Coding of a translation of the two-dimensional torus

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Abstract

Let α be in the two-dimensional torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. Assume that the translation map $T : x \to x + \alpha$ acts ergodically. We present a symbolic coding of the map T which shares several properties with the Sturmian coding of a one-dimensional translation. The symbolic dynamical system is metrically isomorphic to the geometric dynamical system $(\mathbf{T}^2, T)$. The coding is of quadratic growth complexity and 2-balanced. Moreover, there is a geometric underpinning, the coding is related to a fundamental domain for the action of $\mathbf{Z}^2$ on $\mathbf{R}^2$ and also to bounded remainder sets.

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1 Introduction

Let (X, T) be a dynamical system and $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_k\}$ a finite partition of X. Each point x in X can be coded by its itinerary under the action of T :

$$\pi : x \in X \to \omega \in \mathcal{W}^{\mathbf{N}}$$

where $T^n x \in \omega(n)$ for all integer n in $\mathbf{N}$. The map π is called the coding map. This gives rise to a symbolic dynamical system $(\pi(X), S)$ where S is the shift map defined by

$$\omega = (\omega(0), \omega(1), \omega(2), ...) \to S\omega = (\omega(1), \omega(2), \omega(3), ...).$$

The symbolic dynamical system $(\pi(X), S)$ and the geometric dynamical system (X, T) are related by $\pi \circ T = S \circ \pi$ and the simultaneous study of both dynamical systems can be very fruitful. We shall study a very particular case: the translations $T : \mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2 \to \mathbf{T}^2$ defined by

$$T : x \to x + \alpha$$

where $\alpha \in \mathbf{T}^2$. In this setting, there are two well known examples of coding, Sturmian or natural coding of translations of the one-dimensional torus, and Rauzy's coding of some "cubic" translations of the two-dimensional torus. Our aim is to find a coding of a general translation of the two-dimensional torus sharing most properties with these two examples. The algebraic setting of Rauzy's example makes impossible to use directly Rauzy's ideas. It is the reason why we take the one-dimensional Sturmian coding as a starting point. There is an alternative way to introduce a Sturmian sequence associated with an irrational number. This way which seems to be unusual, is inspired by Rauzy's original presentation ([13]) of his example, and is easy to extend to the two-dimensional case. This enables to find a coding which gives rise to a minimal, uniquely ergodic symbolic dynamical system metrically isomorphic to a translation of the two-dimensional torus. Together with these dynamical properties, we also show some geometric and combinatorial properties of this coding.

2 The two examples

We recall the facts which make clear the march leading to our two-dimensional coding as well as the aims we have in mind.

1. Rauzy's coding. Let ξ be the unique real root of the equation $x + x^2 + x^3 = 1$ and set $\alpha = (\xi, \xi^2)$. Let $u = (u_n)_{n \in \mathbf{N}}$ the unique fixed point in $\{0, 1, 2\}^{\mathbf{N}}$ of the Tribonacci substitution τ:

$$\tau(0) = 01,\ \tau(1) = 02,\ \tau(2) = 0,$$

$$u = 01020100102....$$

Set

$$P_0 = 0,\ P_n = \sum_{i=1}^{n} u_i e_i$$

where $e_0 = (0, 0)$ $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The *Rauzy fractal* is the set

$$\mathcal{R} = \overline{\{n\alpha - P_n : n \in \mathbf{N}\}}$$

which is split into three parts

$$\mathcal{R}_i = \overline{\{n\alpha - P_n : u_n = i\}},\ i = 0, 1, 2.$$

G. Rauzy has shown that $\mathcal{R}_0, \mathcal{R}_1$ and $\mathcal{R}_2$ enjoy some nice properties [13], [8]:

- *The interiors Ω_0, Ω_1 and Ω_2 of $\mathcal{R}_0, \mathcal{R}_1$ and $\mathcal{R}_2$ respectively, are disjoint and their boundaries are of zero Lebesgue measure.*

