Elementary Partitions of Line Segments in the Plane

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Abstract. Given a set S of line segments in the plane, we introduce a new family of partitions of the convex hull of S called elementary partitions of S. The set of faces of such a partition is a maximal set of disjoint triangles that cut S at, and only at, their vertices. Surprisingly, several properties of point set triangulations extend to elementary partitions. Thus, the number of their faces is an invariant of S. In the same way, if S is in general position, there exists a unique elementary partition of S whose faces are inscribable in circles whose interiors do not intersect S. This partition, called elementary Delaunay partition, is dual to the segment Voronoi diagram. The main result of this paper is that the local optimality which characterizes point set Delaunay triangulations [8] extends to elementary Delaunay partitions. A similar result holds for elementary partitions with same topology as the Delaunay one.

1 Introduction

The Voronoi diagram of a set S of sites in the d-dimensional Euclidean space $\mathcal{E}$ partitions $\mathcal{E}$ into regions, one per site; the region for a site s consists of all points closer to s than to any other site. In very recent years, particular attention has been paid to the study of the segment Voronoi diagram in three dimensions [10], [14], [7], ... However, the topology of this diagram is really known only for a set of three lines [6]. The investigation for the point set Voronoi diagram has been fairly facilitated by the well understanding of its dual, the Delaunay diagram. Recall that, if no $d+1$ points of S are cospherical, the Delaunay diagram of S is the unique triangulation of S whose tetrahedra are inscribable in empty spheres, that is, spheres whose interiors do not intersect S. Among all the triangulations of S, the Delaunay diagram of S has many optimality properties, some of them extending in any dimension [12], [13]. Till now, no such properties have been given, even in the plane, for the dual of the segment Voronoi diagram which has been introduced by Chew and Kedem [3].

In this paper, we introduce a new family of diagrams, called elementary partitions, which decompose the convex hull of a set S of points and line segments in the plane. The set of faces of an elementary partition of S is a maximal set of disjoint triangles such that the vertices of each triangle belong to three distinct sites of S and no other point of the triangle belongs to S. The edges of the elementary partition are the (possibly two-dimensional) connected components of the convex hull of S when the sites and open faces are removed. The aim of this paper is, after having shown that the dual of the segment Voronoi diagram is a particular elementary partition, to characterize by local properties this diagram among the set of elementary partitions.

In the two first sections, we study geometrical and topological properties of elementary partitions. Especially, we show that every edge of an elementary partition of S is incident to two sites of S. It follows that the number of faces and edges of an elementary partition of S is linear with the number of sites of S and is an invariant of S.

In the next section, we prove that there exists one and only one elementary partition of S whose faces are inscribable in empty circles. We show that this partition, called elementary Delaunay partition, is the dual, introduced by Chew and Kedem, of the segment Voronoi diagram.

The point set Delaunay triangulation admits an important local characterization which is used to prove many of its optimality properties: A triangulation is Delaunay if and only if every couple of faces sharing a common edge is in Delaunay position with respect to its four defining sites [8]. The main result of this paper is that this property also characterizes the elementary Delaunay partition among all the elementary partitions of a set of line segments. We also give another local property that characterizes the set of elementary partitions having the same topological structure as the elementary Delaunay partition. These properties enable to test in linear time whether an elementary partition is equal or topologically equivalent to the Delaunay partition. By duality, they also enable to check the correctness of the topological structure constructed by a program that should build a segment Voronoi diagram. For more details on efficient program checkers in computational geometry see, for example, [4] and [9].

2 Geometrical Properties of Elementary Partitions

Let S be a finite set of $n \geq 2$ disjoint closed segments in the plane, which we call sites. Throughout this paper, a closed segment may possibly be reduced to a single point. We say that a circle is tangent to a site s if s meets the circle and s does not meet its interior. The sites of S are supposed to be in general position, that is, we suppose that no three segment endpoints are collinear and that no circle is tangent to four sites.

