotes the transpose of a matrix M) together with $\mathbb{Z}^n$ has rank strictly less than $m+n$, then there exists $\underline{x}$ in $\mathbb{Z}^m$ with $|\underline{x}|_\infty$ arbitrarily large, such that $\|A\underline{x}\| = 0$ and, consequently, for any real number w and any sufficiently large $X > 1$, the system of inequalities

$$\|A\underline{x}\| \leq X^{-w}, \quad 0 < |\underline{x}|_\infty \leq X$$

has a solution $\underline{x}$ in $\mathbb{Z}^m$. In several of the questions considered below, we have to exclude this degenerate situation, thus we are led to introduce the set $\mathcal{M}^*_{n,m}(\mathbb{R})$ of $n \times m$ matrices for which the associated subgroup G has rank $m+n$.

When $m = n = 1$, that is, when $A = (\xi)$ for some irrational real number ξ, it is not difficult to show that Dirichlet's Theorem can be improved if, and only if, ξ is badly approximable (or, equivalently, ξ has bounded partial quotients in its continued fraction expansion); see [19] and [13] for a precise statement. Furthermore, by using the theory of continued fractions, one can prove that, for any irrational real number ξ, there are arbitrarily large integers X such that the system of inequalities

$$\|x\xi\| \leq \frac{1}{2X} \quad \text{and} \quad 0 < x \leq X \tag{1.2}$$

has no integer solutions; see Proposition 2.2.4 of [5].

Since the set of badly approximable numbers has Lebesgue measure zero and Hausdorff dimension 1, this implies that the set of 1×1 matrices A for which Dirichlet's Theorem can be improved has Lebesgue measure zero and Hausdorff dimension 1. The latter assertion has been extended as follows.

Theorem A. *For any positive integers m, n, the set of real $n \times m$ matrices for which Dirichlet's Theorem can be improved has mn-dimensional Lebesgue measure zero and Hausdorff dimension mn.*

The first assertion of Theorem A has been established by Davenport and Schmidt [14] when $\min\{m, n\} = 1$. According to Kleinbock and Weiss [21], their proof can be generalized to $n \times m$ matrices. Actually, a more general result is proved in [21].

As for the latter assertion of Theorem A, Davenport and Schmidt [13] showed that, for $(m, n) = (1, 2)$ or $(2, 1)$, Dirichlet's Theorem can be improved for the $n \times m$ matrix A if A is badly approximable. They noted on page 117 that this assertion is true for arbitrary integers

m, n. Combined with a result of Schmidt [28] on the size of the set of badly approximable matrices, this gives the latter assertion of Theorem A.

We introduce the related notion of singular and regular matrices, which goes back to Khintchine [20].

Definition 3. Let m, n be positive integers and A a real $n \times m$ matrix. We say that the matrix A is singular if, for every positive real number c, the system of inequalities (1.1) has a solution $\underline{x}$ in $\mathbb{Z}^m$ for any sufficiently large X. A matrix which is not singular is called regular.

Khintchine [20] proved that the set of singular $n \times m$ matrices A has mn-dimensional Lebesgue measure zero; see also [8], page 92.

A natural question is then to determine the Hausdorff dimension of the set of singular $n \times m$ real matrices A. The case $n = m = 1$ is easy: there is no irrational real number ξ such that the matrix (ξ) is singular (recall that (1.2) has no integer solutions for arbitrarily large values of X). The case $n = 2, m = 1$ was recently solved by Cheung [10]. For an integer $n \geq 2$, we often use the terminology n-dimensional (column) vector instead of $n \times 1$ matrix.

Theorem B. *The Hausdorff dimension of the set of singular two-dimensional vectors is equal to* $\frac{4}{3}$.

Cheung's result was very recently extended to n-dimensional vectors, for an arbitrary integer $n \geq 2$, by Cheung and Chevallier [11].

Theorem C. *For every integer $n \geq 2$, the Hausdorff dimension of the set of singular n-dimensional vectors is equal to* $\frac{n^2}{n+1}$.

However, the following question remains unsolved.

Problem 1. *Let m, n be integers at least equal to 2. What is the Hausdorff dimension of the set of singular $n \times m$ matrices ?*

Kadyrov *et al.* [18] established that this dimension is bounded from above by $\frac{mn(m+n-1)}{m+n}$ and it is conjectured that there is in fact equality.

We can further discriminate between the singular matrices by introducing exponents of *uniform* Diophantine approximation. We keep the notation from [6].

Definition 4. Let n and m be positive integers and let A be a real $n \times m$ matrix. We denote by $\widehat{\omega}_{n,m}(A)$ the supremum of the real numbers w for which, *for all sufficiently large* positive real numbers X, the system of inequalities

$$\|A\underline{x}\| \leq X^{-w}, \quad 0 < |\underline{x}|_\infty \leq X \tag{1.3}$$

has a solution $\underline{x}$ in $\mathbb{Z}^m$.

For ω in $(0, +\infty]$, let $\mathrm{Sing}_{n,m}(\omega)$ (resp. $\mathrm{Sing}^*_{n,m}(\omega)$) denote the set of matrices A in $\mathcal{M}_{n,m}(\mathbb{R})$ (resp., in $\mathcal{M}^*_{n,m}(\mathbb{R})$) such that

$$\widehat{\omega}_{n,m}(A) \geq \omega,$$

and $\underline{\mathrm{Sing}}_{n,m}(\omega)$ (resp. $\underline{\mathrm{Sing}}^*_{n,m}(\omega)$) denote the set of matrices A in $\mathcal{M}_{n,m}(\mathbb{R})$ (resp., in $\mathcal{M}^*_{n,m}(\mathbb{R})$) such that (1.3) holds for sufficiently large real numbers X. Observe that the set $\underline{\mathrm{Sing}}_{n,m}(\omega)$ is included in $\mathrm{Sing}_{n,m}(\omega)$ and depends on the choice of the norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ whereas $\mathrm{Sing}_{n,m}(\omega)$ does not.

For a real $n \times m$ matrix A, Dirichlet's Theorem implies that

$$\widehat{\omega}_{n,m}(A) \geq \frac{m}{n}. \tag{1.4}$$

Furthermore, we have equality in (1.4) for almost all matrices A, with respect to the Lebesgue measure on $\mathbb{R}^{mn}$, as follows from the Borel–Cantelli Lemma. Any real matrix A satisfying $\widehat{\omega}_{n,m}(A) > \frac{m}{n}$ is singular, and there exist singular matrices A with $\widehat{\omega}_{n,m}(A) = \frac{m}{n}$.

Since, for any real irrational number ξ, there are arbitrarily large integers X for which the system of inequalities (1.2) has no solutions, we deduce that, for any $n \geq 1$, any real $n \times 1$ matrix A satisfies $\widehat{\omega}_{n,1}(A) \leq 1$. Khintchine [19] established that, for any integer $n \geq 2$, there exist matrices A such that $\widehat{\omega}_{n,1}(A) = 1$ and, for any integer $m \geq 2$ and any integer $n \geq 1$, there exist matrices A such that $\widehat{\omega}_{n,m}(A) = +\infty$.

The following problem complements Problem 1. It has been considered by R. C. Baker [1, 2], Yavid [31], and Rynne [27, 26].

Problem 2. *Let m, n be positive integers. Let ω be in $[\frac{m}{n}, +\infty]$ with $\omega \leq 1$ if $m = 1$. What is the Hausdorff dimension of the set of $n \times m$ matrices A in $\mathcal{M}^*_{n,m}(\mathbb{R})$ satisfying $\widehat{\omega}_{n,m}(A) \geq \omega$ (resp. $\widehat{\omega}_{n,m}(A) = \omega$)?*

Before stating our new results, which deal with the case $(n, m) = (2, 1)$, we summarize what is known towards the resolution of Problem 2.

We first point out a result of Jarník [16] asserting that any real 1×2 matrix A in $\mathcal{M}^*_{1,2}(\mathbb{R})$ satisfies

$$\widehat{\omega}_{2,1}({}^tA) = 1 - \frac{1}{\widehat{\omega}_{1,2}(A)}. \tag{1.5}$$

Thus, the cases $(n, m) = (1, 2)$ and $(n, m) = (2, 1)$ are equivalent.

Let $\tau > 2$ be a real number. Baker [1, 2] proved that

$$\frac{2}{\tau} \leq \dim_H \mathrm{Sing}^*_{1,2}(\tau) \leq \frac{6}{\tau + 1}, \tag{1.6}$$

thus

$$\dim_H \mathrm{Sing}^*_{1,2}(+\infty) = 0.$$

Bugeaud and Laurent [7] observed that a direct combination of (1.5) with a result of Dodson [15] yields the slightly sharper upper bound

$$\dim_H \mathrm{Sing}^*_{1,2}(\tau) \leq \frac{3\tau}{\tau^2 - \tau + 1}, \tag{1.7}$$

which was improved to $(2\tau + 2)/(\tau^2 - \tau + 1)$ by Laurent in an unpublished manuscript. We deduce from (1.5) that (1.6) and (1.7) give, for $\mu \geq 1/2$,

$$2(1 - \mu) \leq \dim_H \mathrm{Sing}^*_{2,1}(\mu) \leq \frac{3(1 - \mu)}{\mu^2 - \mu + 1}. \tag{1.8}$$

Observe that for $\mu = 1/2$ the right hand-side of (1.8) is equal to 2, while Theorem B implies that $\lim_{\mu\to\frac{1}{2}} \dim \mathrm{Sing}^*_{2,1}(\mu) \le 4/3$. This shows that the right hand inequality in (1.8) is certainly not best possible for $\mu > (\sqrt{105}-5)/8 = 0.655\ldots$

For $m \ge 3$, combining results of Baker [2] and Rynne [26], one gets that

$$m - 2 + \frac{m}{\tau} \le \dim_H \mathrm{Sing}^*_{1,m}(\tau) \le m - 2 + 2\frac{m+1}{\tau+1}$$

holds for any real number $\tau > m$, thus

$$\dim_H \mathrm{Sing}^*_{1,m}(+\infty) = m - 2,$$

for $m \ge 2$.

As far as we are aware, there is no contribution towards Problem 2 when $\min\{m, n\} \ge 2$.

1.2 New results

The purpose of the present paper is to address Problem 2 for the pair $(n, m) = (2, 1)$. Our first result improves the right hand inequality in (1.8) for every value of μ in $(1/2, 1)$.

Theorem 1. *For any real number μ in $(1/2, \sqrt{2}/2]$, we have*

$$\dim_H \mathrm{Sing}^*_{2,1}(\mu) \le \frac{(3-2\mu)(1-\mu)}{\mu^2 - \mu + 1}.$$

For any real number μ in $[\sqrt{2}/2, 1)$, we have

$$\dim_H \mathrm{Sing}^*_{2,1}(\mu) \le 2(1-\mu).$$

Observe that our upper bound for $\dim_H \mathrm{Sing}^*_{2,1}(\mu)$ is a continuous function of μ in $(1/2, 1)$.

Combined with (1.8), Theorem 1 yields the exact value of the dimension when μ is sufficiently large.

Corollary 2. *For any real number μ in $[\sqrt{2}/2, 1)$, we have*

$$\dim_H \mathrm{Sing}^*_{2,1}(\mu) = 2(1-\mu).$$

Our second result improves the left hand inequality in (1.8) for every value of μ in $(\sqrt{2}/2, 1)$.

Theorem 3. *For any real number μ in $(1/2, \sqrt{2}/2)$, we have*

$$\dim_H \underline{\mathrm{Sing}}^*_{2,1}(\mu) \ge (1-\mu) \sup_{b>0} \frac{(2b^2 + 2b\mu + b + (2-\mu)(2\mu-1))}{(b+2\mu-1)(\mu^2-\mu+b+1)},$$

and thus

$$\dim_H \underline{\mathrm{Sing}}^*_{2,1}(\mu) \ge 2(1-\mu).$$

A combination of Theorems 1 and 3 yields the following corollary.

Corollary 4. *We have*

$$\lim_{\mu\to 1/2,\, \mu>\frac{1}{2}} \dim_H \underline{\mathrm{Sing}}^*_{2,1}(\mu) = \frac{4}{3}.$$

By Theorem B, the set of singular two-dimensional vectors has dimension $\frac{4}{3}$. Corollary 4 shows that there is no jump of Hausdorff dimension.

Remark 1.1. For a fixed μ in $(\frac{1}{2}, \frac{1}{\sqrt{2}})$, it is not difficult to compute the positive real number b_0 giving the maximum of the rational fraction

$$b \longmapsto \frac{(2b^2 + 2b\mu + b + (2-\mu)(2\mu-1))}{(b+2\mu-1)\,(\mu^2-\mu+b+1)}.$$

It satisfies a quadratic equation. Unfortunately, the lower bound we obtain does not match with the upper bound established in Theorem 1.

Remark 1.2. For real numbers $\mu, \tau \geq 1/2$, denote by $\underline{\mathrm{Sing}}_{2,1}(\mu,\tau)$ the set of matrices A in $\underline{\mathrm{Sing}}_{2,1}(\mu)$ such that there are arbitrarily large real numbers X for which the system of inequalities

$$\|A\underline{x}\| \leq X^{-\tau}, \quad 0 < |\underline{x}|_\infty \leq X$$

has a solution $\underline{x}$ in $\mathbb{Z}$.

The proof of Theorem 3 enables us to state a more precise result, namely

$$\dim_H \underline{\mathrm{Sing}}^*_{2,1}(\mu,\tau) \geq (1-\mu)\frac{(2b^2 + 2b\mu + b + (2-\mu)(2\mu-1))}{(b+2\mu-1)\,(\mu^2-\mu+b+1)},$$

where $\tau = \frac{1}{(1-\mu)(b+1)}\,(\mu^2-\mu+b+1) - 1$ and b is any positive real number (this is a consequence of Lemma 25).

Remark 1.3. It is very likely that

$$\dim_H\{A \in \mathcal{M}^*_{2,1}(\mathbb{R}) : \widehat{\omega}_{2,1}(A) = \mu\} = 2(1-\mu)$$

for every μ in $[\sqrt{2}/2, 1)$. However, this does not follow from our results and it seems to us that a proof would require additional ideas.

Finally, we also prove a result about the packing dimension.