- *$\mathcal{R}$ is a "fundamental domain" for the action of $\mathbf{Z}^2$ on $\mathbf{R}^2$: $\cup_{a \in \mathbf{Z}^2}(a + \mathcal{R}) = \mathbf{R}^2$ and the sets $a + \Omega_0 \cup \Omega_1 \cup \Omega_2$, $a \in \mathbf{Z}^2$, are disjoint.*

- *The translation $x \in \mathbf{T}^2 \to x + \alpha$ can be seen as a domain exchange: there exist three lattice vectors $u_0, u_1, u_2 \in \mathbf{Z}^2$ such that*

$$\mathcal{R}_1 + u_0 + \alpha \subset \mathcal{R},$$
$$\mathcal{R}_2 + u_1 + \alpha \subset \mathcal{R},$$
$$\mathcal{R}_3 + u_2 + \alpha \subset \mathcal{R}.$$

- The coding map π associated to $\mathcal{W} = \{\Omega_0, \Omega_1, \Omega_2\}$ can be defined almost everywhere and is one to one. Moreover the coding map π is onto the dynamical system generated by the sequence u:

$$Y = \overline{\{S^n u : n \in \mathbf{N}\}}.$$

- (Y, S) is uniquely ergodic, and π is an isomorphism of the measurable dynamical systems $(\mathbf{T}^2, T)$ and (Y, S).
- For all integer $n \geq 0$, the complexity of the infinite sequence u is $p_u(n) = 2n + 1$ (see below the definition of the complexity).
- The three sets $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are bounded remainder sets (see below the definition of a bounded remainder set).

2. Sturmian sequences. Let α be in $\mathbf{R}\backslash\mathbf{Q}$. Sturmian sequences are codings of translations $T : x \in \mathbf{T}^1 \to x + \alpha$ with respect to the partition $\mathcal{W} = \{[0, 1-\alpha[, [1-\alpha, 1[\}$ (or $\{]0, 1-\alpha],]1-\alpha, 1]\}$). This gives rise to a symbolic minimal and uniquely ergodic dynamical system (Y, S) which is metrically isomorphic to $(\mathbf{T}^1, \mu, T)$ where μ is the Lebesgue measure on $\mathbf{T}^1$ (see [8], [3]). This geometric definition of Sturmian sequences is equivalent to a combinatorial property:
A sequence $\omega \in \{0, 1\}^{\mathbf{N}}$ is Sturmian if and only if for all integer $n \geq 0$, the complexity $p_\omega(n)$ is exactly $n + 1$.
Sturmian sequences have been extensively studied (see [8]). Let us now show how to introduce the Sturmian partition $\{[0, 1-\alpha[, [1-\alpha, 1[\}$ in a way which reminds Rauzy's example:
Let (p_n) be the sequence of integers defined inductively by $p_0 = 0$ and p_{n+1} is the closest integer from $(n+1)\alpha$ among p_n and $p_n + 1$. It is not difficult to show that $(p_{n+1} - p_n)_{n\in\mathbf{N}}$ is a Sturmian sequence which is the coding of 0 according to the partition mod 0:

$$\overline{\{n\alpha - p_n : p_{n+1} - p_n = 0\}} = [-\frac{1}{2}, \frac{1}{2} - \alpha],$$
$$\overline{\{n\alpha - p_n : p_{n+1} - p_n = 1\}} = [\frac{1}{2} - \alpha, \frac{1}{2}].$$

We shall use exactly the same idea to associate to each α in the two-dimensional torus, a partition with three pieces.

3 Statements of results

We recall some definitions. The first two are standard combinatorial definitions.