Definition 1. *An elementary partition P of S is a partition of the convex hull $conv(S)$ of S in disjoint sites, edges and faces such that:*
(i) Every face of P is an open triangle whose vertices belong to three distinct sites of S and whose open edges do not intersect S,
(ii) No face can be added without intersecting another one,
(iii) The edges of P are the connected components of $conv(S) \setminus (F \cup S)$, where F is the set of faces of P.

Note that the edges of an elementary partition are not necessarily one-dimensional and that they may even not be convex (see Figures 1 and 2(a)).

We first show that such a partition always exists, that is, for any set S, there is a finite number of faces verifying Definition 1.

Proposition 1. *Every set S of $n \geq 2$ sites admits at least one elementary partition.*

Proof. If $n = 2$, S admits a unique elementary partition which is reduced to the edge $conv(S) \setminus S$. Suppose now that $n = 3$ and that S admits an elementary partition with at least two faces $t1$ and $t2$. It is easy to see that this is only possible if one of the triangles has exactly one edge on the boundary of $conv(t1 \cup t2)$ and if the other triangle has either exactly one edge or exactly one vertex on this boundary. In both cases, if the triangles are oriented in counter-clockwise direction, the three sites are encountered in two distinct orders. This shows that, in an elementary partition of $n \geq 3$ sites, at most two faces can have their vertices on the three same sites. Thus, the total number of faces of such a partition is bounded. $\square$

Theorem 1. *The closure of every edge of an elementary partition of S intersects exactly two sites of S.*

Proof. (i) Call S-polygon, any closed two-dimensional subset A of $conv(S)$, equal to the closure of its interior, such that $A \setminus S$ is connected and the boundary of A is composed of a finite number of line segments that are, either closed and contained in S, or open and such that their interiors do not intersect S and their endpoints belong to S.

Call triangulation of A, any partition T of A in triangles whose vertices belong to S, whose interiors do not cut S, and whose open sides either do not cut S or are contained in S.

Let $\Delta_T(A)$ be the (possibly empty) set of triangles of T having one side in S. We show, by induction on the number $|\Delta_T(A)|$ of triangles of $\Delta_T(A)$, that, if A intersects at least three sites of S, T contains at least one triangle whose vertices belong to three distinct sites of S.

Obviously, if $\Delta_T(A) = \emptyset$, the vertices of every triangle of T belong to three distinct sites. Suppose now the result true for every triangulation T with $|\Delta_T(A)| < k$ $(k \geq 1)$.

For every triangulation T of A with $|\Delta_T(A)| = k$ and for every closed triangle t of $\Delta_T(A)$, the closure $\overline{A \setminus t}$ of $A \setminus t$ intersects the same sites as A. If $A \setminus t$ is connected, $A' = \overline{A \setminus t}$ is a S-polygon and, otherwise, $A \setminus t$ has two connected components, one at least whose closure is a S-polygon. In the latter case, each of the S-polygons intersects the two sites to which the vertices of t belong, and it follows that at least one of these S-polygons intersects at least three sites. Let A' be this S-polygon. In both cases, if T' is the restriction of T to A', $|\Delta_{T'}(A')| < |\Delta_T(A)|$. Thus, by induction hypothesis, T' contains at least one triangle whose vertices belong to three distinct sites of S. It is the same for T.

(ii) Every edge e of an elementary partition P of S is a connected component of $conv(S) \setminus (F \cup S)$, where F is the set of faces of P. It follows that e is
– either an open line segment connecting two points of S,
– or a two-dimensional connected subset of $conv(S)$, equal to the closure of its interior, whose boundary is composed of at least two closed line segments included

in S and of some open line segments that are sides either of triangles of F or of $conv(S)$.

In the second case, $\overline{e}$ is a S-polygon. Since we cannot place in $\overline{e}$ a triangle whose vertices belong to three distinct sites of S (such a triangle would belong to F), it follows from (i), that $\overline{e}$ intersects exactly two sites. □

The shape of the edges of an elementary partition P of S follows directly from the proof above. The closure of an edge either is reduced to a line segment joining two points in two distinct sites of S, or is a triangle with one side and its opposite vertex in S, or is a (possibly non-convex) quadrilateral with two opposite sides in S (see Figure 1). Moreover, every edge of P contains
– either two sides of two triangles of P,
– or one side of one triangle of P and one side of $conv(S)$ that is not a site,
– or two such sides of $conv(S)$.