Proposition 5. *For every real number μ in $(\frac{1}{2}, 1)$, we have*

$$\dim_P \underline{\mathrm{Sing}}^*_{2,1}(\mu) \geq \sup_{b>0} \frac{(2b^2 + 2b\mu + b + (2-\mu)(2\mu-1))}{(\mu+1+2b)\,(b+2\mu-1)},$$

thus, in particular,

$$\dim_P \underline{\mathrm{Sing}}^*_{2,1}(\mu) > 1.$$

Remark 1.4. Using Theorem 3 and Proposition 5 and some numerical experiments it is easy to see that

$$\dim_P \underline{\mathrm{Sing}}^*_{2,1}(\mu) > \dim_H \underline{\mathrm{Sing}}^*_{2,1}(\mu)$$

for $\mu \geq 0.565\ldots$ However Theorem 3 and Proposition 5 are not strong enough to get the strict inequality for $\mu \leq 0.565\ldots$

Sketch of the proofs

Since the proofs deal only with the sets $\mathrm{Sing}_{2,1}(\mu)$, we will drop the subscript ${}_{2,1}$ when there is no ambiguity. For convenience, we replace column vectors by row vectors. We use also the following notation. We take θ in $\mathbb{R}^2$ and consider elements $x = (p, q)$ in $\mathbb{Z}^2 \times \mathbb{Z}_{\geq 1}$, where $p = (p_1, p_2)$ is a pair of integers. Then, $\frac{p}{q}$ denotes the pair $(\frac{p_1}{q}, \frac{p_2}{q})$. We also write $|x| = q$.

The strategy of our proofs follows closely the one of [11]. As in this work, the guideline for the proofs relies on two simple results. For each primitive vector $x = (p, q)$ of the lattice $\mathbb{Z}^3$ with p in $\mathbb{Z}^2$ (we keep this notation throughout this paper) and q in $\mathbb{Z}_{>0}$, let $\lambda_1(x)$ denote the length of the shortest vector of the lattice $\Lambda_x = \mathbb{Z}^2 + \mathbb{Z}\frac{p}{q}$. Roughly, the first result is: θ in $\mathbb{R}^2$ is in $\underline{\mathrm{Sing}}(\mu)$ if and only if for n large enough,

$$\lambda_1(x_n) \leq |x_n|^{-\mu}$$

where $x_n = (p_n, q_n)$ is the n-th term of the the sequence of best approximation vectors of θ and $|x_n| = q_n$ (see Section 3 and Corollary 12 for an exact statement). The second result is a multidimensional extension of Legendre's Theorem about convergents of ordinary continued fraction expansions: if $x = (p, q)$ is a best approximation vector of θ, then $\theta \in B(\frac{p}{q}, \frac{2\lambda_1(x)}{|x|})$ and conversely, if $\theta \in B(\frac{p}{q}, \frac{\lambda_1(x)}{2|x|})$, then x is a best approximation vector of θ (see Lemma 10). Then we use the standard strategy for computing the Hausdorff dimension of Cantor sets defined by a nested tree of intervals. Precisely, defining the children of an interval as the immediate successors with respect to the partial order induced by inclusion of intervals, the diameter of one interval raised to the power s has to be compared with the sum over all the children intervals of their diameters raised to the power s.

For the upper bound, consider a set $\sigma_\mu(x)$ for each primitive vector $x = (p, q)$ in $\mathbb{Z}^2 \times \mathbb{Z}_{>0}$ with $\lambda_1(x) \leq |x|^{-\mu}$. This set plays the role of the children of x. The first idea is to take for $\sigma_\mu(x)$ the set of all possible primitive vectors y in $\mathbb{Z}^2 \times \mathbb{Z}_{>0}$ with $\lambda_1(y) \leq |y|^{-\mu}$ such that x and y are two consecutive best approximation vectors of some θ in $\mathbb{R}^d$. If for all x,

$$\sum_{y=(u,v)\in\sigma_\mu(x)} \left(\operatorname{diam} B\left(\frac{u}{v}, \frac{2\lambda_1(y)}{|y|}\right)\right)^s \leq \left(\operatorname{diam} B\left(\frac{p}{q}, \frac{2\lambda_1(x)}{|x|}\right)\right)^s$$

then the Hausdorff dimension of $\underline{\mathrm{Sing}}(\mu)$ is at most equal to s. We make this statement more precise by using self-similar covering introduced by the second author (see [10] and Theorem 6). However the above inequality does not hold and as in [10] and [11] we modify the definition of the set $\sigma_\mu(x)$ with an "acceleration" by considering only a subsequence of the sequence of best approximations (see Definition 7). Note that the subsequence is not the same as that in [11]. Another point is that it is better to use a radius larger than $\frac{2\lambda_1(x)}{|x|}$ (see Corollary 17), for it avoids the second acceleration used in [10]. The choice of a good radius is more delicate than in [11]. With these ingredients the proof of the upper bound follows readily; see Section 4.

The lower bound is trickier. The idea is to find a Cantor set included in $\underline{\mathrm{Sing}}^*(\mu)$. This Cantor set has an "inhomogeneous" tree structure. For each $x = (p, q)$ such that $\lambda_1(x) \leq |x|^{-\mu}$, we define a finite set $\sigma(x)$ and a ball $B(x)$ such that for all $z = (u, v)$ in $\sigma(x)$, we have both

$\lambda_1(z) \leq |z|^{-\mu}$ and

$$B(z) \subset B\Big(\frac{u}{v}, \frac{\lambda_1(z)}{2\,|z|}\Big) \subset B(x).$$

The above inclusions ensure that x and z are best approximation vectors of all θ in $B(z)$ which in turn will be helpful to show that the Cantor set defined by the sets $\sigma(x)$ and the balls $B(x)$ is included in $\underline{\mathrm{Sing}}^*(\mu)$ (see Proposition 26). Then, the inequality

$$\sum_{z \in \sigma(x)} (\mathrm{diam}\, B\,(z))^s \geq (\mathrm{diam}\, B\,(x))^s$$

together with a condition about the distribution in $B(x)$ of the points z in $\sigma(x)$ imply that the Hausdorff dimension of $\underline{\mathrm{Sing}}^*(\mu)$ is at least equal to s. However, this program is not straightforward because the condition about the distribution of the elements of $\sigma(x)$ used in [11] does not work in our context (see Theorem 3.6 of [11]).

To overcome this problem, we use a more flexible condition which is an adaptation of the mass distribution principle to self-similar covering; see Theorem 7. This more flexible condition, together with a careful study of the geometric positions of the points of $\sigma(x)$ in the ball $B(x)$ (see Lemmas 19 and 27), finally lead to the lower bound.

1.3 Questions and problems

In this subsection, we gather some suggestions for further research closely related to the present work.

Maybe, it is possible to adapt the methods of [14, 21] to solve the following problem, which seems to be rather difficult.

Problem 3. *Let c be a real number with $0 < c < 1$. What is the Hausdorff dimension of the set of $n \times m$ matrices such that (1.1) has a solution $\underline{x}$ in $\mathbb{Z}^m$ for any sufficiently large X? Is this a continuous function of c?*

All the results quoted above are concerned with approximation of independent quantities in the sense that we assume that the entries of the matrices A are independent. It is a notorious fact that questions of approximation of dependent quantities are much more delicate. An emblematic example in the case of $n \times 1$ matrices is given by the Veronese curve $(\xi, \xi^2, \ldots, \xi^n)$. At present, we do not know the Hausdorff dimension of the set of real numbers ξ such that the pair (ξ, ξ^2) is singular. In 2004 Roy [25] showed that this set is nonempty. In the oppposite direction, Shah [29, 30] has obtained several striking results on the size of sets of matrices with dependent entries for which Dirichlet's Theorem cannot be improved.

Problem 4. *Let $n \geq 2$ be an integer. What is the Hausdorff dimension of the set of real numbers ξ such that $(\xi, \xi^2, \ldots, \xi^n)$ is singular?*

The latter problem is deeply connected with the following famous conjecture of Wirsing on approximation to real numbers by algebraic numbers of bounded degree. Recall that the height of an algebraic number α, denoted by $H(\alpha)$, is the maximum of the absolute values of the coefficients of its minimal defining polynomial over $\mathbb{Z}$.

Problem 5. *(Wirsing) Let $n \geq 2$ be an integer and ξ be a transcendental real number. For any positive ε, there exist algebraic numbers α of degree at most n and of arbitrarily large height such that*

$$|\xi - \alpha| < H(\alpha)^{-n-1+\varepsilon}.$$

It follows from results established in [6] that the Hausdorff dimension of the set of counterexamples to the Wirsing conjecture on the approximation by algebraic numbers of degree at most n is at most equal to the Hausdorff dimension of the set of real numbers ξ such that $(\xi, \xi^2, \dots, \xi^n)$ is singular. See Chapter 3 of [4] for a survey of known results towards Wirsing's conjecture.

A further line of research is Diophantine approximation on fractal sets. Rather than assuming that A is an arbitrary real $n \times m$ matrix, we restrict our attention to matrices in a given fractal set.

Problem 6. *What is the Hausdorff dimension of the set of singular pairs whose entries belong to the middle third Cantor set?*

Our results on the packing dimension motivate the following questions.

Problem 7. *Is the packing dimension of* $\mathrm{Sing}^*(\mu)$ *strictly greater than the Hausdorff dimension for all $\mu > 1/2$? What is the value of the packing dimension of the set of singular pairs? Is it equal to its Hausdorff dimension, that is, to $4/3$?*

2 Definitions and results about self-similar coverings

Definition 5. Let Y be a metric space. A *self-similar structure* on Y is a triple (J, σ, B) where J is countable, σ is a subset of J^2, and B is a map from J into the set of bounded subsets of Y. A *σ-admissible sequence* is a sequence $(x_n)_{n\in\mathbb{N}}$ in J such that

(i) for all integers n, $(x_n, x_{n+1}) \in \sigma$.

Let X be a subset of Y. A *self-similar covering* of X is a self-similar structure (J, σ, B) such that, for all θ in X, there exists a σ-admissible sequence $(x_n)_{\in\mathbb{N}}$ in J satisfying

(ii) $\lim_{n\to\infty} \operatorname{diam} B(x_n) = 0$,

(iii) $\bigcap_{n\in\mathbb{N}} B(x_n) = \{\theta\}$.

The set covered by a self-similar structure (J, σ, B) is the set all θ in Y with the two properties above.

Notation. We denote by $\sigma(x)$ the set of y in J such that $(x, y) \in \sigma$.

Definition 6. By a strictly nested self-similar structure we mean a self-similar structure (J, σ, B) that satisfies $\lim_{n\to\infty} \operatorname{diam} B(x_n) = 0$, for all σ-admissible sequence $(x_n)_{n\in\mathbb{N}}$, and $B(y) \subset B(x)$, for all x in J and all y in $\sigma(x)$.

2.1 Upper bound for the Hausdorff dimension

We quote a result from [10].

Theorem 6. *([10]) Let Y be a metric space, let X be a subset of Y that admits a self-similar covering (J, σ, B) and let s be a positive real number. If*

$$\sum_{y \in \sigma(x)} \operatorname{diam} B(y)^s \leq \operatorname{diam} B(x)^s,$$

holds for all x in J, then $\dim_H X \leq s$.

2.2 Lower bound for the Hausdorff dimension

There already exist results providing lower bounds for the Hausdorff dimension of self similar structures, see [10] or [11]. However these results are not suitable for our purpose. An adaptation of the mass distribution principle to self similar structures leads to a more flexible statement.

Let (J, σ, B) be a self-similar structure on a complete metric space (Y, d). For a subset F of Y and x in J, we set

$$\sigma_F(x) = \{y \in \sigma(x) : F \cap B(y) \neq \emptyset\}.$$

Theorem 7. *Let (J, σ, B) be a strictly nested self-similar structure on a complete metric space (Y, d). Suppose that, for all $x \in J$, the set $B(x)$ is bounded and closed. Let s be a positive real number and suppose that*

i. for all x in J, $\operatorname{diam} B(x) > 0$ and $\sum_{y \in \sigma(x)} (\operatorname{diam} B(y))^s \geq (\operatorname{diam} B(x))^s$,

ii. for all x in J, the sets $B(y)$, $y \in \sigma(x)$, are disjoint,

iii. there exists a constant C such that for all x in J and all subsets F in Y such that

$$\delta(x) = \min_{y \neq y' \in \sigma(x)} d(B(y), B(y')) \leq \operatorname{diam} F \leq \operatorname{diam} B(x),$$

we have

$$\frac{\sum_{y \in \sigma_F(x)} (\operatorname{diam} B(y))^s}{(\operatorname{diam} F)^s} \leq C \frac{\sum_{y \in \sigma(x)} (\operatorname{diam} B(y))^s}{(\operatorname{diam} B(x))^s},$$

Then $\dim_H E \geq s$ and the Hausdorff dimension of the set covered by (J, σ, B) is $\geq s$.

We need an auxiliary Lemma. Let (J, σ, B) be a self-similar structure on a complete metric space (Y, d). For x_0 in J, we consider the set Ω_{x_0} of all admissible sequences starting at x_0 and, for a finite admissible sequence $a_0 = x_0, a_1 \ldots, a_n$ in J, we denote by

$$[a_1, \ldots, a_n] = \{(x_n)_{n \in \mathbb{N}} \in \Omega_{x_0} : x_i = a_i,\ i = 1, \ldots, n\}$$

the associated cylinder. We endow Ω_{x_0} with the topology induced by the product topology on $J^{\mathbb{N}}$.

Lemma 8. *Let (J, σ, B) be a strictly nested self-similar structure on a complete metric space (Y, d). Suppose that, for all $x \in J$, the set $B(x)$ is bounded and closed. Then Ω_{x_0} is a compact subset of $J^{\mathbb{N}}$ and for all sequence $(x_n)_{n\in\mathbb{N}}$ in Ω_{x_0} there exists a unique point a in the intersection of the closed sets $B(x_n)$, $n \in \mathbb{N}$. Furthermore the map $\varphi : \Omega_{x_0} \to Y$ defined by $\varphi((x_n)_{n\in\mathbb{N}}) = a$ is continuous and the sequence*

$$D_n = \max\{\operatorname{diam} \varphi([x_1, \ldots, x_n]) : x_1, \ldots, x_n \in J\}$$

goes to zero when n goes to infinity.

Proof of the Lemma. The only thing which is not clear is the last point. Consider the sequence of functions $(d_k)_{k\geq 1}$ defined by

$$d_k((x_n)_{n\in\mathbb{N}}) = \operatorname{diam} \varphi([x_1, \ldots, x_k])$$

for a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω_{x_0}. By the definition of the topology, each d_k is continuous on the compact set Ω_{x_0}. Clearly the sequence $(d_k)_k$ is non-increasing and by assumption $\lim_{k\to\infty} d_k((x_n)_{n\in\mathbb{N}}) \leq \lim_{k\to\infty} \operatorname{diam} B(x_k) = 0$ for all $(x_n)_{n\in\mathbb{N}}$ in Ω_{x_0}, hence by Dini's theorem, the sequence $(d_k)_{k\geq 1}$ converges uniformly to zero. □

Proof of Theorem 7. We keep the notations of the Lemma. The set $E := \varphi(\Omega_{x_0})$ is a compact subset of Y. It is enough to prove that there exists a probability measure ν on Y supported by E such that for every Borel subset F of Y, we have

$$\nu(F) \leq C(\operatorname{diam} F)^s$$

for some absolute constant C.