Definition 1 *1. Let $\mathcal{A}$ be a finite set (an alphabet). The complexity function of a sequence ω in $\mathcal{A}^{\mathbf{N}}$ is the map $p_\omega : \mathbf{N}^* \to \mathbf{N}$ defined by,*

$$p_\omega(n) = \text{card}\{(\omega(k), ..., \omega(k + n - 1)) : k \geq 0\}.$$

2. Let C be a real number and let a be an element of $\mathcal{A}$. A sequence ω in $\mathcal{A}^{\mathbf{N}}$ is C-balanced over the letter a if for all integers $p, q \geq 0$ and $l > 0$,

$$\mathrm{card}\{n \in \{p, ..., p+l-1\} : \omega(n) = a\} - \mathrm{card}\{n \in \{q, ..., q+l-1\} : \omega(n) = a\} \leq C.$$

When A and B are two measurable subsets of $\mathbf{R}^d$, the equality $A = B \bmod 0$ means that the symmetric difference of A and B is a set of zero Lebesgue measure. To make our results precise, we explain what we mean by a fundamental domain:

Definition 2 *A measurable subset $\mathcal{D}$ of $\mathbf{R}^d$ is a fundamental domain if*
- $\mathcal{D} + \mathbf{Z}^d = \mathbf{R}^d \bmod 0$,
- the sets $(\mathcal{D} + n)$, $n \in \mathbf{Z}^d$, are disjoint mod 0.
Moreover if $\mathcal{D}$ is a compact set, and if $\overline{\mathcal{D}^o} = \mathcal{D}$, we say that $\mathcal{D}$ is a regular fundamental domain.

Remark. If $\mathcal{D}$ is a fundamental domain, the sets $\mathcal{D}^o + n$, $n \in \mathbf{Z}^d$, are disjoint and if $\mathcal{D}$ is a compact fundamental domain, we have $\mathcal{D} + \mathbf{Z}^d = \mathbf{R}^d$.

The next definition can be found in many sources, see for example [8] or [3].

Definition 3 *Let α be in $\mathbf{R}^d$ and $T : \mathbf{T}^d \to \mathbf{T}^d$ the translation defined by $Tx = x + \alpha$. A sequence u is a natural coding of the translation T if there exist a fundamental domain $\mathcal{D}$ together with a finite partition $\mathcal{D} = \mathcal{D}_1 \cup ... \cup \mathcal{D}_p$ such that for each i in $\{1, ..., p\}$, there exists a lattice vector $e_i \in \mathbf{Z}^d$ with $\mathcal{D}_i + \alpha + e_i \subset \mathcal{D}$, and there exists a point x in $\mathcal{D}$ such that u is the coding of x under the action of T with respect to the partition $\{\mathcal{D}_1, ..., \mathcal{D}_p\}$.*

Let $\alpha = (\alpha_1, \alpha_2)$ be in $\mathbf{R}^2$. Remember that the translation $T : x \in \mathbf{T}^2 \to x + \alpha \in \mathbf{T}^2$ is ergodic if and only if, T is minimal, or if and only if, $1, \alpha_1$ and α_2 are linearly independent over the rational numbers. Our main result is:

Theorem 1 *Let $\alpha = (\alpha_1, \alpha_2)$ be in $\mathbf{R}^2$. Assume that $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 < 1$, and that the translation $T : x \in \mathbf{T}^2 \to x + \alpha \in \mathbf{T}^2$ is ergodic. Set $e_0 = (0, 0)$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and define the sequence $(P_n)_{n \in \mathbf{N}}$ of lattice points by induction: $P_0 = (0, 0)$ and P_{n+1} is the point closest to $(n + 1)\alpha$ among the three points $P_n + e_0$, $P_n + e_1$ and $P_n + e_2$ for the Euclidean distance.*
Denote $\Omega = \{e_0, e_1, e_2\}^{\mathbf{N}}$, $\omega_0(n) = P_{n+1} - P_n$, and Y the orbit closure of ω_0 under the shift map $S : \Omega \to \Omega$.
Then
1. the dynamical system (Y, S) is uniquely ergodic and minimal,
2. the two dynamical systems $(\mathbf{T}^2, T)$ and (Y, S) endowed with their unique invariant probability measures are metrically isomorphic,
3. the sequence ω_0 is a natural coding of the translation T and $\mathcal{D}_\alpha = \overline{\{n\alpha - P_n : n \in \mathbf{N}\}}$ is a regular fundamental domain.