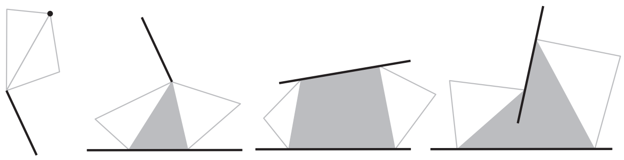


Fig. 1. Examples of edges (grey) connecting two sites in an elementary partition.

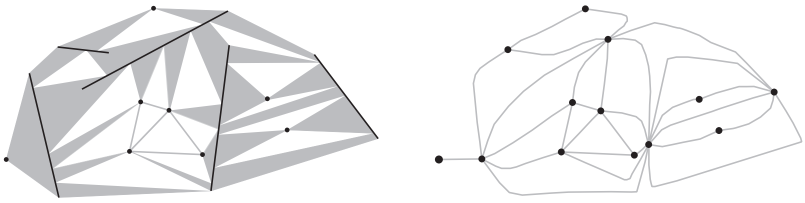
3 Topological Properties of Elementary Partitions

Theorem 1 shows that every edge of an elementary partition P of S "connects" two sites of S. We can thus associate an abstract graph with P such that:

- the vertices of the graph are the sites of S,
- the edges connecting two sites s and t in the graph are the edges of P whose closure intersects s and t.

Proposition 2. *The abstract graph associated with an elementary partition P of S is planar.*

Proof. For every site s of S, let γ_s be a convex closed Jordan curve such that:
– s is inside γ_s (i.e. in the subset of the plane bounded by γ_s),
– $S \setminus s$ is outside γ_s,
– the interior of γ_s intersects only the edges of P whose closures intersect s.
Replace now every site s by a point p_s inside γ_s. For every edge e of P that intersects γ_s, replace the subset of e inside γ_s by a line segment connecting p_s to a point of e on γ_s. While doing this, the order of the edges around s remains unchanged and the reduced edges do not intersect. Once this transformation is fulfilled in every Jordan curve γ_s, replace every reduced edge by a Jordan arc included in it. Finally, we get a planar representation of the abstract graph associated with P (see Figure 2(b)). □



(a) (b)

Fig. 2. An elementary partition (a) (the sites are in black, the edges in grey, and the faces in white) and its associated graph (b).

Theorem 2. *Every elementary partition P of a set S of n sites contains $3n-n'-3$ edges and $2n-n'-2$ faces, where n' is the number of edges of $conv(S)$ that are not sites.*

Proof. Counting the edges and faces of P comes down to counting the edges and bounded faces of the planar representation G constructed in the proof of Proposition 2. Moreover, the unbounded face of G corresponds to the complementary of $conv(S)$. The result is then an immediate consequence of Euler's relation, of the fact that every bounded face of G has three edges, and that the edges adjacent to one (resp. no) bounded faces appear once (resp. twice) while traversing the boundary of the unbounded face of G. □

An interesting consequence of this theorem is that the size of an elementary partition is linear with the number of sites. Moreover, it shows that the number of triangles of the partition is an invariant of the set of sites. This is an extension of a well known property of the triangulations of planar point sets.

Using the planar representation G constructed in the proof of Proposition 2, we can associate a combinatorial map M with the elementary partition P:

- the vertices of M are the vertices of G,
- the edges of M are the edges of G endowed with their two opposite orientations,
- for every vertex s of M, the sequence of oriented edges out of s is ordered in the counter-clockwise direction as in the planar representation G.

The ordering of the edges around the vertices induces a set of directed circuits such that an oriented edge st follows the oriented edge rs in a circuit if sr is the successor of st around s. Every oriented edge of M belongs to one and only one circuit and to every circuit corresponds one face of G, bounded or not. Note that, in general, the same map M is associated with different elementary partitions of S. In order to use M as a data structure to store the elementary partition P, we only need to add the coordinates of the vertices of the triangles of P in the structure: One vertex per oriented edge. Indeed, every oriented edge belongs to the circuit of a unique face and issues from a unique vertex.