A map μ defined on the set of cylinders can be extended to a probability measure on Ω_{x_0} if for all cylinders $[x_1, \ldots, x_n]$ we have the additive formula

$$\sum_{x\in\sigma(x_n)} \mu([x_1, \ldots, x_n, x]) = \mu([x_1, \ldots, x_n]).$$

For all x in J set $M(x) = \sum_{y\in\sigma(x)} (\operatorname{diam} B(y))^s$. The following recursion formulas

$$\mu([x_1]) = \frac{(\operatorname{diam} B(x_1))^s}{M(x_0)},$$
$$\mu([x_1, \ldots, x_{n+1}]) = \frac{(\operatorname{diam} B(x_{n+1}))^s}{M(x_n)} \mu([x_1, \ldots, x_n]),$$

define a measure μ on the set of cylinders. Clearly the additive formula holds, hence μ extends to a probability measure.

Call ν the image of μ by the map φ. The support of ν is included in E.

We want to check that $\nu(F) \leq C(\operatorname{diam} F)^s$ for all Borel subset F of Y. We can suppose that $F \subset E$.

First, let us show by induction that for all cylinders $[x_1, \ldots, x_n]$, we have the inequality,

$$\mu([x_1, \ldots, x_n]) \le \frac{(\operatorname{diam} B(x_n))^s}{(\operatorname{diam} B(x_0))^s}.$$

For all $x_1 \in \sigma(x_0)$,

$$\begin{aligned}\mu([x_1]) &= \frac{(\operatorname{diam} B(x_1))^s}{M(x_0)} \\ &\le \frac{(\operatorname{diam} B(x_1))^s}{(\operatorname{diam} B(x_0))^s},\end{aligned}$$

and since $M(x_n) \ge (\operatorname{diam} B(x_n))^s$,

$$\begin{aligned}\mu([x_1, \ldots, x_{n+1}]) &= \frac{(\operatorname{diam} B(x_{n+1}))^s}{M(x_n)}\mu([x_1, \ldots, x_n]) \\ &\le \frac{(\operatorname{diam} B(x_{n+1}))^s}{(\operatorname{diam} B(x_n))^s} \times \frac{(\operatorname{diam} B(x_n))^s}{(\operatorname{diam} B(x_0))^s} \\ &\le \frac{(\operatorname{diam} B(x_{n+1}))^s}{(\operatorname{diam} B(x_0))^s}.\end{aligned}$$

Let F be a subset of E. If F is reduced to one point $a = \varphi((x_n)_{n\in\mathbb{N}})$, we have to check that $\nu(F) = 0$. By the disjointness assumption φ is one to one and

$$\nu(F) \le \nu(\varphi([x_1, \ldots, x_n])) = \mu([x_1, \ldots, x_n]) \le \frac{(\operatorname{diam} B(x_n))^s}{(\operatorname{diam} B(x_0))^s},$$

which goes to zero because the self-similar covering is strictly nested.

Suppose now that $\operatorname{diam} F > 0$. By the last point of the above lemma there is a cylinder $\mathcal{C} =[x_1, \ldots, x = x_n]$ of maximal length containing the image $\varphi(\mathcal{C})$ ($\mathcal{C}$ can be Ω_{x_0}). By maximality, there exists $y \ne y'$ in $\sigma(x)$ such that F intersects both $B(y)$ and $B(y')$, hence $\operatorname{diam} F \ge \delta(x)$. Therefore,

$$\frac{\sum_{y\in\sigma_F(x)}(\operatorname{diam} B(y))^s}{(\operatorname{diam} F)^s} \le C\frac{\sum_{y\in\sigma(x)}(\operatorname{diam} B(y))^s}{(\operatorname{diam} B(x))^s} = C\frac{M(x)}{(\operatorname{diam} B(x))^s}.$$

By the definition of σ_F, we have

$$\begin{aligned}F &\subset (\cup_{y\in\sigma_F(x)} B(y)), \\ \nu(F) &\le \sum_{y\in\sigma_F(x)} \nu(B(y)),\end{aligned}$$

and, by the definition of ν and by the disjointness assumption,

$$\begin{aligned}\sum_{y\in\sigma_F(x)} \nu(B(y)) &= \sum_{y\in\sigma_F(x)} \mu([x_1, \ldots, x, y]) \\ &= \sum_{y\in\sigma_F(x)} \mu([x_1, \ldots, x])\frac{(\operatorname{diam} B(y))^s}{M(x)}.\end{aligned}$$

Hence, we deduce from the above inequality about cylinders that

$$\nu(F) \leq \sum_{y \in \sigma_F(x)} \frac{(\operatorname{diam} B(x))^s}{(\operatorname{diam} B(x_0))^s} \frac{(\operatorname{diam} B(y))^s}{M(x)} \\ \leq \frac{C}{(\operatorname{diam} B(x_0))^s} (\operatorname{diam} F)^s.$$

□

2.3 Lower bound for the packing dimension

Lemma 9. *Let (J, σ, B) be a strictly nested self-similar structure on a metric space Y and let s be a positive real number. Suppose that we have a map $x \mapsto \widehat{x}$ from J to Y and a map $B' : x \mapsto B'(x) = B(\widehat{x}, r(x))$ from J to the set of closed balls in Y. We also make the following assumptions:*

1. *for all x in J, $\sigma(x)$ is finite,*
2. *there exists $k < 1$ such that $B(x) \subset B(\widehat{x}, kr(x))$ for all x in J,*
3. *for all x in J, the balls $B'(y)$, $y \in \sigma(x)$, are disjoint and included in $B'(x)$,*
4. *for all σ-admissible sequence $(x_n)_{\in \mathbb{N}}$ in J, we have $\lim_{n\to\infty} \operatorname{diam} B'(x_n) = 0$,*
5. *for all x in J, $\operatorname{diam} B'(x) > 0$ and $\sum_{y \in \sigma(x)} (\operatorname{diam} B'(y))^s \geq (\operatorname{diam} B'(x))^s$.*

Then, the packing dimension of the set covered by (J, σ, B) is at least equal to s.

Proof. We keep the notations of the previous section and consider, for $x_0 \in J$, the set Ω_{x_0} of all admissible sequences starting at x_0. We are going to show that

$$\dim_P E \geq s.$$

Let ε be a positive real number. As in the proof of Lemma 8, Dini's theorem implies that

$$\lim_{p\to\infty} \sup\{\operatorname{diam} B'(x_p) : (x_n)_{n\in\mathbb{N}} \in \Omega_{x_0}\} = 0.$$

Therefore, there exists an integer q_ε such that

$$\sup\{\operatorname{diam} B'(x_{q_\varepsilon}) : (x_n)_{n\in\mathbb{N}} \in \Omega_{x_0}\} \leq \varepsilon.$$

For a positive integer q, let J_q be the set of x in J such that there exists a σ-admissible sequence $x_0, x_1, \ldots, x_q$ with $x_q = x$. The disjointness property in item 3 implies that the sets $\sigma(x)$, $x \in J_q$, are disjoint. Hence, we have a disjoint union $J_{q+1} = \cup_{x \in J_q} \sigma(x)$. An easy induction together with item 5 implies that for all q,

$$\sum_{x \in J_q} (\operatorname{diam} B'(x))^s \geq (\operatorname{diam} B'(x_0))^s,$$

hence we would have shown that the ε-packing measure satisfies

$$\mathcal{P}^s_\varepsilon(E) \geq \operatorname{diam} B'(x_0))^s,$$

if the balls $B'(x)$, $x \in J_{q_\varepsilon}$, were centered at points in $E = \varphi(\Omega_{x_0})$. Now, by item 2, the set $\varphi([x_0, \ldots, x_q])$ is included in the ball $B'(\widehat{x}_q, kr(x_q))$, hence there is a point $y(x_q) \in E$ such that the ball $B(y(x_q), (1-k)r(x_q))$ is included in the ball $B(\widehat{x}_q, r(x_q))$. It follows that

$$\sum_{x \in J_q} (\operatorname{diam} B(y(x), (1-k)r(x)))^s \geq ((1-k) \operatorname{diam} B'(x_0))^s,$$

which in turn implies that $\mathcal{P}^{s'}(E) = \infty$ for all $s' < s$. It remains to show that the packing measure $p^{s'}(E)$ does not vanish. This is proved by means of a standard argument. If $(E_i)_{i \in N}$ is any covering of E, then, by Baire's Theorem, one of the closure $F_i = \bar{E}_i$, say F_q, contains a subset of E of nonempty relative interior. It follows that there exists a cylinder $C = [a_0, \ldots, a_j]$ of Ω_{x_0} such that $\varphi(C) \subset F_q$. Now, the previous way of reasoning implies that

$$\mathcal{P}^s(\varphi(C)) \geq ((1-k) \operatorname{diam} B'(a_j))^s,$$

hence, for all $s' < s$,

$$\mathcal{P}^{s'}(F_q) = \mathcal{P}^{s'}(E_q) = \infty$$

and $p^{s'}(E) = \infty$. □

3 Farey Lattices and best approximants

From now on we suppose that $\mathbb{R}^2$ is equipped with the standard Euclidean norm $\|.\|_e$.

Let the set of primitive vectors in $\mathbb{Z}^3$ corresponding to rationals in $\mathbb{Q}^2$ in their "lowest terms representation" be denoted by

$$Q = \{(p_1, p_2, q) \in \mathbb{Z}^3 : \gcd(p_1, p_2, q) = 1, q > 0\}.$$

Given $x = (p, q) \in Q$, where $p \in \mathbb{Z}^2$, we use the notation

$$|x| = q \quad \text{and} \quad \widehat{x} = \frac{p}{q}.$$

For x in Q, let

$$\Lambda_x := \mathbb{Z}^2 + \mathbb{Z}\widehat{x} = \pi_x(\mathbb{Z}^3)$$

where $\pi_x : \mathbb{R}^3 \to \mathbb{R}^2$ is the "projection along the lines parallel to x" given by the formula $\pi_x(m, n) = m - n\widehat{x}$ for $(m, n) \in \mathbb{R}^2 \times \mathbb{R}$. Observe that $\operatorname{vol} \Lambda_x = |x|^{-1}$.

Given a norm on $\mathbb{R}^2$, we denote the successive minima of Λ_x by $\lambda_i(x)$ and the *normalized* successive minima by

$$\widehat{\lambda}_i(x) := |x|^{1/2} \lambda_i(x) \quad \text{for} \quad i = 1, 2.$$

We collect without proof a few lemmas the proof of which can be found in [10] and [11].

3.1 Inequalities of best approximation

The ordinary continued fraction expansion is a very efficient tool for the study of Diophantine exponents of a single real number. In higher dimensions, it is convenient to replace the ordinary continued fraction expansion by the sequence of best Diophantine approximations vectors because a weak form of many properties of the one-dimensional expansion still hold.

Recall that the sequence $(q_n)_{n\geq 0}$ of *best simultaneous approximation denominators* of $\theta \in \mathbb{R}^2$ with respect to the norm $\|\cdot\|_e$ is defined by the recurrence relation

$$q_0 = 1, \quad q_{n+1} = \min\{q \in \mathbb{N} : q > q_n, \operatorname{dist}(q\theta, \mathbb{Z}^2) < \operatorname{dist}(q_n\theta, \mathbb{Z}^2)\}.$$

By definition, the sequence $(q_n)_{n\geq 0}$ is strictly increasing, while the sequence $(r_n)_{n\geq 0}$ where $r_n = \operatorname{dist}(q_n\theta, \mathbb{Z}^2)$, is strictly decreasing. These sequences are infinite if and only if $\theta \in \mathbb{R}^2 \setminus \mathbb{Q}^2$. For each $n \geq 0$, we choose p_n so that $\|q_n\theta - p_n\|_e = r_n$ and set $x_n = (p_n, q_n) \in \mathbb{Z}^2 \times \mathbb{Z}_{>0}$. It is customary to refer to $(x_n)_{n\geq 0}$ as *the sequence of best simultaneous approximation vectors*, even though the choice of p_n need not be unique.[1] See [12, 22, 23, 24] for more about best approximations. In what follows we shall often write best approximation instead of best simultaneous approximation vector.

First we qote a result that generalizes *Legendre's* Theorem: *p/q is a convergent of $\alpha \in \mathbb{R}$ as soon as $|\alpha - p/q| < 1/2q^2$.* Denote by μ_2 the supremum of $\lambda_1(L)$ over all 2-dimensional lattices $L \subset \mathbb{R}^2$ of covolume 1.

Lemma 10 (Thm. 2.11 of [10]). *For $x \in Q$, let $\Delta(x) = \{\theta : \widehat{x}$ is a best approximation of $\theta\}$. If $|x| > \left(\frac{\mu_2}{\lambda_1(\mathbb{Z}^2)}\right)^2$, then*

$$\bar{B}\left(\widehat{x}, \frac{\lambda_1(x)}{2\,|x|}\right) \subset \Delta(x) \subset B\left(\widehat{x}, \frac{2\lambda_1(x)}{|x|}\right),$$

where $\bar{B}$ denote the closed ball.

The unimodular property, $|p_{n+1}q_n - q_{n+1}p_n| = 1$, which hold for two consecutive convergents $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ of the ordinary continued fraction expansion cannot be extended to best Diophantine approximations in higher dimensions (see [12] and [24]). However (i) of Lemma 11 can be seen as a weak form of the unimodular property.

The notation $x \asymp_2 y$ means $\frac{1}{2}y \leq x \leq 2y$.

Lemma 11 ([10], [11]). *Let $x_n = (p_n, q_n), n \geq 0$, be the sequence of best approximation vectors of $\theta \in \mathbb{R}^2$. Then*

(i) $\|\widehat{x}_n - \widehat{x}_{n+1}\|_e < \frac{4\lambda_1(x_{n+1})}{|x_n|}$.

(ii) For all $k \geq 0$, $\|\widehat{x}_n - \widehat{x}_{n+k}\|_e < \frac{4\lambda_1(x_n)}{|x_n|}$.

(iii) For all $y = (p, q) \in \mathbb{Z}^{2+1}$ with $0 < q < |x_n|$, $\|p - q\theta\|_e \asymp_2 \|p - q\widehat{x}_n\|_e$.