Moreover two combinatorial properties hold for the sequence ω_0:
4. the complexity of the sequence ω_0 is of quadratic growth,
5. the sequence ω_0 is 2-balanced over the letters e_1 and e_2.

Bounded remainder sets for a sequence in $[0,1[$ were introduce a long time ago. Hecke proved in 1922 [9] that for a real number α and for an interval $I \subset [0,1[$ whose length $l = p + q\alpha$ belongs to $\mathbf{Z} + \alpha\mathbf{Z}$, the following property holds

$$\forall n \in \mathbf{N},\ |\mathrm{card}\{k \in \{0,...,n-1\} : n\alpha \bmod 1 \in I\} - nl| \leq 2q.$$

This leads to the following definitions.

Definition 4 *1. Let $(x_n)_{n\in\mathbf{N}}$ be a sequence in a set X and let $\mathcal{A}$ be a subset of X. The set $\mathcal{A}$ is a bounded remainder set for the sequence $(x_n)_{n\in\mathbf{N}}$ if there is a real number a and a constant C such that for all integer n,*

$$|\mathrm{card}\{k \in \{0,...,n-1\} : x_k \in \mathcal{A}\} - na| \leq C.$$

2. Let α be in $\mathbf{T}^d$ and let $T : \mathbf{T}^d \to \mathbf{T}^d$ be the translation defined by $Tx = x + \alpha$. A subset $\mathcal{A}$ of $\mathbf{T}^d$ is a bounded remainder set for α if $\mathcal{A}$ is a bounded remainder set for almost all sequences $(T^n x)_{n\in\mathbf{N}}$ with respect to the Lebesgue measure where a and C are independent of x.

By Hecke's result, if α is in $\mathbf{T}$, the sets $[0, 1-\alpha[$ and $[1-\alpha, 1[$ associated to Sturmian's coding, are bounded remainder sets for all sequences $(x + n\alpha \bmod 1)_{n\in\mathbf{N}}$ with the same a and C. Therefore these two sets are bounded remainder sets for α. In Rauzy's example, the same holds for the three sets $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$, and $\alpha = (\xi, \xi^2)$. The converse of Hecke's result was proved by Kesten [10] in 1966, and Liardet [12] extended Kesten's result to boxes $\Pi_{i=1}^d I_i$. Liardet proved that if the product $\Pi_{i=1}^d I_i$ is a bounded remainder set for $\alpha = (\alpha_1, ..., \alpha_d)$ in $\mathbf{T}^d$ and if $1, \alpha_1, ..., \alpha_d$ are linearly independents over $\mathbf{Q}$, then the length l_i of the intervals I_i are equal to 1 except for a single i in $\{1, ..., d\}$ for which the length l_i must be in $\mathbf{Z} + \alpha_i\mathbf{Z}$. This shows that bounded remainder sets for α are rather exceptional sets. It is also well known that bounded remainder sets are related to balanced sequences and this is indeed the case in our work.

Let $\mathbf{p} : \mathbf{R}^2 \to \mathbf{R}^2/\mathbf{Z}^2$ be the canonical projection.

Corollary 5 *Let $(P_n)_{n\in\mathbf{N}}$ be defined as in the previous theorem. Then the three sets*

$$\mathbf{p}(\overline{\{n\alpha - P_n : P_{n+1} - P_n = e_i\}}),\ i = 0, 1, 2,$$

are bounded remainder sets. Moreover, they have disjoint nonempty interiors and their boundaries are a finite union of segments.

In fact the conclusion of the proposition is even stronger for the boundaries of the sets we are dealing with, are made of a finite number of segments; lemma 12 below, allows us to remove the "almost all" in the definition of bounded remainder sets.

Remark. It is easily deduced from the proof that

$$\mu(\mathbf{p}(\overline{\{n\alpha - P_n : P_{n+1} - P_n = e_i\}})) = \alpha_i, \ i = 1, 2,$$

where μ is the Lebesgue measure on the two-dimensional torus.