An elementary partition of a set S of n sites can thus be stored using $O(n)$ space. Furthermore, it can be shown that every constrained triangulation of S is a refinement of an elementary partition of S. There exists a sweep-line algorithm to construct a constrained triangulation in $O(n \log n)$ time [5] and the algorithm can easily be adapted to construct an elementary partition also in $O(n \log n)$ time.

4 Elementary Delaunay Partition and Segment Voronoi Diagram

We want to prove the existence of a special elementary partition, which we call the elementary Delaunay partition. By the way, we show that the elementary Delaunay partition of S is dual to the segment Voronoi diagram of S. Our proof uses some properties of the segment Voronoi diagram of a set of line segments in the plane, which can be found in [1], [2], and [11].

Let now F be the set of triangles of the plane such that the vertices of each triangle belong to three distinct sites of S and such that the interior of the circumcircle of each triangle does not intersect S.

Theorem 3. *(i) The triangles of F are the faces of an elementary partition P of S that we call the elementary Delaunay partition.*

(ii) The combinatorial map M associated to P is dual to the segment Voronoi diagram of S.

Proof. Since the interior of the circumcircle of every triangle of F is empty, two such triangles cannot intersect. Thus, they are faces of an elementary partition. On the one hand, the number of vertices of the Voronoi diagram $Vor(S)$ of S is known and by Theorem 2, it is the same as the number of triangles of an elementary partition of S. On the other hand, each vertex of the Voronoi diagram corresponds to one triangle of F. Therefore, the number of triangles of F is maximal which means that F is the set of triangles of an elementary partition P. Furthermore, by definition of the Voronoi diagram, there is a one to one correspondence between the regions of $Vor(S)$ and the sites which are, by definition, the vertices of M.
It remains to study the edges of M and of $Vor(S)$. Let a be an edge of $Vor(S)$ incident to the two Voronoi regions of s and t. Each point p in a is the center of an empty circle C_p touching the two sites s and t at the points p_s and p_t. Such an open segment $p_s p_t$ and a triangle f of F never meet, since both the interior of the circumcircle of f and the interior of the circle C_p cannot intersect the sites. Thus, for each p in a, the open segment $p_s p_t$ is included in an edge of the elementary partition P. Furthermore, the union E_a of all the open segments $p_s p_t$, $p \in a$, is a connected subset of $conv(S)$, therefore E_a is included in a single edge e of P. Since the sites s and t intersect the closure of e, e is incident to s and t. The last thing to show is that for each edge e of P there is exactly one edge a of $Vor(S)$ such that $E_a \subset e$. Since the numbers of edges of P and of $Vor(S)$ are equal, it suffices to prove that for each edge e of P there is at least an edge a such that $E_a \subset e$. Let C be the set of open segments c which are either an open side of a triangle of F or an open side of $conv(S)$ which are not sites. For each c in C there is an edge a of $Vor(S)$ such that $c \subset E_a$ and for each edge e of P there is a c in C such that $c \subset e$, therefore, all edges of P contains a set E_a where a is an edge of $Vor(S)$. $\square$

It is easy to see that the elementary Delaunay partition of S defined in this theorem is equivalent to the dual of $Vor(S)$ introduced by Chew and Kedem, which they called the edge Delaunay triangulation of S [3]. Using algorithms that construct segment Voronoi diagrams, the elementary Delaunay partition can be computed in $O(n \log n)$ time [11].