[1] It is unique as soon as q_n is large enough, e.g. if $q_n > (4\mu_2/\lambda_1(\mathbb{Z}^2))^2$. See [22] or Remark 2.13 of [10].

The previous lemma allows to almost characterize the set $\underline{\mathrm{Sing}}(\mu)$ with best approximation vectors.

Corollary 12. *Let $\mu' > \mu > 0$ and let θ be in $\mathbb{R}^2$. Call $x_n = (p_n, q_n), n \geq 0$, the sequence of best approximation vectors of $\theta \in \mathbb{R}^2$. If $\theta \in \underline{\mathrm{Sing}}(\mu')$, then for all n large enough*

$$\lambda_1(x_n) \leq \|q_{n-1}\widehat{x}_n - p_{n-1}\|_e \leq |x_n|^{-\mu}.$$

Conversely, if

$$\lambda_1(x_n) \leq |x_n|^{-\mu'}$$

for all n large enough, then $\theta \in \underline{\mathrm{Sing}}(\mu)$.

Proof. By Lemma 11 (iii), if $\theta \in \underline{\mathrm{Sing}}(\mu')$, then for all n large enough

$$\begin{aligned}\lambda_1(x_n) &\leq \|q_{n-1}\widehat{x}_n - p_{n-1}\|_e \\ &\leq 2\|q_{n-1}\theta - p_{n-1}\|_e \\ &\leq 2(q_n - 1)^{-\mu'} \\ &\leq |x_n|^{-\mu}.\end{aligned}$$

Conversely, if $\lambda_1(x_n) \leq |x_n|^{-\mu'}$, then by Lemma 11 (iii) and (i), for all $q_{n-1} \leq q < q_n$, we have

$$\begin{aligned}d(\{\theta, \ldots, q\theta\}, \mathbb{Z}^2) &= \|q_{n-1}\theta - p_{n-1}\|_e \\ &\leq 2\|q_{n-1}\widehat{x}_n - p_{n-1}\|_e \\ &\leq 8\lambda_1(x_n) \\ &\leq 8q_n^{-\mu'} \leq q^{-\mu},\end{aligned}$$

when n is large enough. □

3.2 The subspace H_x

Call $x_n = (p_n, q_n)$, $n \in \mathbb{N}$, the sequence of best approximation vectors of $\theta \in \mathbb{R}^2$. Corollary 12 shows that if θ is in $\underline{\mathrm{Sing}}(\mu)$ with $\mu > \frac{1}{2}$, then $\widehat{\lambda}(x_n) \to 0$ when n goes to ∞. It follows that the shortest vector of the lattice Λ_{x_n} is very small compare to $\lambda_2(x_n)$ when n is large. So, at the scale of the second minimum, the lattice Λ_{x_n} looks like an evenly spaced union of lines parallel to the shortest vector, with very closed points evenly spaced in these lines. This picture is helpful and shows that the line defined by the shortest vector should play an important role. The subspace H_x defined below could have been defined with the shortest vector of the lattice Λ_x. However as in [11] we use the volume instead of the length because it works in any dimension.

For each x in Q we fix once and for all a co-dimension one sub-lattice of Λ_x of minimal volume and call it Λ'_x. Let $H_x = \pi_x^{-1}H'_x$ where H'_x is the real span of Λ'_x. Thus,

$$\Lambda'_x = \Lambda_x \cap H'_x.$$

The two Lemmas below are easy and proved in [11].

Lemma 13. *Let x and y be in Q. Then, $y \in H_x$ if and only if $\widehat{y} \in \widehat{x} + H'_x$.*

Lemma 14. *Let x and y be in Q. Suppose that $|x| \leq |y|$, $y \in H_x$, and $\|\widehat{x} - \widehat{y}\|_e \leq \frac{4\lambda_1(x)}{|x|}$. Then $\lambda_2(x) \asymp \lambda_2(y)$.*

3.3 The first minimum of Λ_y

In one dimension, when $\widehat{x}_n = \frac{p_n}{q_n}$ and $\widehat{x}_{n+1} = \frac{p_{n+1}}{q_{n+1}}$ are two consecutive convergents of a real number, the unimodular property of the ordinary continued fraction algorithm implies the two equivalent properties:

(i) $\pi_{x_n}(p_{n+1}, q_{n+1}) = p_{n+1} - q_{n+1}\widehat{x}_n$ is one of the two primitive elements of the lattice Λ_{x_n},

(ii) $\pi_{x_{n+1}}(p_n, q_n) = p_n - q_n\widehat{x}_{n+1}$ is a shortest vector of $\Lambda_{x_{n+1}}$.

In higher dimensions, lattices have infinitly many primitive elements. So, a priori, given two consecutive best approximation vectors x and $y \in Q$ there are infinitely many possible primitive elements $\alpha \in \Lambda_x$ that could be the projection $\alpha = \pi_x(y)$. Moreover property (i) no longer imply property (ii). Lemma 15 below give an additional condition which, together with (i), implies (ii).

Given $x \in Q$ and a primitive element α in Λ_x, we let

$$\Lambda_{\alpha^\perp} = \pi_\alpha^\perp(\Lambda_x),$$

where $\pi_\alpha^\perp$ is the orthogonal projection of $\mathbb{R}^2$ onto the subspace $\alpha^\perp$ of vectors of $\mathbb{R}^2$ orthogonal to α.

For any $y \in Q$ such that $\pi_x(y) = \alpha$, the 1-volume of $\Lambda_{\alpha^\perp}$ satisfies

$$\mathrm{vol}(\Lambda_{\alpha^\perp}) = \frac{\mathrm{vol}(\Lambda_x)}{\|\alpha\|_e} = \frac{1}{\|\alpha\|_e\,|x|} = \frac{1}{|x \wedge y|}.$$

Here, the quantity $|y \wedge z|$ is the 2-volume of the orthogonal projection of $y \wedge z \in \Lambda^2\mathbb{R}^3$ onto the subspace spanned by $e_1 \wedge e_3$ and $e_2 \wedge e_3$. Equivalently, (see §2 of [9])

$$|y \wedge z| = |y|\,|z|\,d(\widehat{y}, \widehat{z}).$$

Denote the first minimum of $\Lambda_{\alpha^\perp}$ by $\lambda_1(\alpha)$. The following lemma was proved in [11].

Lemma 15. *Let $x \in Q$ and α be a primitive element of Λ_x. Suppose that y is an element in Q such that $\pi_x(y) = \alpha$. Then $\frac{|x\wedge y|}{|y|} \leq \lambda_1(\alpha)$ implies $\lambda_1(y) = \frac{|x\wedge y|}{|y|} = \|\pi_y(x)\|_e$.*

4 Upper bound for the Hausdorff dimension

Let $\mu' > \mu > 0$ be two real numbers. We want to define a self-similar covering (J, σ, B) of the set $\underline{\mathrm{Sing}}^*(\mu')$. Since the sequence of best approximation vectors $(x_n)_{n\in\mathbb{N}}$ of any $\theta \in \mathbb{R}^2$ converges to θ, it is natural to choose a self similar structure such that all the sequences of best approximations vectors of the $\theta \in \underline{\mathrm{Sing}}^*(\mu')$ are admissible. Moreover, according to Corollary 12, all the best approximation vectors of $\theta \in \underline{\mathrm{Sing}}^*(\mu')$ are in the set

$$Q_\mu = \{x \in Q : \lambda_1(x) \leq |x|^{-\mu}\},$$

hence $J = Q_\mu$ is a natural choice. The maps σ and B are more difficult to defined. Using the extension to higher dimensions, of Legendre's Theorem (Lemma 10) it is tempting to defined the map B with $B(x) = B\left(\widehat{x}, \frac{2\lambda_1(x)}{|x|}\right)$. However, by a result of Jarník [17], if the uniform exponent $\widehat{\omega}_{1,2}(\theta)$ is $\geq \mu$, then the standard exponent of approximation $\omega_{1,2}(\theta)$ is larger than

$$\frac{\mu^2}{1-\mu}.$$

Therefore using subsequences of sequences of best approximation vectors, it should be possible to define the sets $B(x)$ with smaller diameters. The precise definition involves the subspaces H_x defined section 3.2.

Notation.
$E(x) = \{y \in Q_\mu : |y| > |x|\,,\ y \notin H_x,\ \|\pi_y(x)\|_e \leq \frac{1}{|y|^\mu},\ \pi_x(y)$ is primitive in $\Lambda_x\}$,
$D(y) = \{z \in Q_\mu : |z| \geq |y|\,,\ z \in H_y,\ \|\widehat{y} - \widehat{z}\|_e \leq 4\frac{\lambda_1(y)}{|y|}\}$.

Definition 7. We set $\sigma_\mu(x) = \cup_{y \in E(x)} D(y)$ and $B_{\mu,c}(x) = B(\widehat{x}, \frac{c}{(\lambda_2(x)^\mu |x|)^{\frac{1}{1-\mu}}})$.

Remark 4.1. In [11], the roles of D and E were permuted and $\sigma(x)$ was defined as

$$\sigma_\mu(x) = \cup_{y \in D(x)} E(y).$$

Remark 4.2. When $\lambda_1(x) \leq |x|^{-\mu}$, using the second Minkowski Theorem, it is easy to see that the radius of the ball $B_{\mu,c}(x)$ is $\ll$

$$|x|^{-(1+\frac{\mu^2}{1-\mu})}$$

which is precisely what is expected from the result of Jarník quoted above.

Theorem 1 is a consequence of the following two lemmata.

Lemma 16. *When c is large enough, $(Q_\mu, \sigma_\mu, B_{\mu,c})$ is a self-similar covering of $\underline{\mathrm{Sing}}^*(\mu')$ for all $\mu' > \mu$.*

Proof. Let $\theta \in \underline{\mathrm{Sing}}^*(\mu')$ and let $((p_n, q_n))_{n \geq 0}$ be the sequence of best approximations of θ. For $n \geq 0$, set $x_n = (p_n, q_n)$. By Corollary 12 and removing the first best approximation vectors if necessary, we can suppose that $x_n \in Q_\mu$ for all n. Consider a subsequence $(x_{n_i})_{i \geq 0}$ such that for all $i \geq 1$,

$$x_{n_i+1} \notin H_{x_{n_i}},\ x_{n_i+1}, \ldots, x_{n_{i+1}} \in H_{x_{n_i+1}},\ x_{n_{i+1}+1} \notin H_{x_{n_i+1}}.$$

Such a subsequence exists since the sequence $(x_n)_{n \geq 0}$ must leave each subspace H_{x_k}: otherwise the coordinates of the point θ together with 1 would be rationally dependent. Observe that

$$H_{x_{n_i+1}} = H_{x_{n_i+2}} = \ldots = H_{x_{n_{i+1}}} \neq H_{x_{n_{i+1}+1}}.$$

Let i be an integer. Set $x = (p,q) = x_{n_i}$, $y = (u,v) = x_{n_i+1}$ and $z = x_{n_{i+1}}$. We have $y \notin H_x$ and, by Corollary 12,

$$\|q\widehat{y} - p\|_e \leq \frac{1}{|y|^{\mu}}.$$

Since x and y are consecutive best approximation vectors, $\pi_x(y)$ is primitive in Λ_x, hence $y \in E(x)$. Let (e_1, e_2) be a reduced basis of Λ_x and $\alpha = \pi_x(y)$. Since $y \notin H_x$ we have $\alpha = ae_1 + be_2$, where b is a nonzero integer. We have

$$\frac{\|\alpha\|_e\,|x|}{|y|} = \frac{|x \wedge y|}{|y|} = \|q\widehat{y} - p\|_e \leq |y|^{-\mu},$$

hence

$$|y| \geq (\|\alpha\|_e\,|x|)^{\frac{1}{1-\mu}}$$

and

$$\frac{|y|}{|x|} \geq (\|\alpha\|_e\,|x|^{\mu})^{\frac{1}{1-\mu}}.$$

It follows that $y = \alpha + kx$, where the real number k satisfies $|k| \geq (\|\alpha\|_e\,|x|^{\mu})^{\frac{1}{1-\mu}}$. Moreover,

$$\widehat{y} = \widehat{x} + \frac{\alpha}{|y|}.$$

Since $\|\alpha\|_e \gg \lambda_2(x)$, we get

$$\begin{aligned} d(\widehat{x}, \widehat{y}) &\ll \frac{\|\alpha\|_e}{(\|\alpha\|_e\,|x|)^{\frac{1}{1-\mu}}} = \frac{1}{(\|\alpha\|_e^{\mu}\,|x|)^{\frac{1}{1-\mu}}} \\ &\ll \frac{1}{(\lambda_2(x)^{\mu}\,|x|)^{\frac{1}{1-\mu}}}. \end{aligned}$$

Furthermore, $\theta \in B(\widehat{y}, \frac{2\lambda_1(y)}{|y|})$ and

$$\begin{aligned} \frac{\lambda_1(y)}{|y|} &\ll \frac{1}{|y|^{1+\mu}} \leq \frac{1}{(\|\alpha\|_e\,|x|)^{\frac{1+\mu}{1-\mu}}} \\ &\ll \frac{1}{(\lambda_2(x)^{\mu}\,|x|)^{\frac{1}{1-\mu}}} \times \frac{1}{(\lambda_2(x)\,|x|^{\mu})^{\frac{1}{1-\mu}}}. \end{aligned}$$

Since $\mu \geq \frac{1}{2}$, we deduce from Minkowski's Theorem that

$$\lambda_2(x)\,|x|^{\mu} \geq \lambda_2(x)\,|x|^{1-\mu} \gg 1,$$

which implies that θ is in $B(\widehat{x}, \frac{c}{(\lambda_2(x)^{\mu}|x|)^{\frac{1}{1-\mu}}})$ when c is large enough. The last thing to check is that $z \in D(y)$, but this follows from Lemma 11 (ii). □

It appears that in some cases, it is better to use a larger radius for the balls $B_{c,\mu}$. This observation has already been done in [11]. Since $\lambda_2(x) \gg |x|^{\mu-1}$ for $x \in Q_\mu$, a convex interpolation between the exponents of $\lambda_2(x)$ and $|x|^{\mu-1}$ yields

Corollary 17. *For $\gamma \in [0,1]$ and $x \in Q_\mu$ set*

$$B_{\mu,\gamma}(x) = B(x) = B\Big(\widehat{x}, \frac{c}{(\lambda_2(x)^{(1-\gamma)\mu} |x|^{(\mu-1)\mu\gamma+1})^{\frac{1}{1-\mu}}}\Big).$$

When c is large enough, $(Q_\mu, \sigma_\mu, B_{\mu,\gamma})$ is a self-similar covering of $\underline{\mathrm{Sing}}^*(\mu')$ for all $\mu < \mu'$.