4 Ingredients of the proof of Theorem 1

4.1 General results

One part of the proof of theorem 1 rests on several independent results. The first two are very likely to be "folklore"results. They connect the properties of the geometric dynamical system $(\mathbf{T}^d, Tx = x + \alpha)$ and of the partition with the properties of the symbolic dynamical system associated with the partition. There are various ways to formulate these properties for a homeomorphism T of a compact metric space X. Most of the time, the partition needs to "separate" the points of X under the action of T (see [6] section 15). In the next definition, the aperiodicity condition will imply that the points of $\mathbf{T}^d$ are separated under the action of the translation $Tx = x + \alpha$.

Definition 6 *Let $(G, +)$ be a topological Abelian group and let $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_n\}$ be a finite partition of the group G.*
*1. The partition $\mathcal{W}$ is **aperiodic** if for all a in $G \backslash \{0\}$ there is an x in G such that x is in the interior $\mathcal{W}_i^o$ for some i and $x + a$ is not in the closure $\overline{\mathcal{W}_i}$ of the same $\mathcal{W}_i$.*
*2. The partition is **regular** if for all i, $\mathcal{W}_i$ is included in the closure of its interior.*

In the following, we shall assume that $\Omega = \mathcal{W}^{\mathbf{N}}$ is endowed with the product topology ($\mathcal{W}$ is finite and endowed with the discrete topology).

Theorem A. *Let $T : \mathbf{T}^2 \to \mathbf{T}^2$ be an ergodic translation of the two-dimensional torus and $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_k\}$ be a finite, regular, and aperiodic partition of $\mathbf{T}^2$. Set $\mathcal{G} = \mathbf{T}^2 \backslash \bigcup_{n \in \mathbf{N}} \bigcup_{i=1}^k T^{-n} \partial W_i$ and $Y = \overline{\pi(\mathcal{G})}$ where π is the coding map associated with the partition $\mathcal{W}$.*
Then the coding map $\pi : \mathbf{T}^2 \to \Omega$ (see the introduction) is one-to-one and there exists a continuous map $\phi : Y \to \mathbf{T}^2$ such that
i. $\phi \circ \pi(x) = x$ for all x in $\mathcal{G}$,
ii. $(\mathbf{T}^2, T)$ is a topological factor of (Y, S): $\phi(Y) = \mathbf{T}^2$, $\phi \circ S = T \circ \phi$, furthermore the topological dynamical system (Y, S) is minimal.

Remark. The space Y depends on the $\mathcal{W}_i^o$, $i = 1, ..., k$ rather on the $\mathcal{W}_i$, $i = 1, ..., k$.

Theorem B. *Let μ denote the Lebesgue measure on $\mathbf{T}^2$. Let $T : \mathbf{T}^2 \to \mathbf{T}^2$ be an ergodic translation of the two-dimensional torus and let $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_k\}$ be a finite, regular, and aperiodic partition of $\mathbf{T}^2$. If the boundaries $\partial\mathcal{W}_i$, $i = 1, ..., k$, are μ-negligible, then the topological dynamical system (Y, S) is uniquely ergodic and π is an isomorphism of the two measurable dynamical systems $(\mathbf{T}^2, T, \mu)$ and (Y, S, ν) where ν is the unique S-invariant probability on Y.*

We shall give the complete proof of these two theorems in the appendix.

Remark. Theorem A and B can be formulated in a more abstract setting of Abelian compact metric groups.

The complexity of symbolic sequences arising from translation in compact groups has already been studied. Steineder and Winkler [16] proved the following result. Let X be a compact group, let $\mathcal{W}$ be a subset of X whose boundary is of zero measure for the Haar measure and x and g be in X. Then the complexity of the sequence ω defined by $\omega(n) = 1$ if $x + ng \in \mathcal{W}$ and $\omega(n) = 0$ if $x + ng \in X\backslash\mathcal{W}$ is always subexponential. In the case of the torus $X = \mathbf{T}^d$ and of a box $\mathcal{W} = \Pi_{i=1}^d I_i$ they proved that $\lim_{n\to\infty} n^{-d} p_\omega(n) = 2^d \Pi l_i^{d-1}$ where l_i is the length of the interval I_i, $i = 1, ..., n$. In the two-dimensional case, the next result give the behavior of the complexity of the coding for polygonal subsets rather than for boxes.