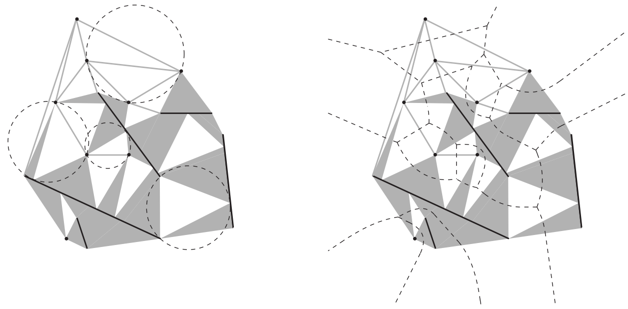


Fig. 3. An elementary Delaunay partition (a) and an illustration of the duality (b).

5 Legality in Elementary Partitions

An interesting property of the Delaunay triangulation of a planar point set is the legal edge property. Consider an edge of a point set triangulation and its two adjacent triangles. The edge is illegal if a vertex of one of these triangles lies inside the circumcircle of the other triangle. It is well known that the Delaunay triangulation of a point set is the unique triangulation of this point set whithout illegal edge. In the following, we are going to prove a similar property for elementary partitions.

Definition 2. *An egde of a given elementary partition is legal if the circumcircles of its adjacent triangles contain no point of the sites adjacent to these triangles in their interiors.*

Theorem 4. *The elementary Delaunay partition of S is the unique elementary partition of S whose edges are all legal.*

Proof. Obviously, the elementary Delaunay partition has no illegal edge. Let P be an elementary partition which is not Delaunay and let f be a face of P whose circumcircle c_f contains a point of S in its interior d_f. We have to prove that P has an illegal edge. Let x be a point in f and p a point in d_f lying on a site. We can assume that the interior of the segment xp does not intersect S. Denote by k the number of edges crossed by the segment xp. Note that $k \geq 1$, for, by definition, p can neither be in f, nor in an edge adjacent to f. Denote e the first edge crossed by xp, g the other face adjacent to e, c_g its circumcircle, d_g the interior of c_g, ab the side of g contained in e, and u the site which contains the vertex of g that is not a vertex of e. If $k = 1$, p lies on u and therefore the edge crossed by xp is illegal. Now suppose that, if xp crosses k edges then at least one of them is illegal. We have to prove that if xp crosses $(k+1)$ edges then P has an illegal edge. If the edge e is illegal we are done. Otherwise the points a and b cannot be in the disk d_f, and the point $y = ab \cap xp$ is in d_f (see Figure 4). Therefore, the segment ab splits d_f into two parts. Denote d_1 the part containing the face f, and d_2 the other part. The disk d_g must contain at least d_1 or d_2 and since e is legal it can not contain d_1. It follows that the segment yp is in d_g and crosses one edge less than xp. Using the induction hypothesis, we conclude that P has an illegal edge. □

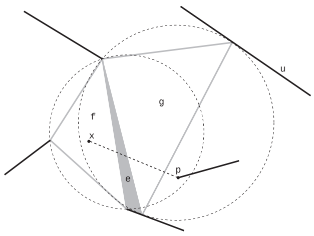


Fig. 4. Illustration of the proof of the Theorem 4.

Definition 3. *Let f be a face of an elementary partition of S. The tangency triangle of f is the triangle such that:*

- *its vertices are on the same three sites as the vertices of f,*
- *if f and its tangency triangle are traversed in counter-clockwise direction, they encounter these three sites in the same order,*
- *the interior of its circumcircle does not intersect these three sites.*

Definition 4. *Let M be a map associated with an elementary partition of S. An edge e of M is legal in the two following cases:*

1. *e is adjacent to at most one internal triangle.*
2. *e is adjacent to two internal triangles and the following property holds. Denote T_1 and T_2 these two internal triangles and denote t, r, u, v the sites such that t, r, u are incident to T_1 and r, t, v are incident to T_2 in counter-clockwise direction. Let $t_1r_1u_1$ and $r_2t_2v_2$ be the tangency triangles of T_1 and T_2 with $t_i \in t$, $r_i \in r$, $u_1 \in u$, and $v_2 \in v$. Then the polygon $t_1t_2r_2r_1$ is either reduced to a segment, or a counter-clockwise oriented simple polygon (with three or four edges), and the circumcircles' interiors of $t_1r_1u_1$ and $r_2t_2v_2$ do not intersect the sites t, r, u, v.*

Theorem 5. *Let M be a map associated with an elementary partition P of S. Suppose that all the edges of this map are legal, then M is also the map associated with the elementary Delaunay partition of S.*

Proof. We want to prove that the collection of the tangency triangles gives rise to the elementary Delaunay partition. Making use of the previous Theorem, we see that the only thing to prove is that the interiors of the tangency triangles T' are pairwise disjoint and do not meet any site $s \in S$.