Lemma 18. *Let a and b be real numbers with $b > 2$ and $\frac{b-1}{1-\mu} - a > 2$. Then, for $x \in Q_\mu$ with $|x|$ large enough, we get*

$$\sum_{z\in\sigma_\mu(x)} \frac{1}{\lambda_2(z)^a |z|^b} \ll \frac{1}{\lambda_2(x)^A |x|^B},$$

where $A = \frac{b-1}{1-\mu} - a - 2$ and $B = \mu\frac{b-1}{1-\mu} - a - 1 + b$.

Proof. **Step 1.** For $z \in D(y)$, we have $\lambda_2(z) \asymp \lambda_2(y)$ because $z \in H_y$. It follows that

$$\begin{aligned} S_1(y) &= \sum_{z\in D(y)} \frac{1}{\lambda_2(z)^a |z|^b} \\ &\asymp \sum_{z\in D(y)} \frac{1}{\lambda_2(y)^a |z|^b}. \end{aligned}$$

Since the number of elements of

$$D_k(y) = \{z \in D(y) : k\,|y| \le |z| < (k+1)\,|z|\}$$

is $\ll k$, we have

$$\begin{aligned} S_1(y) &\asymp \sum_{k\ge1} \sum_{z\in D_k(y)} \frac{1}{\lambda_2(y)^a |z|^b} = \frac{1}{\lambda_2(y)^a |y|^b} \sum_{k\ge1} \sum_{z\in D_k(y)} \Big(\frac{|y|}{|z|}\Big)^b \\ &\ll \frac{1}{\lambda_2(y)^a |y|^b} \sum_{k\ge1} \frac{1}{k^{b-1}}. \end{aligned}$$

Since $b > 2$, we get

$$S_1(y) \ll \frac{1}{\lambda_2(y)^a |y|^b}.$$

Step 2. By the definition of $\sigma_\mu(x)$ and by step 1, we have

$$\begin{aligned} S(x) &= \sum_{z\in\sigma_\mu(x)} \frac{1}{\lambda_2(z)^a |z|^b} = \sum_{y\in E(x)} \sum_{z\in D(y)} \frac{1}{\lambda_2(z)^a |z|^b} \\ &\ll \sum_{y\in E(x)} \frac{1}{\lambda_2(y)^a |y|^b} \\ &= \sum_{\substack{\alpha\in\Lambda_x\setminus H'_x \\ \alpha \text{ primitive}}} \sum_{y\in E(x)\,:\,\pi_x(y)=\alpha} \frac{1}{\lambda_2(y)^a |y|^b}. \end{aligned}$$

By the definition of $E(x)$, if $y \in E(x)$, then we have $\|\pi_y(x)\|_e \leq \frac{2}{|y|^\mu}$ and

$$\|\alpha\|_e = \frac{|x \wedge y|}{|x|} = \frac{\|p - q\widehat{y}\|_e\,|y|}{|x|} = \frac{\|\pi_y(x)\|_e\,|y|}{|x|},$$

hence

$$\|\alpha\|_e\,|x| \leq 2\,|y|^{1-\mu}$$

and

$$\frac{|y|}{|x|} \geq (\frac{1}{2}\,\|\alpha\|_e\,|x|^\mu)^{\frac{1}{1-\mu}}.$$

Since $\frac{|x|\|\alpha\|_e}{|y|} = \|\pi_y(x)\|_e \geq \lambda_1(y)$, we deduce from Minkowski's Theorem that

$$\lambda_2(y) \gg \frac{1}{\|\alpha\|_e\,|x|}$$

holds for all $y \in E(x)$ such that $\pi_x(y) = \alpha$. Call $\lambda_1(\alpha)$ the first minimum of the orthogonal projection of Λ_x on the line orthogonal to α. By Lemma 15, if $\frac{|x\wedge y|}{|y|} < \lambda_1(\alpha)$ then $\lambda_1(y) = \frac{|x\wedge y|}{|y|} = \frac{|x|\|\alpha\|_e}{|y|}$, which implies that $\lambda_2(y) \asymp \frac{1}{\|\alpha\|_e|x|}$. Now $\lambda_1(\alpha) = \frac{1}{\|\alpha\|_e|x|}$ and $\mu > \frac{1}{2}$, hence, for $|x|$ large enough,

$$\begin{aligned}
|y| > (\frac{1}{2}\,\|\alpha\|_e\,|x|)^{\frac{1}{1-\mu}} &\Rightarrow |y| > (\|\alpha\|_e\,|x|)^2 \\
&\Rightarrow \frac{\|\alpha\|_e\,|x|}{|y|} < \frac{1}{\|\alpha\|_e\,|x|} \\
&\Rightarrow \frac{|x \wedge y|}{|y|} < \lambda_1(\alpha).
\end{aligned}$$

It follows that

$$\begin{aligned}
S &\ll \sum_{\alpha\in\Lambda_x\setminus H'_x}\ \sum_{y\in E(x):\pi_x(y)=\alpha} \frac{1}{(\frac{1}{|x|\|\alpha\|_e})^a\,|y|^b} \\
&\asymp \sum_{\alpha\in\Lambda_x\setminus H'_x}\ \sum_{k\geq(\frac{1}{2}\|\alpha\|_e|x|^\mu)^{\frac{1}{1-\mu}}} \frac{(|x|\,\|\alpha\|_e)^a}{|x|^b\,k^b} \\
&\ll \sum_{\alpha\in\Lambda_x\setminus H'_x} \frac{(|x|\,\|\alpha\|_e)^a}{|x|^b\,(\frac{1}{2}\,\|\alpha\|_e\,|x|^\mu)^{\frac{b-1}{1-\mu}}} \\
&\ll \sum_{\|\alpha\|_e\geq\lambda_2(x)} \frac{1}{|x|^{\mu\frac{b-1}{1-\mu}-a+b}\,\|\alpha\|_e^{\frac{b-1}{1-\mu}-a}}.
\end{aligned}$$

Now $\frac{b-1}{1-\mu} - a > 2$ if $s > \frac{3-2\mu}{1-\mu+\mu^2}(1-\mu)$. Therefore, by Lemma 2.5 of [11],

$$\begin{aligned}
S &\ll \frac{1}{|x|^{\mu\frac{b-1}{1-\mu}-a+b}\,\mathrm{vol}\,\Lambda_x\lambda_2(x)^{\frac{b-1}{1-\mu}-a-2}} \\
&= \frac{1}{\lambda_2(x)^A\,|x|^B},
\end{aligned}$$

where $B = \mu\frac{b-1}{1-\mu} - a - 1 + b$ and $A = \frac{b-1}{1-\mu} - a - 2$. □

Completion of proof of Theorem 1. Let μ_0 be in $(\frac{1}{2}, 1)$.

Case 1. Assume that $\mu_0 > \frac{1}{\sqrt{2}}$. By Lemma 16, $(Q_\mu, \sigma_\mu, B_\mu)$ is a self-similar covering of $\mathrm{Sing}^*(\mu_0)$ for all μ such that $\frac{1}{\sqrt{2}} < \mu < \mu_0$. Let $s > 2(1-\mu)$. Set $t = \frac{s}{1-\mu}$, $a = \mu t$ and $b = t$. For $x \in Q_\mu$, set

$$S(x) = \sum_{z \in \sigma_\mu(x)} (\operatorname{diam} B(z))^s.$$

With these notations, $(\operatorname{diam} B(x))^s = \frac{c^s}{\lambda_2(x)^a |x|^b}$ for all $x \in \mathbb{Q}_\mu$, hence by the above Lemma, we have

$$\frac{S(x)}{(\operatorname{diam} B(x))^s} \ll \frac{1}{\lambda_2(x)^{A-a} |x|^{B-b}}.$$

Straightforward calculations give

$$\begin{aligned} A - a &= \frac{b-1}{1-\mu} - 2a - 2 \\ &= \frac{1}{1-\mu}\big(t(1 - 2\mu + 2\mu^2) + 2\mu - 3\big) \end{aligned}$$

and

$$\begin{aligned} B - b &= \mu\frac{b-1}{1-\mu} - a - 1 \\ &= \frac{t\mu^2 - 1}{1-\mu}. \end{aligned}$$

By assumption $t > 2$ and $\mu^2 > \frac{1}{2}$, so $B - b$ is positive. If $A - a < 0$, then $S(x) \leq (\operatorname{diam} B(x))^s$ when $|x|$ is large enough. Otherwise we use that $\lambda_2(x) \gg |x|^{\mu-1}$ and we get

$$\frac{1}{\lambda_2(x)^{A-a} |x|^{B-b}} \ll \frac{1}{|x|^C},$$

with

$$\begin{aligned} C &= (\mu - 1)(A - a) + (B - b) \\ &= \frac{2\mu - 1}{1-\mu}\big(t(1 - \mu + \mu^2) + \mu - 2\big) \\ &> \frac{2\mu - 1}{1-\mu}(-\mu + 2\mu^2) > 0. \end{aligned}$$

We conclude that $S(x) \leq (\operatorname{diam} B(x))^s$ when $|x|$ is large enough. Therefore, by Theorem 6,

$$\dim_H \mathrm{Sing}^*(\mu_0) \leq s$$

and since this holds for all $s > 2(1-\mu)$ and all $\frac{1}{\sqrt{2}} < \mu < \mu_0$, we obtain

$$\dim_H \mathrm{Sing}^*(\mu_0) \leq 2(1-\mu_0).$$

Case 2. Assume that $\mu_0 < \frac{1}{\sqrt{2}}$. We use Corollary 17 instead of Lemma 16 with $\mu < \mu_0$ and a suitable choice of γ. Set $t = \frac{s}{1-\mu}$, $a = (1-\gamma)\mu t$ and $b = (1+\gamma(\mu-1)\mu)t$. The idea is to find a value of γ such that the constraints

$$b > 2,\ \frac{b-1}{1-\mu} - a > 2,\ B - b > 0$$

are satisfy with t minimal. This leads to the value $\gamma = \frac{1-2\mu^2}{\mu(1-\mu)(3-2\mu)}$. In fact with the value $t = \frac{3-2\mu}{1-\mu+\mu^2}$ we find $b = 2$, $\frac{b-1}{1-\mu} - a = 2$, and $B - b = 0$. It follows that if $t > \frac{3-2\mu}{1-\mu+\mu^2}$ the three strict inequalities hold. The last thing to check is that with this value of γ and $t > \frac{3-2\mu}{1-\mu+\mu^2}$ we have $A - a \leq 0$. Now, if $t = \frac{3-2\mu}{1-\mu+\mu^2}$ we have $A - a = \frac{1}{\mu-1}(2\mu-1) < 0$, hence $A - a < 0$ for t close to $\frac{3-2\mu}{1-\mu+\mu^2}$ which implies that $S(x) \leq 1$ for $|x|$ large enough. □

5 Lower bounds for the Hausdorff dimension: tools

5.1 The counting/diameter function

We will use Theorem 7 when all the diameters of the sets $B(z)$, $z \in \sigma(x)$, have the same order. In that case we can replace the sums $\sum_{z\in\sigma_F(x)} (\operatorname{diam} B(z))^s$ in condition (iii) of Theorem 7 by an equivalent sum

$$(\operatorname{diam} B(z))^s \times \operatorname{card}\{z \in \sigma(x) : B(z) \cap F \neq \emptyset\}.$$

So we are reduced to bound $\operatorname{card}\{z \in \sigma(x) : B(z) \cap F \neq \emptyset\}$ from above with $(\operatorname{diam} F)^s$. This will be done when the $z \in \sigma(x)$ are on line segments through some points in almost lattice positions. The next lemma allows us to bound from above $\sum_{z\in\sigma_F(x)} \frac{(\operatorname{diam} B(z))^s}{(\operatorname{diam} F)^s}$ in such a situation.

Definition 8. Let $C_0 \geq 1$, $H > 0$ and $V > 0$ be real numbers. A C_0-distorted $H \times V$-tiling of a subset $\mathcal{B}$ in $\mathbb{R}^2$ is a finite collection of subsets $\mathcal{R}_i$, $i \in I$, such that

1. each $\mathcal{R}_i$ is included in $\mathcal{B}$,
2. the intersection of $\mathcal{R}_i$ and $\mathcal{R}_j$ has measure zero for all $i \neq j$,
3. each $\mathcal{R}_i$ contains a rectangle of horizontal length $\frac{1}{C_0}H$ and of vertical length $\frac{1}{C_0}V$,
4. each $\mathcal{R}_i$ is contained in a rectangle of horizontal length $C_0 H$ and of vertical length $C_0 V$.

Assumptions of Lemma 19. Let $C_0 \geq 1$ be a real number, let $R_0 > R_1 > R_2 > R_3$ and H, V be real numbers such that

$$\frac{R_0}{C_0} \geq H, \quad V \geq \frac{R_1}{C_0},$$

and let $\mathcal{E}$ be a finite subset of $\mathbb{R}^2$. Assume that $(\mathcal{R}_y)_{y\in\mathcal{E}}$ is a C_0-distorted $H\times V$ tiling of the ball $B(x,R_0)$ such that each set $\mathcal{R}_y$ contains the corresponding y of $\mathcal{E}$. Furthermore assume that, for each $y\in\mathcal{E}$, the ball $B(y,R_1)$ contains a set of balls $B(z_1,R_3),\dots,B(z_{k_y},R_3)$, $k_y\le\lfloor\frac{2R_1}{R_2}\rfloor$, which are disjoint and whose centers z_i are in a same line going through y, the distance between consecutive centers being at least R_2. Call $\mathcal{D}_y$ the set of all the z_i and set

$$\mathcal{S}=\cup_{y\in\mathcal{E}}\mathcal{D}_y.$$

Lemma 19. *Set* $f(r)=\max_{a\in\mathbb{R}^2}\frac{\operatorname{card}\mathcal{S}\cap B(a,r)}{r^s}$.

1. *If* $1\le s\le 2$, *then*

$$\max_{R_3\le r\le R_0} f(r)\le 72C_0^4\max\Big\{\frac{1}{R_3^s},\frac{R_1R_0^2}{VHR_2}\times\frac{1}{R_0^s}\Big\}.$$

2. *If* $s<1$, *then*

$$\max_{R_3\le r\le R_0} f(r)\le 72C_0^4\max\Big\{\frac{1}{R_3^s},\frac{R_1}{R_2R_1^s},\frac{R_1R_0^2}{VHR_2}\times\frac{1}{R_0^s}\Big\}.$$

Proof. We can assume that $V\le H$.