Theorem 2 *Let $T : \mathbf{T}^2 \to \mathbf{T}^2$ be an ergodic translation of the two-dimensional torus and $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_k\}$ be a finite partition of $\mathbf{T}^2$. If the partition $\mathcal{W}$ is regular, aperiodic, and if the boundaries of each $\mathcal{W}_i$ is a finite union of segments then there exist two positive constants c and C such that for all ω in Y (see Theorem A)*

$$cn^2 \leq p_\omega(n) \leq Cn^2.$$

In view of the lemma 12 below, for all x in $\mathbf{T}^2$, the trajectory $(T^n x)_{n\in\mathbf{N}}$ may stay only a finite time in the $\partial\mathcal{W}_i$; this shows that the complexity $p_{\pi(x)}$ is of quadratic growth for all x in $\mathbf{T}^2$.

4.2 Stable sets

In order to prove Theorem 1, we introduce a class of partitions which agrees with Theorems A, B and 2. Propositions 8, 9 and 10 below summarize the properties of this class of partitions. Proposition 8 may be of independent interest.

Definition 7 *Let α be in $\mathbf{R}^d$and let $\mathcal{W}_1, ..., \mathcal{W}_p$ be compact subsets of $\mathbf{R}^d$. Denote by $\mathcal{K}$ the union $\cup_{i=1}^p \mathcal{W}_i$.*
*1. The set of compact subsets $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_p\}$ is α-**stable** if $\mathcal{K}$ is not empty, and if for each i in $\{1, ..., p\}$, there exists a lattice vector $e_i \in \mathbf{Z}^d$ such that $\mathcal{W}_i + \alpha + e_i \subset \mathcal{K}$.*

2. The set of compact subsets $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_p\}$ *is* ***regular*** *if:*
- for i *in* $\{1, ..., p\}$, $\mathcal{W}_i = \overline{\mathcal{W}_i^o}$
- for all $i \neq j$ *in* $\{1, ..., p\}$, $\mathcal{W}_i^o \cap \mathcal{W}_j = \emptyset$.
- for all i *in* $\{1, ..., p\}$, *the Lebesgue measure of the boundary of* $\mathcal{W}_i$ *is* 0.

Let us illustrate this definition by a simple example in one dimension. Take $\alpha \in [0, 1/2]$, $\mathcal{W}_1 = [0, 1]$ and $\mathcal{W}_2 = [1, 3/2]$. Since $\mathcal{W}_1 + \alpha \subset [0, 3/2]$ and $\mathcal{W}_2 + \alpha - 1 \subset [0, 3/2]$, $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2\}$ is α-stable.

Notation. Let α be in $\mathbf{R}^d$ and let $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_p\}$ be an α-stable set of compact subsets. Fix $e_1, ..., e_p$ in $\mathbf{Z}^d$ such that $\mathcal{W}_i + \alpha + e_i \subset \mathcal{K}$. We denote $F_{\mathcal{W}}$ the map defined on all subsets of $\mathbf{R}^d$ by

$$\forall A \subset \mathbf{R}^d,\ F_{\mathcal{W}}(A) = \cup_{i=1}^p (A \cap \mathcal{W}_i + \alpha + e_i).$$

The stability of a set of compact subsets is a crucial property linked to pieces exchange and to natural coding. Given closed subsets $\mathcal{V}_1, ..., \mathcal{V}_p$ of $\mathbf{R}^d$, it is always possible to define a map by

$$\forall A \subset \mathbf{R}^d,\ F(A) = \cup_{i=1}^p (A \cap \mathcal{V}_i + \alpha + e_i).$$