The main idea is to use a result of Devillers et al. [4] that can be stated as follows. Let be a map C and let be a set of smooth curves representing C in the plane. Suppose that:
i. *all the curves representing the circuits of C are simple curves,*
ii. *the ordering of the edges out of every vertex s of C corresponds to the counter-clockwise order of the curves incident to s in the representation,*
then the representation is planar. Actually, the result of Devillers et al. is stated with segments instead of smooth curves but an approximation argument leads to the same result for smooth curves. We shall use this result with a new map M' and not directly with the map M associated with P.

Step 1. We construct a representation of a new map M'.

Choose $\varepsilon > 0$ sufficiently small. For each site $s \in S$, consider the simple closed curve $\gamma_{s,\varepsilon}$ made of all the points at distance ε from s. Choose the counter-clockwise orientation for the curve $\gamma_{s,\varepsilon}$.

Let T be a triangle of P incident to the sites s, t, and u in counter-clockwise direction. Choose three points $p_{T,s}$, $p_{T,t}$, and $p_{T,u}$ in the interior of T such that $p_{T,s} \in \gamma_{s,\varepsilon}$, $p_{T,t} \in \gamma_{t,\varepsilon}$, and $p_{T,u} \in \gamma_{u,\varepsilon}$. Next, join the points $p_{T,s}$, $p_{T,t}$, $p_{T,u}$ by three smooth simple and disjoint curves $\gamma_{T,s,t}$, $\gamma_{T,t,u}$, and $\gamma_{T,u,s}$ lying in the interior of T at a distance smaller than 2ε from the boundary of T. These three curves can be chosen so that they do not meet the closed curves $\gamma_{s,\varepsilon}$, $\gamma_{t,\varepsilon}$, and $\gamma_{u,\varepsilon}$ except at their endpoints. The same can be done with the external face of P: $\mathbf{R}^2 \setminus conv(S)$. In the following, $\gamma_{T,s,t}$ will denote the curve going from $p_{T,s}$ to $p_{T,t}$ and $\gamma_{T,t,s}$ the curve going from $p_{T,t}$ to $p_{T,s}$, i.e, the same geometric curve but with reverse orientation.

Let s be a site and $T_0, ..., T_{k-1}$ the faces of P incident to s in the counter-clockwise direction (around s) (one of these faces could be the external face). We split the closed curve $\gamma_{s,\varepsilon}$ into k simple non overlapping curves $\gamma_{s,T_i,T_{i+1}}$ going from the point p_{s,T_i} to the point $p_{s,T_{i+1}}$ ($i = i \, mod \, k$). As before, γ_{s,T_{i+1},T_i} is the same geometric curve as $\gamma_{s,T_i,T_{i+1}}$ but with the reverse orientation.

When ε is sufficiently small, this defines a geometric planar graph (see Figure 5) which is a geometric representation Γ of the map M'.

Step 2. We associate to M' a second geometric representation. We proceed as in step 1 with the tangency triangles T' instead of the triangles T of P. In each triangle T' choose a point p'_{T,s_i} on each curve $\gamma_{s_i,\varepsilon}$, $s_i \in S$ which intersects T', and choose a sequence of simple disjoint curves $\gamma'_{T,s_i,s_{i+1}}$ which join these points without intersecting $\gamma_{s_i,\varepsilon}$, $s_i \in S$. Only the definition of the curves turning around the sites has to be changed compared to step 1.