Observe first that a $4C_0H\times 4C_0V$ rectangle can meet at most $36C_0^4$ tiles $\mathcal{R}_y$ because the union of all these tiles is included in a $6C_0H\times 6C_0V$ rectangle and these tiles have an area at least equal to $C_0^{-2}HV$. Next, if a ball $B(a,r)$ meets a ball $B(y,R_1)$ with $y\in\mathcal{E}$, then the ball $B(a,r+R_1)$ meets the tile $\mathcal{R}_y$. Since a ball $B(a,r+R_1)$ with $r\le C_0V$ is included in a $4C_0H\times 4C_0V$ rectangle, it follows that a ball $B(a,r)$ with $r\le R_1$ meets at most $36C_0^4$ balls $B(y,R_1)$, $y\in\mathcal{E}$.

Case 1. $R_3\le r\le R_2$.
Since, for a given y in $\mathcal{E}$, a ball $B(a,r)$ contains at most two points z in $\mathcal{D}_y$, by the above observation we have

$$f(r)\le 72C_0^4\times r^{-s}=g(r),$$

which is a decreasing function of r.

Case 2. $R_2\le r\le R_1$.
Since, for a given y in $\mathcal{E}$, a ball $B(a,r)$ contains at most $\frac{2r}{R_2}$ points z in $\mathcal{D}_y$, by the above observation we have

$$f(r)\le\frac{36C_0^4}{r^s}\times\frac{2\times r}{R_2}=72C_0^4\times\frac{r^{1-s}}{R_2}=g(r),$$

which is an increasing function of r if $s\le 1$, and a decreasing function otherwise.

Case 3. $R_1\le r\le C_0V$.
By the above observation we have

$$f(r)\le 36C_0^4\times\frac{2R_1}{R_2}\times r^{-s}=g(r),$$

which is a decreasing function of r.

Case 4. $C_0V \le r \le C_0H$.

We need first to refine the above observation. A $2(r+R_1)\times 2(r+R_1)$ square is included in a $4C_0H \times \frac{4r}{C_0V}C_0V$ rectangle and all the tiles meeting this rectangle are included in a $(\frac{4r}{C_0V}+2)C_0V \times 6C_0H$ rectangle. It follows that the $2(r+R_1)\times 2(r+R_1)$ square meets at most

$$(\frac{6r}{C_0V}\times 6)C_0^2\frac{VH}{C_0^{-2}VH} = 36C_0^4 \times \frac{r}{C_0V}$$

tiles $\mathcal{R}_y$. Hence

$$f(r) \le 36C_0^4 \times \frac{r}{C_0V}\times\frac{2R_1}{R_2}\times r^{-s} = 72C_0^3\frac{R_1}{VR_2}r^{1-s} = g(r),$$

which is an increasing function of r if $s \le 1$, and a decreasing function otherwise.

Case 5. $C_0H \le r \le R_0$.

The number of tiles meets by $2(r+R_1)\times 2(r+R_1)$ square is at most $\frac{36C_0^2r^2}{HV}$, hence

$$f(r) \le \frac{\frac{36C_0^2r^2}{HV}\times\frac{2R_1}{R_2}}{r^s} = 72C_0^2 \times \frac{R_1}{VHR_2}r^{2-s} = g(r)$$

which is a decreasing function of r.

Conclusion.

If $s \ge 1$, then $f(r) \le g(r) \le \max(g(R_3), g(R_0)) \le 72C_0^4 \max\{\frac{1}{R_3^s}, \frac{R_1R_0^2}{VHR_2}\times\frac{1}{R_0^s}\}$.

If $s \le 1$, the maximum of g might be reach in $r = R_1$. □

The above lemma will be used with an s chosen so that $\operatorname{card}\mathcal{S}\times R_3^s \ge R_0^s$. Thanks to Theorem 7, it gives a lower bound for the Hausdorff dimension of the image of Ω_{x_0} when $s \ge 1$. If $s \le 1$, it will be necessary to check that

$$1 \gg \frac{g(R_0)}{g(R_1)}.$$

5.2 A first step in the definition of the self-similar structure: definition of σ and of Q_σ

Let $\mu > \frac{1}{2}$ be fixed. We want to define a self-similar structure (J, σ, B) that covers a subset of $\underline{\mathrm{Sing}}^*(\mu)$. In this subsection we only define J and σ.

We denote by $c_1, c_2, \dots$ some constants that will be chosen later. These constants might depend on μ. The constants involved in $\ll$, $\gg$, or in $\asymp$ depends only on μ but not on $c_1, c_2, \dots$

For each x in Q let $u_1 = u_1(x), u_2 = u_2(x)$ be a reduced basis of Λ_x (by reduction, we mean the Gauss reduction). The vector subspace H'_x is spanned by u_1 and x (see the definition of H_x in Section 3).

Let $E_1(x)$ be the set of $y = \alpha + kx \in Q$ with $\alpha = \alpha_m = \pi_x(y) = mu_1 + u_2$ in the "first level" $u_2 + H'_x$, $\|mu_1\|_e \leq \lambda_2(x)$ and $|y| \in (\|\alpha\|_e |x|)^{\frac{1}{1-\mu}}[c_0, 2c_0]$ where $c_0 = 32^{\frac{1}{1-\mu}}$. This value of c_0 will be used in the proof of Proposition 26.

Fix b a positive real number. Let y be in $E_1(x)$. Let $D_1(y)$ be the set of z in Q such that $|z| \geq |y|\,,\ z \in H_y$,

$$\frac{1}{2}|y|^b \leq \frac{|z|}{|y|} \leq |y|^b\,,$$
$$\|\widehat{y} - \widehat{z}\|_e \leq c_1 \frac{\lambda_1(y)}{|y|},$$

where $c_1 \leq \frac{1}{4}$ is small enough and will be chosen after Lemma 27.

For each x in Q, set

$$\sigma(x) = \cup_{y \in E_1(x)} D_1(y)$$

and

$$J = Q_\sigma = \bigcup_{x \in Q,\, |x| \geq c} \sigma(x),$$

where c is a constant. The remaining Propositions and Lemmas hold when $|x|$ is large enough, so the constant c will be chosen in order all these results hold.

5.3 A few Calculations

Let x be in Q_σ, y be in $E_1(x)$ and z be in $D_1(y)$. Since $\pi_x(y) = \alpha = mu_1 + u_2$ with $\|mu_1\|_e \leq \lambda_2(x)$, we have $\|\alpha\|_e \asymp \lambda_2(x)$. So if we can evaluate $\lambda_2(x)$, we will then be able to estimate the height of y, the height of z and also $\lambda_1(x)$. However, estimating $\lambda_2(x)$ is not possible directly and we have to estimate $\lambda_1(y)$ first.

5.3.1 Minima of Λ_y

Lemma 20. *Let x be in Q_σ and y in $E_1(x)$. Then $\lambda_1(y) = \frac{|x \wedge y|}{|x|} \asymp |y|^{-\mu}$ and $\lambda_2(y) \asymp |y|^{\mu - 1}$ when $|x|$ is large enough.*

Proof. Let $\lambda_1(\alpha)$ denote the first minimum of the orthogonal projection of Λ_x on the line orthogonal to α. By Lemma 15, if $\frac{|x \wedge y|}{|y|} \leq \lambda_1(\alpha)$, then $\lambda_1(y) = \frac{|x \wedge y|}{|y|} = \|\alpha\|_e \frac{|x|}{|y|}$. Now $\lambda_1(\alpha) = \frac{\det \Lambda_x}{\|\alpha\|_e} = \frac{1}{\|\alpha\|_e |x|} \geq \frac{|x \wedge y|}{|y|}$ is equivalent to

$$|y| \geq (\|\alpha\|_e |x|)^2$$

and, by definition of $E_1(x)$, we get $|y| \geq c_0(\|\alpha\|_e |x|)^{\frac{1}{1-\mu}} \geq (\|\alpha\|_e |x|)^2$ (note that $\|\alpha\|_e |x| > 1$ when $|x|$ is large enough), therefore $\lambda_1(y) = \frac{|x \wedge y|}{|y|}$ when $|x|$ is large enough.

It follows that

$$\lambda_1(y) = \|\pi_y(x)\|_e = \|\alpha\|_e \frac{|x|}{|y|} \asymp (|x|\, \|\alpha\|_e)^{1 - \frac{1}{1-\mu}} \asymp |y|^{-\mu}$$

and, by Minkowski's Theorem,

$$\lambda_2(y) \asymp \frac{1}{|y|\,\lambda_1(y)} \asymp |y|^{\mu-1}.$$

□

5.3.2 Minima of Λ_x and Λ_z

Lemma 21. *Let x be in Q_σ and z in $\sigma(x)$. We have $\lambda_1(z) \asymp |z|^{-\frac{\mu+b}{1+b}}$ and $\lambda_2(z) \asymp |z|^{\frac{\mu-1}{1+b}}$ when $|x|$ is large enough. Consequently, $\lambda_1(x) \asymp |x|^{-\frac{\mu+b}{1+b}}$ and $\lambda_2(x) \asymp |x|^{\frac{\mu-1}{1+b}}$.*

Proof. Let y be in $E_1(x)$ such that z is in $D_1(y)$. By definition of $D_1(y)$, we get $|z| \asymp |y|^{1+b}$. By Lemma 14 and by the definition of $D_1(y)$, and then by Lemma 20, we have

$$\begin{aligned}
\lambda_2(z) &\asymp \lambda_2(y) \asymp |y|^{\mu-1} \asymp |z|^{\frac{\mu-1}{1+b}}, \\
\lambda_1(z) &\asymp |z|^{-1-\frac{\mu-1}{1+b}} = |z|^{-\frac{\mu+b}{1+b}}.
\end{aligned}$$

□

5.3.3 Distance from $\widehat{x}$ to $\widehat{y}$

Lemma 22. *Let x be in Q_σ and y in $E_1(x)$. Then, when $|x|$ is large enough, we have*

$$d(\widehat{x}, \widehat{y}) \asymp |x|^{r_0},$$

where

$$r_0 = -\frac{\mu^2 - \mu + b + 1}{(1-\mu)\,(b+1)}.$$

Proof. Let $\alpha = \pi_x(y)$. Since $y = \alpha + \frac{|y|}{|x|}x$, we get

$$d(\widehat{x}, \widehat{y}) = \frac{\|\alpha\|_e}{|y|} \asymp \frac{\|\alpha\|_e}{(|x|\,\|\alpha\|_e)^{\frac{1}{1-\mu}}} = \frac{1}{(|x|\,\|\alpha\|_e^{\mu})^{\frac{1}{1-\mu}}} \asymp \frac{1}{(|x|\,\lambda_2^{\mu}(x))^{\frac{1}{1-\mu}}}.$$

Therefore, by Lemma 21, we have

$$\begin{aligned}
d(\widehat{x}, \widehat{y}) &\asymp \frac{1}{(|x|\,|x|^{\frac{\mu-1}{1+b}\mu})^{\frac{1}{1-\mu}}} \\
&\asymp \frac{1}{|x|^{\frac{1}{1-\mu}(1+\frac{\mu(\mu-1)}{1+b})}} = |x|^{r_0}.
\end{aligned}$$

□

5.3.4 Growth rate of $|y|$

Lemma 23. *Let x be in Q_σ and y in $E_1(x)$. Then, when $|x|$ is large enough, we get*

$$|y| = |x|^{e_y},$$

where

$$e_y = \frac{\mu + b}{(1-\mu)(1+b)}.$$

Proof. By the definition of $E_1(x)$ and by Lemma 21, we have

$$|y| \asymp (|x|\,\lambda_2(x))^{\frac{1}{1-\mu}} \asymp (|x|\,|x|^{\frac{\mu-1}{1+b}})^{\frac{1}{1-\mu}}.$$

□

5.4 A nested self-similar structure

We want to define a self-similar structure (J, σ, B). Since $J = Q_\sigma$ and σ have already been defined, it remains only for us to define the map B.

5.4.1 Definition of $B(x)$

For each $x \in Q_\sigma$, set

$$B(x) = B(\widehat{x}, c_2\,|x|^{r_0}),$$

where the constant r_0 is defined in Lemma 22. The constant c_2 will be chosen in the proof of Lemma 27.

Lemma 24. *For x in Q_σ,*

$$B(x) \subset B\Big(\widehat{x}, \frac{\lambda_1(x)}{2\,|x|}\Big),$$

when $|x|$ large enough, and therefore x is a best approximation vector of all θ in $B(x)$.

Proof. By Lemma 21, $\lambda_1(x) \asymp |x|^{-\frac{\mu+b}{1+b}}$ and

$$-\frac{\mu+b}{1+b} - 1 + r_0 = -\frac{\mu+2b+1}{1+b} + \frac{\mu^2-\mu+b+1}{(1-\mu)\,(b+1)} > 0,$$

Therefore $c_2\,|x|^{r_0} \leq \frac{\lambda_1(x)}{2|x|}$ when $|x|$ is large enough. By Lemma 10, it follows that x is a best approximation vector of all θ in $B(x)$. □

5.4.2 Nestedness

Lemma 25. *Let x be in Q_σ. Then for all $z \in \sigma(x)$, $B(z) \subset B(x)$ and $z \in Q_\sigma$ when $|x|$ is large enough. Moreover,*

$$B(z) \subset B\Big(\widehat{y}, \frac{\lambda_1(y)}{2\,|y|}\Big) \subset B\Big(\widehat{y}, \frac{2\lambda_1(y)}{|y|}\Big) \subset B(x),$$

where y is the element of $E_1(x)$ such that $z \in D_1(y)$.

Proof. By Lemma 22,

$$d(\widehat{x}, \widehat{y}) \ll |x|^{r_0},$$

where $r_0 = -\frac{\mu^2-\mu+b+1}{(1-\mu)(b+1)}$. Moreover, by Lemmas 20 and 23, $\frac{\lambda_1(y)}{|y|} \asymp |x|^{-\frac{(\mu+b)(1+\mu)}{(1-\mu)(1+b)}}$ and

$$r_0 + \frac{(\mu+b)(1+\mu)}{(1-\mu)(1+b)} = \frac{2\mu - 1 + b\mu}{(1-\mu)(1+b)} > 0.$$

Therefore

$$B(\widehat{y}, \frac{2\lambda_1(y)}{|y|}) \subset B(x),$$

for all $y \in E_1(x)$ when $|x|$ is large enough.

Now let y be in $E_1(x)$ and z be in $D_1(y)$. By definition we have $d(\widehat{y}, \widehat{z}) \leq \frac{1}{4}\frac{\lambda_1(y)}{|y|}$, so in order to prove that $B(z) \subset B(\widehat{y}, \frac{\lambda_1(y)}{2|y|}) \subset B(x)$ it is enough to prove that $c_2\,|z|^{r_0} \leq \frac{1}{4}\frac{\lambda_1(y)}{|y|}$.