The property $F(\mathcal{K}) \subset \mathcal{K}$ means that $\mathcal{W} = \{\mathcal{V}_1 \cap \mathcal{K}, ..., \mathcal{V}_p \cap \mathcal{K}\}$ is α-stable, but it is not always easy to find a compact set $\mathcal{K}$ with this property. In the one-dimensional case, the sets $\mathcal{V}_1 =]-\infty, -\frac{1}{2} + \alpha]$, $\mathcal{V}_2 = [-\frac{1}{2} + \alpha, +\infty[$ and $\mathcal{K} = [-\frac{1}{2}, \frac{1}{2}]$ give rise to the "Sturmian" α-stable set of compacts subsets $\mathcal{W} = \{\mathcal{V}_1 \cap \mathcal{K} = [-\frac{1}{2}, -\frac{1}{2} + \alpha],\ \mathcal{V}_2 \cap \mathcal{K} = [-\frac{1}{2} + \alpha, \frac{1}{2}]\}$. In the two-dimensional case, the product of the α_1-stable set of compacts subsets $\mathcal{W}^1 = \{[-\frac{1}{2}, -\frac{1}{2} + \alpha_1],\ [-\frac{1}{2} + \alpha_1, \frac{1}{2}]\}$ and of the the α_2-stable set of compacts subsets $\mathcal{W}^2 = \{[-\frac{1}{2}, -\frac{1}{2} + \alpha_2],\ [-\frac{1}{2} + \alpha_2, \frac{1}{2}]\}$ gives an (α_1, α_2)-stable set $\mathcal{W} = \{\mathcal{W}^i \times \mathcal{W}^j,\ 1 \leq i, j \leq 2\}$ with 4 elements. However, we wish to find an α-stable set of compact subsets with only three elements because it implies that the associated coding sequences are C-balanced.

Each of the three next propositions corresponds to one property of the Rauzy fractal $\mathcal{R}$:
- $\mathcal{R}$ is a fundamental domain,
- there is a pieces exchange which induces the translation $x \in \mathbf{T}^2 \to x + \alpha \in \mathbf{T}^2$,
- the dynamical system $(\mathbf{T}^2, T)$ is isomorphic to a symbolic dynamical system.

In the propositions 8, 9, and 10 we assume the following hypothesis: α is an element of $\mathbf{R}^d$ such that the translation $x \in \mathbf{T}^d \to x + \alpha$ is ergodic, $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_p\}$ is a regular α-stable set of compact subsets of $\mathbf{R}^d$, and x_0 is in $\mathcal{K} = \cup_{i=1}^p \mathcal{W}_i$. Set

$$\mathcal{D}_\alpha = \overline{\cup_{n \in \mathbf{N}} F_{\mathcal{W}}^n(\{x_0\})}.$$

Proposition 8 *If the Lebesgue measure of* $\mathcal{K}$ *is strictly less than* 2, *then* $\mathcal{D}_\alpha$ *is a fundamental domain.*

For $i = 1, ..., p$, set
$$\mathcal{R}_i = \overline{\cup F^n_{\mathcal{W}}(\{x_0\})}$$
where the union is taken over all the integer n such that $F^n_{\mathcal{W}}(\{x_0\}) \subset \mathcal{W}_i$.

Proposition 9 *1. Suppose that the Lebesgue measure of $\mathcal{K}$ is strictly less than 2 and suppose that*
- $x_0 + \mathbf{Z}\alpha + \mathbf{Z}^d$ and $\mathcal{N} = \cup_{i=1}^p \partial\mathcal{W}_i$ have empty intersection,
- there exists an integer k such that $F^k_{\mathcal{W}}(\mathcal{K}) \cap (x_0 + \mathbf{Z}^d) = \{x_0\}$.
Then $\mathcal{D}_\alpha$ is a regular fundamental domain, $F_{\mathcal{W}}(\mathcal{D}_\alpha) = \mathcal{D}_\alpha$, for all i in $\{1, ..., p\}$,
$$\mathcal{R}_i = \mathcal{W}_i \cap \mathcal{D}_\alpha,$$
and
$$\overline{\mathcal{R}^o_i} = \mathcal{R}_i.$$
2. Moreover, if for each i in $\{1, ..., p\}$, the boundary of $\mathcal{W}_i$ is a finite union of segments then the boundary of $\mathcal{R}_i$, $i = 1, ..., p$, is a finite union of segments too.