Let s be a site and $T_0, ..., T_{k-1}$ the faces of P incident to s in counter-clockwise direction. For each i in $\{0, ..., k-1\}$ consider the curve $\gamma'_{s,T_i,T_{i+1}}$ going in counter-clockwise direction from $p'_{T_i,s}$ to $p'_{T_{i+1},s}$ on $\gamma_{s,\varepsilon}$. The collection of curves $\gamma'_{T,s,t}$, $\gamma'_{s,T_i,T_{i+1}}$ defines a new geometric representation Γ' of the map M'.

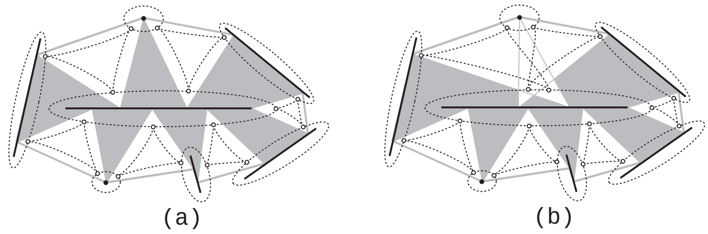


Fig. 5. (a): Illustration of the step 1 of Theorem 5. (b): Illustration of the step 2 of Theorem 5, with an illegal edge.

Step 3. To prove the Theorem it is enough to show that this new geometric representation of the map M' is planar. Indeed, letting ε going to 0, it will show that the tangency triangles T' give rise to an elementary partition of S.

Since the tangency triangles T' have the same orientation as the triangles T of P, the geometric ordering of the edges of Γ' around each vertex agrees with the geometric ordering of the edges of Γ, therefore condition ii of Devillers et al. holds for Γ'. It remains to show that the representation of the circuits of M' are simple closed curves. The map M' has three kinds of circuits:
- circuits with three edges corresponding to the faces of P, and the external circuit corresponding to the boundary of $conv(S)$.
- circuits with four edges corresponding to the edges of P,
- circuits around each site of P.
The representations of the first kind of circuits are simple curves by construction. The representations of the circuits associated with the edges are simple curves, for, all edges of M are legal. The third kind is more difficult to handle.

Let s be a site and $T_0, ..., T_{k-1}$ the faces of P incident to s in counter-clockwise direction. For sake of simplicity, we suppose that these faces are all internal triangles. Fix an arbitrary non zero vector $\overrightarrow{U}$ and an origin O lying on the site s. For a smooth oriented curve $\gamma : [a, b] \to \mathbf{R}^2$ not containing O, denote $var(\gamma)$ the variation of the angle $\measuredangle(\overrightarrow{U}, \overrightarrow{O\gamma(t)})$ along the curve γ. Since the curves $\gamma'_{s,T_i,T_{i+1}}$ are all oriented in counter-clockwise direction, it is enough to prove that

$$var(\gamma'_{s,T_0,T_1}) + var(\gamma'_{s,T_1,T_2}) + ... + var(\gamma'_{s,T_{k-1},T_0}) = 2\pi.$$

Denote by $t_0, ..., t_{k-1}$ the sites ($\neq s$) such that, for $i \in \{0, ..., k-1\}$, T_{i-1} and T_i are incident to t_i.
The representation α'_i, for each $i \in \{0, ..., k-1\}$, of the circuit corresponding to the face T_i is the closed curve beginning at $p'_{T_i,s}$ made of the three curves γ'_{T_i,s,t_i}, $\gamma'_{T_i,t_i,t_{i+1}}$, and $\gamma'_{T_i,t_{i+1},s}$. It is clear that the curves α'_i do not enclosed the origin O; therefore $var(\alpha'_i) = var(\gamma'_{T_i,s,t_i}) + var(\gamma'_{T_i,t_i,t_{i+1}}) + var(\gamma'_{T_i,t_{i+1},s}) = 0$.
The representation β'_i, for each $i \in \{0, ..., k-1\}$, of the circuit corresponding to the edge incident to T_{i-1} and T_i is the closed curve beginning at $p'_{T_{i-1},s}$ made of the four curves γ'_{T_{i-1},s,t_i}, $\gamma'_{t_i,T_{i-1},T_i}$, $\gamma'_{T_i,t_i,s}$, $\gamma'_{s,T_i,T_{i-1}}$. The legality of the edges

of the partition implies that the curves β'_i do not enclosed the origin O; therefore, $var(\beta'_i) = var(\gamma'_{T_{i-1},s,t_i}) + var(\gamma'_{t_i,T_{i-1},T_i}) + var(\gamma'_{T_i,t_i,s}) + var(\gamma'_{s,T_i,T_{i-1}}) = 0$.