Since $\lambda_1(y) \asymp |y|^{-\mu}$ (by Lemma 20), we are reduced to check that

$$c_2\,|y|^{(1+b)r_0} \asymp c_2\,|z|^{r_0} \leq \frac{1}{4\,|y|^{1+\mu}},$$

which holds when $|y|$ is large enough because

$$-(1+b)r_0 - (1+\mu) = \frac{1}{1-\mu}\left((2\mu-1)\mu + b\right) > 0.$$

$\square$

Proposition 26. *The self-similar structure (Q_σ, σ, B) is strictly nested and covers a subset of $\underline{\mathrm{Sing}}^*(\mu)$.*

Proof. The nestedness is ensured by the previous Lemma and the fact that $\lim_{n\to\infty} \operatorname{diam} B(x_n) = 0$ for all admissible sequence $(x_n)_{n\geq 0}$ is an immediate consequence of the inequality $|z| > |x|$ for all $z \in \sigma(x)$.

Let $(x_n)_{n\geq 0}$ be an admissible sequence and let θ be the unique point in $\cap_{n\geq 0} B(x_n)$. We have to show that $\theta \in \underline{\mathrm{Sing}}(\mu)$ and that $\mathbb{Z}\theta + \mathbb{Z}^2$ is everywhere dense in $\mathbb{R}^2$. Let Q be an integer. We want to prove that there exists an integer $q \leq Q$ such that

$$d(q\theta, \mathbb{Z}^2) \leq Q^{-\mu}.$$

Let n be the integer defined by $|x_n| \le Q < |x_{n+1}|$. Set $x = x_n$. By the definition of σ, there exists $y \in E_1(x)$ such that $z = x_{n+1} \in D_1(y)$.

Case 1: $|x_n| \le Q < |y|$.

By the above lemma $\theta \in B(z) \subset B(\widehat{y}, \frac{\lambda_1(y)}{2|y|})$, hence y is a best approximation vector of θ. By Lemma 11 (iii), for all (p,q) in $\mathbb{Z}^2$ with $0 < q < |y|$, we have

$$\|p - q\theta\|_e \le 2\,\|p - q\widehat{y}\|_e .$$

Now by the definition of $E_1(x_n)$ we have $|y| \ge c_0(\|\alpha\|_e\,|x_n|)^{\frac{1}{1-\mu}}$ where $\alpha = \pi_x(y)$. This implies that

$$\|\widehat{y} - \widehat{x}\|_e \le \frac{|y|^{-\mu}}{c_0^{1-\mu}\,|x|}$$

and therefore, by Lemma 11 (iii), the constant c_0 can be chosen large enough so that

$$\|p - q\theta\|_e \le 2\,\|p - q\widehat{y}\|_e \le |y|^{-\mu} \le Q^{-\mu}.$$

where $x = (p,q)$.

Case 2: $|y| \le Q < |z|$.

Since $d(\widehat{z}, \widehat{y}) \le \frac{1}{4}\frac{\lambda_1(y)}{|y|} \le \frac{\lambda_1(y)}{2|y|}$, by Lemma 10, y is a best approximation vector of $\widehat{z}$. Let $y_0 = y = (p_0,q_0)$, $y_1 = (p_1,q_1), \ldots, y_k = (p_k,q_k) = z$ be all the intermediate best approximation vectors of $\widehat{z}$. Since $\Lambda_z = \pi_z(\mathbb{Z}^3) \subset H'_y + \Lambda'_y$ and since by Lemma 20, $\lambda_1(y)$ is small compared to $\lambda_2(y)$ when $|x|$ is large enough, the intermediate best approximation vectors are all in H_y and $\Lambda_{y_i} \subset \mathcal{H} = H_y + \Lambda_y$, $i = 0, \ldots, k$. It follows that for each $i < k$ we have

$$\lambda_1(y_i)e(\mathcal{H}) = \frac{1}{|y_i|},$$

where $e(\mathcal{H})$ is the distance between two consecutive lines of $\mathcal{H}$. Since $e(\mathcal{H}) \ge \frac{\sqrt{3}}{2}\lambda_2(y)$, and $\lambda_1(y)\lambda_2(y) \ge \frac{1}{|y|}$ (the minima are associated with an Euclidean norm),

$$\lambda_1(y_i) = \frac{1}{e(\mathcal{H})\,|y_i|} \le \frac{2}{\sqrt{3}}\frac{\lambda_1(y)\,|y|}{|y_i|} \le 2\frac{1}{c_0^{1-\mu}}\,|y|^{-\mu}\,\frac{|y|}{|y_i|} \le 2\frac{1}{c_0^{1-\mu}}\,|y_i|^{-\mu}.$$

Hence, by Lemma 11 (i) and (iii), for $|y_{i-1}| \le Q < |y_i|$, $i = 1, \ldots, k$,

$$\begin{aligned} d(\{\widehat{z}, \ldots, Q\widehat{z}\}, \mathbb{Z}^2) &\le 2d(\{\widehat{y}_i, \ldots, Q\widehat{y}_i\}, \mathbb{Z}^2) \\ &\le 2\,\|q_{i-1}\widehat{y}_i - p_{i-1}\|_e \le 8\lambda_1(y_i) \\ &\le 16\frac{1}{c_0^{1-\mu}}\,|y_i|^{-\mu} \le 16\frac{1}{c_0^{1-\mu}}Q^{-\mu}. \end{aligned}$$

Now, by Lemma 24, z is a best approximation vector of θ, hence

$$d(\{\theta, \ldots, Q\theta\}, \mathbb{Z}^2) \le 2d(\{\widehat{z}, \ldots, Q\widehat{z}\}, \mathbb{Z}^2) \le 32\frac{1}{c_0^{1-\mu}}Q^{-\mu},$$

which is equal to $Q^{-\mu}$ by the choice of c_0.

It remains to see why $\mathbb{Z}\theta + \mathbb{Z}^2$ is everywhere dense. This simply follows from the fact that $e(\mathcal{H}) \le \lambda_2(y)$ and that $\lambda_2(y)$ tends to 0 for $y \in E_1(x_n)$ when n goes to infinity. □

5.5 A distorted tiling associated with the set of $\widehat{y}$, for y in $E_1(x)$.

Let x be in $\mathbb{Q}_\sigma$ and let $u_1,\ u_2$ be the reduced basis of Λ_x. For each y in $E_1(x)$ there are unique integers m and a, and $0 \le r < 1$ such that $y = mu_1 + u_2 + (a+r)x$. For given integers m and a, consider the trapezoid $T(m,a)$ with extreme points

$$\widehat{x} + \frac{mu_1 + u_2}{a\,|x|},\ \widehat{x} + \frac{(m+1)u_1 + u_2}{a\,|x|},\ \widehat{x} + \frac{mu_1 + u_2}{(a+1)\,|x|},\ \widehat{x} + \frac{(m+1)u_1 + u_2}{(a+1)\,|x|}.$$

For $y = mu_1 + u_2 + (a+r)x$ in $E_1(x)$ we set

$$\mathcal{R}_y = \mathcal{R}_{\widehat{y}} = T(m,a).$$

Let $\mathcal{E}(x)$ denote the set of $\widehat{y}$, y in $E_1(x)$.

Lemma 27. *There exists a constant C_0 such that for all x in Q_σ with $|x|$ large enough and c_2 large enough, the collection $\mathcal{R}_{\widehat{y}}$, $\widehat{y}$ in $\mathcal{E}(x)$, is a C_0-distorted $H \times V$-tiling of $B(x)$ with*

$$H = |x|^h = |x|^{-\frac{(2-\mu)(b+\mu)}{(1-\mu)(b+1)}}$$
$$V = |x|^v = |x|^{-\frac{(1+\mu)(b+\mu)}{(1-\mu)(b+1)}}.$$

Moreover, every $\widehat{y}$ in $\mathcal{E}(x)$ lies on a vertical side of $\mathcal{R}_y$ and the minimal distance $\rho(x)$ between two elements in $\mathcal{E}(x)$ is $\gg V \asymp \frac{\lambda_1(y)}{|y|}$.

Proof. Observe that the trapezoid $T(m,a)$ lies between the vertical lines

$$V_m = \widehat{x} + \mathbb{R}(mu_1 + u_2),\ \text{and}\ V_{m+1} = \widehat{x} + \mathbb{R}((m+1)u_1 + u_2)$$

and between the horizontal lines

$$H_a = \widehat{x} + \frac{u_2}{a\,|x|} + \mathbb{R}u_1,\ \text{and}\ H_{a+1} = \widehat{x} + \frac{u_2}{(a+1)\,|x|} + \mathbb{R}u_1.$$

Therefore, the trapezoids $\mathcal{R}_{\widehat{y}}$ have intersections of Lebesgue measure zero. Observe that for y in $E_1(x)$, by Lemma 23, $a \asymp \frac{|y|}{|x|} \asymp |x|^{e_y - 1}$. On the one hand, by definition of $E_1(x)$ and Lemma 20, the distance between two consecutive horizontal lines is

$$\asymp \frac{\lambda_2(x)}{a^2\,|x|} \asymp \frac{|x|\,\lambda_2(x)}{|y|^2} \asymp \frac{|x \wedge y|}{|y|^2} = \frac{\lambda_1(y)}{|y|} \asymp |y|^{-1-\mu} \asymp |x|^v.$$

On the other hand, the distance between the two vertical segments of $T(m,a)$ is $\asymp \frac{\lambda_1(x)}{|y|}$ which is $\asymp |x|^h$ by Lemmas 23 and 21. Since $h > v$, $\operatorname{diam} \mathcal{R}_{\widehat{y}} \asymp H$ and since $h < r_0$, we see that all the trapezoids $\mathcal{R}_{\widehat{y}}$ are included in $B(x)$ when c_2 is large enough.

Since $\widehat{y} = \widehat{x} + \frac{mu_1+u_2}{(a+r)|x|}$, $\widehat{y}$ is in the left vertical side of $\mathcal{R}_{\widehat{y}}$ and the nearest element of $\mathcal{E}(x)$ is in the same vertical line at a distance $\asymp V$. Therefore $\rho(x) \asymp \frac{\lambda_1(y)}{|y|}$. □

5.5.1 Choice of the constants c, c_1 and c_2

The constant c_2 is chosen according to Lemma 27. With this choice, we determine the constant c_1. Since $\rho(x) \asymp \frac{\lambda_1(y)}{|y|}$, it is possible to take c_1 small enough in order that for all z in $D_1(y)$,

$$d(\widehat{y}, \widehat{z}) \leq c_1 \frac{\lambda_1(y)}{|y|} \leq \frac{1}{4}\rho(x).$$

The choice of the constant c involved in the definition of Q_σ is done at the very end taking all the "$|x|$ large enough" into account.

5.5.2 Distance between the points $\widehat{z}$ for z in $D_1(y)$

Lemma 28. *Let x be in Q_σ and y be in $E_1(x)$. If $z \neq z'$ are in $D_1(y)$, then*

$$d(\widehat{z}, \widehat{z'}) \geq \frac{\lambda_1(y)}{2\,|y|^{1+2b}} \geq 3 \max_{u \in \sigma(x)} \operatorname{diam} B(u)$$

when $|x|$ is large enough. Hence the balls $B(z)$, $z \in \sigma(x)$, are disjoint.

Proof. Choose a generator u_y of $\Lambda'_y = \Lambda_y \cap H_y$ and y' in $\mathbb{Z}^3 \cap H_y$ such that $\pi_y(y') = u_y$ and $|y'| \leq \frac{1}{2}|y|$. We have $y' = u_y + ry$ with $|r| \leq \frac{1}{2}$ and $\mathbb{Z}^3 \cap H_y = \mathbb{Z}y + \mathbb{Z}y'$.

Let $z = ay' + ky$ be in $Q \cap H_y$. We have $z = au_y + (ar+k)y$, hence

$$\widehat{z} = \frac{a}{(ar+k)} \frac{u_y}{|y|} + \widehat{y},$$

and, since z is primitive, the pair (a,k) is primitive in $\mathbb{Z}^2$.

Now, if $z = ay' + ky$ is in $D_1(y)$ then

$$|a|\,\lambda_1(y) = \|au_y\|_e = \|\pi_y(z)\|_e \leq \frac{1}{4}\frac{\lambda_1(y)}{|y|} \times |z|,$$

hence $|a| \leq \frac{1}{4}\frac{|z|}{|y|}$. Moreover, $|z| \leq |k|\,|y| + \frac{1}{2}|a|\,|y|$, thus

$$|k| \geq \frac{1}{|y|}(|z| - \frac{1}{2}|a|\,|y|) = \frac{|z|}{|y|}(1 - \frac{1}{8}) \geq \frac{1}{2}\frac{|z|}{|y|}.$$

Let $z = ay' + ky$ and $z' = a'y' + k'y$ be two distinct points in $D_1(y)$. We have

$$\begin{aligned} \widehat{z} - \widehat{z'} &= (\frac{a}{(ar+k)} - \frac{a'}{(a'r+k')})\frac{u_y}{|y|} \\ &= (\frac{ak' - a'k}{(ar+k)(a'r+k')})\frac{u_y}{|y|}. \end{aligned}$$

Since (a,k) and (a',k') are primitive, we have $ak' - a'k \neq 0$. It follows that

$$d(\widehat{z}, \widehat{z'}) \geq \frac{\lambda_1(y)}{|y|} \times \frac{1}{\frac{|z|}{|y|} \times \frac{|z'|}{|y|}} \geq \frac{\lambda_1(y)}{|y|^{1+2b}}.$$

It remains to see that for all z,

$$\operatorname{diam} B(z) \leq \frac{1}{3}\frac{\lambda_1(y)}{|y|^{1+2b}} \asymp \frac{1}{|y|^{1+2b+\mu}}.$$

Since $\operatorname{diam} B(z) \asymp |z|^{r_0} \asymp |y|^{(1+b)r_0}$ and since

$$-(1+b)r_0 - (1+2b+\mu) = \frac{(b+\mu)(2\mu-1)}{1-\mu} > 0$$

we have $\operatorname{diam} B(z) \leq \frac{1}{3}\frac{\lambda_1(y)}{|y|^{1+2b}}$.