Now, summing $var(\beta'_i) + var(\alpha'_i)$, $i \in \{0,...,k-1\}$, all the terms of the form $var(\gamma'_{T_{i-1},s,t_i})$, $var(\gamma'_{T_{i-1},t_i,s})$ cancel out as well as the terms $var(\gamma'_{T_i,s,t_i})$, $var(\gamma'_{T_i,t_i,s})$. Indeed these terms are the variations of the angle along the same geometric curve but with opposite orientations. Thus

$$\sum_{i=0}^{k-1} var(\gamma'_{s,T_{i-1},T_i}) = \sum_{i=0}^{k-1}(var(\gamma'_{t_i,T_{i-1},T_i}) + var(\gamma'_{T_i,t_i,t_{i+1}})).$$

The previous calculus has been done for the representation Γ' but it works for the representation Γ as well. Therefore,

$$2\pi = \sum_{i=0}^{k-1} var(\gamma_{s,T_{i-1},T_i}) = \sum_{i=0}^{k-1}(var(\gamma_{t_i,T_{i-1},T_i}) + var(\gamma_{T_i,t_i,t_{i+1}})).$$

The last thing to see is that both sums $\sum_{i=0}^{k-1}(var(\gamma'_{t_i,T_{i-1},T_i}) + var(\gamma'_{T_i,t_i,t_{i+1}}))$ and $\sum_{i=0}^{k-1}(var(\gamma_{t_i,T_{i-1},T_i}) + var(\gamma_{T_i,t_i,t_{i+1}}))$ are equal. Indeed, on each curve $\gamma_{t_i,\varepsilon}$ choose a curve δ_i going from $p_{t_i,T_{i-1}}$ to $p'_{t_i,T_{i-1}}$. Since the triangles T_i and T'_i have the same orientation, the successive curves

$$\gamma_{t_i,T_{i-1},T_i},\ \gamma_{T_i,t_i,t_{i+1}},\ \delta_{i+1},\ \gamma'_{T_i,t_{i+1},t_i},\ \gamma'_{t_i,T_i,T_{i-1}},\ -\delta_i$$

form a closed curve not enclosing the origin O, thus sum over i of the variation leads to desired equality. □

It is possible to derive directly from this theorem an algorithm that checks whether a random elementary partition has the same topology as the Delaunay one. For each edge, the algorithm computes in constant time the geometric representation of this edge and its two adjacent tangency triangles, then it checks in constant time the conditions of the legal edge property. From Theorem 2 the number of edges is in $O(n)$, then this algorithm runs in linear time.

6 Conclusion

In this paper we have introduced a new family of partitions of the convex hull of a set of line segments in the plane that we called elementary partitions. We have shown that several properties of point set triangulations extend to elementary partitions. For example, there exists an elementary partition which is dual to the segment Voronoi diagram, the elementary Delaunay partition. We have shown that this partition is the unique elementary partition which is locally Delaunay in all its edges. This result extends a fundamental property of the point set Delaunay triangulation, which is used to prove different optimality properties of the Delaunay triangulation among all point set triangulations. Especially, the Delaunay triangulation is the most regular one, in the way that it maximizes the minimum angle of its triangles

[15]. This property also enables to give a so-called flip algorithm, to transform any triangulation in the Delaunay triangulation by a sequence of local improvements. The local characterization of the elementary Delaunay partition, should allow us to extend some optimality properties, as well as the flip algorithm, to this partition. We also hope that the extension of elementary partitions to line segments in higher dimensions will help to better understand the topological structure of the segment Voronoi diagram in higher dimensions.

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