The last thing to see is that the balls $B(z)$, $z \in \sigma(x)$ are disjoint. Recall that the constant c_1 has been chosen in order that for all z in $D_1(y)$, $d(\widehat{y}, \widehat{z}) \leq \frac{1}{4}\rho(x)$, the minimal distance between two points $\widehat{y}$, where y is in E_1. It follows that the balls $B(z)$ are disjoint provided that

$$\max_{z\in\sigma(x)} \operatorname{diam} B(z) < \frac{1}{4}\rho(x).$$

This latter inequality holds because $\rho(x) \asymp \frac{\lambda_1(y)}{|y|}$ and $\frac{\lambda_1(y)}{|y|^{1+2b}} \geq \operatorname{diam} B(z)$. □

5.5.3 Number of points in $\sigma(x)$

Lemma 29. *Let x be in Q_σ and y in $E_1(x)$. Then*

$$\operatorname{card} D_1(y) \asymp |x|^{2be_y} = |x|^{d_1},$$
$$\operatorname{card} E_1(x) \asymp |x|^{2\frac{\mu-1}{1+b}+e_y} = |x|^{e_1},$$

and

$$\operatorname{card} \sigma(x) \asymp |x|^{n_x},$$

where

$$n_x = \frac{1}{(1-\mu)(b+1)}\left(2b^2 + 2b\mu + b + (2\mu-1)(2-\mu)\right)$$

when $|x|$ is large enough.

Proof. It is not difficult to see that the number of points z in $H_y \cap \mathbb{Z}^3$ such that

$$\|\widehat{y} - \widehat{z}\|_e \leq c_1\frac{\lambda_1(y)}{|y|} \quad \text{and} \quad \frac{1}{2}|y|^b \leq \frac{|z|}{|y|} \leq |y|^b$$

is $\asymp |y|^{2b}$. Indeed, the condition $\|\widehat{y} - \widehat{z}\|_e \leq c_1\frac{\lambda_1(y)}{|y|}$ is equivalent to $\|\pi_y(z)\|_e \leq c_1\frac{|z|}{|y|}\lambda_1(y)$ which means that there are $\asymp \frac{|z|}{|y|} \asymp |y|^b$ possible values for $\alpha = \pi_y(z)$. Moreover, the set of integer points on each of the lines $\pi_y^{-1}(\alpha)$ is a translate of $\mathbb{Z}y$ and therefore there are $|y|^b$ possible z for each α. The fact that many of such z are primitive is less clear; in fact, by Lemma 7.11 of [11], we have

$$\operatorname{card} D_1(y) \asymp |y|^{2b} \asymp |x|^{2be_y}.$$

Clearly,

$$\operatorname{card} E_1(x) \asymp \frac{\lambda_2(x)}{\lambda_1(x)} \times \frac{|y|}{|x|} \asymp \lambda_2^2(x)\, |x|^{e_y} \asymp |x|^{2\frac{\mu-1}{1+b}+e_y} .$$

It then follows that the number of points in $\sigma(x)$ satisfies

$$\operatorname{card} \sigma(x) \asymp |x|^{2be_y} \times |x|^{2\frac{\mu-1}{1+b}+e_y} = |x|^{n_x} ,$$

where

$$\begin{aligned} n_x &= 2\frac{\mu-1}{1+b} + (1+2b)\frac{\mu+b}{(1-\mu)(1+b)} \\ &= \frac{1}{(1-\mu)\,(b+1)} \left(2b^2 + 2b\mu + b + (2\mu-1)\,(2-\mu)\right) . \end{aligned}$$

□

6 Lower bounds for the Hausdorff dimension: proofs

Proof of Theorem 3. Let s be a positive real number. Suppose that the following conditions hold:

- $\sum_{z\in\sigma(x)} (\operatorname{diam} B(z))^s \geq (\operatorname{diam} B(x))^s$,
- $R_1^s \operatorname{card} E_1(x) \gg (\operatorname{diam} B(x))^s$ where $R_1 = \max_{y\in E_1(x)} \frac{\lambda_1(y)}{|y|} \asymp |x|^{-e_y(\mu+1)} \asymp |x|^{r_1}$,

for all x in Q_σ with $|x|$ large enough.

Let us show that such an s is a lower bound for $\dim_H \underline{\mathrm{Sing}}^*(\mu)$. We want to use Theorem 7 with the self-similar structure (Q_σ, σ, B) which is strictly nested and covers a subset of $\underline{\mathrm{Sing}}^*(\mu)$ by Proposition 26. The first condition above is just the first hypothesis of Theorem 7 and the second condition of Theorem 7 is implied by Lemma 28. So it remains to check the last hypothesis of Theorem 7. For this last condition, we use Lemma 19 with $R_0 = c_2\, |x|^{r_0}$ and the sets $\mathcal{B} = B(\widehat{x}, R_0)$, $\mathcal{E} = \widehat{E}_1(x)$, $\mathcal{D}_{\widehat{y}} = \widehat{D}_1(y)$, $\mathcal{S} = \cup_{\widehat{y}\in\mathcal{E}} \mathcal{D}_{\widehat{y}}$, R_1 defined above, and

$$R_2 = c_3\, |x|^{-e_y(\mu+1+2b)} \asymp |x|^{r_2} ,\ R_3 = |x|^{e_z r_0} = |x|^{\frac{\mu+b}{(1-\mu)} r_0} = |x|^{r_3} .$$

Let us first check the inequalities between $R_0, \ldots, R_3, H \asymp |x|^h, V \asymp |x|^v$. Looking at the exponents we find

$$\begin{aligned} r_0 &= -\frac{\mu^2-\mu+b+1}{(1-\mu)\,(b+1)} > h = -\frac{(2-\mu)(b+\mu)}{(1-\mu)(b+1)} > v = r_1 = -\frac{(1+\mu)(b+\mu)}{(1-\mu)(b+1)} > \\ r_2 &= -\frac{(\mu+1+2b)(b+\mu)}{(1-\mu)(b+1)} > r_3 = -\frac{(b+\mu)}{(1-\mu)} \times \frac{\mu^2-\mu+b+1}{(1-\mu)\,(b+1)}, \end{aligned}$$

which show that the assumptions of Lemma 19 about the numbers R_0, R_1, R_2, R_3, H and V are satisfied. Moreover, by Lemma 28 and for c_3 small enough, we have

$$d(z,z') \geq \frac{\lambda_1(y)}{2\,|y|^{1+2b}} \geq R_2,$$

for all $z \neq z'$ in $D_1(y)$. Together with Lemma 27, this imply that all the assumptions of Lemma 19 hold.

With the notations of Lemma 19, we get

$$f(r) \ll \max\Big\{\frac{1}{R_3^s}, \frac{R_1}{R_2R_1^s}, \frac{R_0^2R_1}{VHR_2} \times \frac{1}{R_0^s}\Big\}.$$

By Lemma 27 (or 29), $\frac{R_0^2}{VH} \asymp \operatorname{card} E_1(x)$ and, since $\frac{R_1}{R_2} \asymp |y|^{2b} \asymp \operatorname{card} D_1(y)$, we see that

$$\frac{R_0^2R_1}{VHR_2} \times \frac{1}{R_0^s} \asymp \frac{\operatorname{card}\sigma(x)}{(\operatorname{diam} B(x))^s}.$$

With the first assumption $\sum_{z\in\sigma(x)}(\operatorname{diam} B(z))^s \geq (\operatorname{diam} B(x))^s$ we get

$$\frac{1}{R_3^s} \ll \frac{R_0^2R_1}{VHR_2} \times \frac{1}{R_0^s}.$$

With the second assumption $R_1^s \operatorname{card} E_1(x) \gg (\operatorname{diam} B(x))^s$, we get

$$\frac{R_1}{R_2R_1^s} \asymp \frac{R_0^2}{VH}\frac{1}{R_1^s \operatorname{card} E_1(x)} \times \frac{R_1}{R_2} \ll \frac{R_0^2}{VH}\frac{1}{R_0^s} \times \frac{R_1}{R_2}.$$

Therefore, for all r in $[R_3, R_0]$, we have

$$\frac{\operatorname{card}\sigma(x)\cap B(a,r)}{r^s} \ll \frac{\operatorname{card}\sigma(x)}{(\operatorname{diam} B(x))^s}$$

and so, with $F = B(a,r)$,

$$\sum_{z\in\sigma_F(x)} \frac{(\operatorname{diam} B(z))^s}{(\operatorname{diam} F)^s} \ll \sum_{z\in\sigma(x)} \frac{(\operatorname{diam} B(z))^s}{(\operatorname{diam} B(x))^s}.$$

By applying Theorem 7 we conclude that the Hausdorff dimension of $\underline{\operatorname{Sing}}^*(\mu)$ is at least equal to s.

The idea is now to show that the assumption $\sum_{z\in\sigma(x)}(\operatorname{diam} B(z))^s \geq (\operatorname{diam} B(x))^s$ is more restrictive than the other assumption $R_1^s \operatorname{card} E_1(x) \gg (\operatorname{diam} B(x))^s$.

The condition

$$sr_3 + n_x > sr_0$$

implies the first assumption and the condition

$$sr_1 + e_1 > sr_0$$

implies the second assumption. The first condition is equivalent to $s < \frac{n_x}{r_0 - r_3} = s_1$ and the second is equivalent to $s < \frac{e_1}{r_0 - r_1} = s_2$. Therefore, to prove that s_1 is a lower bound for the Hausdorff dimension of $\underline{\text{Sing}}^*(\mu)$, it is enough to check that $s_1 < s_2$ for all μ in $(\frac{1}{2}, 1)$ and all positive b.

Tedious calculations give

$$s_1(b) = \frac{(1-\mu)\left(2b^2 + 2b\mu + b + (2-\mu)(2\mu-1)\right)}{(b + 2\mu - 1)\left(\mu^2 - \mu + b + 1\right)},$$
$$s_2(b) = \frac{1}{2\mu + b\mu - 1}\left(-2\mu^2 + 5\mu + b - 2\right),$$

and

$$s_2(b) - s_1(b) = (b+\mu)\frac{\left(2\mu^2 - 2\mu + 1\right)b^2 + \left(4\mu^2 - 2\mu\right)b + \mu\left(2-\mu\right)\left(2\mu - 1\right)^2}{\left(b - \mu + \mu^2 + 1\right)\left(b + 2\mu - 1\right)\left(2\mu + b\mu - 1\right),}$$

which is > 0 for μ in $[\frac{1}{2}, 1)$ and $b > 0$. It follows that

$$\dim_H \underline{\text{Sing}}^*(\mu) \geq s_1(b),$$

and the proof is complete. □

Proof of corollaries 2 and 4. Clearly,

$$\lim_{b\to\infty} s_1(b) = 2(1-\mu)$$

It follows that

$$\dim_H \underline{\text{Sing}}^*(\mu) \geq \lim_{b\to\infty} s_1(b) = 2(1-\mu).$$

Next we can compute the derivative of the function $b \mapsto s_1(b)$. The numerator of this derivative is

$$Num(b) = (1-\mu)((2\mu^2 - 1)b^2 + (8\mu^3 - 8\mu^2 + 2)b + (6\mu^4 - 7\mu^3 + 3\mu - 1)).$$

When $\mu > \frac{1}{\sqrt{2}}$, $Num(b)$ is positive for all positive b, hence the maximum of s_1 is reached when b goes to infinity. When $\mu < \frac{1}{\sqrt{2}}$, $Num(b)$ vanishes at the value

$$b_0 = \frac{1}{1 - 2\mu^2}\left(\mu - 4\mu^2 + 4\mu^3 + \sqrt{(1-\mu)^3(2\mu-1)(2\mu - 2\mu^2 + 1)}\right),$$

which is positive. Since $Num(b)$ is negative for b large this implies that $s_1(b_0)$ is larger than the limit at infinity and therefore the Hausdorff dimension exceeds $2(1-\mu)$.

Let us look at the limit when μ tends to $\frac{1}{2}$. With $b = \beta(2\mu - 1)$, we obtain

$$s_1 = s_1(\mu, \beta) = \frac{(1-\mu)(2\beta^2(2\mu-1) + 2\beta\mu + \beta + 2 - \mu)}{(\mu^2 - \mu + 1 + \beta(2\mu - 1))(\beta + 1)}$$

Therefore for all $\beta > 0$,

$$\lim_{\mu \to \frac{1}{2}} \dim_H \underline{\mathrm{Sing}}^*(\mu) \geq \frac{\frac{1}{2}(2\beta + \frac{3}{2})}{\frac{3}{4}(\beta+1)}.$$

Letting β going to infinity this implies

$$\lim_{\mu \to \frac{1}{2}} \dim_H \underline{\mathrm{Sing}}^*(\mu) \geq \frac{4}{3}.$$

□

Proof of Proposition 5. We keep the notation of the proof of Theorem 3. We want to use Lemma 9 with the strictly nested self-similar structure (Q_σ, σ, B) and the map $x \mapsto \widehat{x}$. We need to define the map B'. For x in Q_σ, we set

$$B'(x) = B(\widehat{x}, c_4 \left|x\right|^{-\frac{\mu+1+2b}{1+b}}).$$

Since, for $z \in \sigma(x)$,

$$\left|z\right|^{-\frac{\mu+1+2b}{1+b}} \asymp \left|x\right|^{-e_y(1+\mu+2b)} \asymp \frac{\lambda_1(y)}{2\left|y\right|^{1+2b}} \asymp R_2,$$

by using Lemma 28, we see that the balls $B'(z)$, $z \in \sigma(x)$, are disjoint when c_4 is small enough. Moreover, since R_2 is small compared to $\frac{\lambda_1(y)}{|y|}$, Lemma 25 implies that for all $x \in Q_\sigma$, all $y \in E_1(x)$ and $z \in D_1(y)$, we have

$$B'(z) \subset B(\widehat{y}, \frac{2\lambda_1(y)}{|y|}) \subset B(x) \subset B'(x),$$

hence the third assumption of Lemma 9 holds. The second assumption of this lemma holds because R_3 is small compared to R_2. The fifth assumption, namely

$$\sum_{z \in \sigma(x)} (\mathrm{diam}\, B'(z))^s \geq (\mathrm{diam}\, B'(x))^s,$$

needs to be checked. Since

$$\mathrm{diam}\, B'(x) \asymp (\mathrm{diam}\, B(x))^{\frac{\mu+1+2b}{|r_0|(1+b)}},$$

the fifth assumption holds provided that

$$\frac{\mu+1+2b}{|r_0|(1+b)} s \leq s_1.$$

Therefore,

$$\begin{aligned}\dim_P \underline{\mathrm{Sing}}^*(\mu) &\geq \frac{|r_0|(1+b)}{\mu+1+2b} s_1 \\ &= \frac{(2b^2 + 2b\mu + b + (2-\mu)(2\mu-1))}{(\mu+1+2b)\,(b+2\mu-1)}.\end{aligned}$$

Letting b going to infinity, we obtain

$$\dim_P \underline{\mathrm{Sing}}^*(\mu) \geq 1.$$

□

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