Finally, we need a technical condition about the partition $\mathcal{W}$ which implies that the sets $\mathcal{W}_i \cap \mathcal{D}_\alpha$, $i = 1, ..., p$, give rise to an aperiodic partition of the torus (see definition 6). This condition is given in the next proposition which, together with Theorem B, will show that the translation $x \in \mathbf{T}^d \to x + \alpha \in \mathbf{T}^d$ is isomorphic to the desired symbolic dynamical system.

Notation. $\mathbf{p} : \mathbf{R}^d \to \mathbf{T}^d$ denotes the canonical projection.

Proposition 10 *Assume the hypothesis of proposition 9.1 and suppose moreover that there exists an open subset $\mathcal{T}$ included in $\mathcal{K}$ with the following properties:*
- $(\mathcal{T} + \mathbf{Z}^d \backslash \{0\}) \cap \mathcal{K} = \emptyset$,
- for all a in $\mathbf{R}^d \backslash \mathbf{Z}^d$ there exist i in $\{1, ..., p\}$ and x in $\mathcal{T} \cap \mathcal{W}^o_i$ such that $x + a \notin \mathcal{W}_i + \mathbf{Z}^d$.
Then
$$\{\mathbf{p}(\mathcal{R}_1), \mathbf{p}(\mathcal{R}_2) \backslash \mathbf{p}(\mathcal{R}_1), ..., \mathbf{p}(\mathcal{R}_p) \backslash \cup_{i=1}^{p-1} \mathbf{p}(\mathcal{R}_i)\}$$
is a finite regular and aperiodic partition of $\mathbf{T}^d$.

While the proof of proposition 8 is interesting, the proofs of propositions 9 and 10 are somewhat tedious. We shall use these propositions with a very simple set $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2\, \mathcal{W}_3\}$ for which these propositions may seem to be too general. But we think that a direct proof of theorem 1 avoiding propositions 9 and 10, would be quite technical. In fact, the set $\mathcal{W}$ is given explicitly, but the resulting fundamental domain $\mathcal{D}_\alpha$ and the resulting partition depend on α and are rather difficult to foresee.

5 Proof of Theorem 2

Definition 11 *A **segment** of* $\mathbf{T}^d$ *is the projection of a segment* $[a,b]$ *of* $\mathbf{R}^d$.

Notation. for an integer n, $\pi_n : \mathbf{T}^2 \to \mathcal{W}^{\{0,\dots,n\}}$ denote the map defined by $\pi(x)(k) = \mathcal{W}_i$ if $T^k x \in \mathcal{W}_i$.

Since by Theorem 2, (Y,S) is minimal, the set of finite subwords of any ω in Y does not depend on the choice of $\omega \in Y$. It follows that for all ω, $\omega' \in Y$ and for all $n \in \mathbf{N}$, $p_\omega(n) = p_{\omega'}(n)$. Therefore, it is enough to choose an x_0 in $\mathcal{G}$ and to compute the complexity of $\omega_0 = \pi(x_0)$.

We shall first prove the upper bound. Set

$$U_n = \mathbf{T}^2 \backslash \bigcup_{j=0}^{n-1} \bigcup_{i=1}^{k} T^{-j} \partial \mathcal{W}_i.$$

Note that if U is a connected component of U_n then the coding of all the elements x in U begin by the same word $\omega_U = \omega(0)...\omega(n-1)$ of length n. Since for every integer $m \geq 0$, $T^m x_0$ belongs to a connected component of U_n, the number of subwords of ω_0 of length n is less than the number of connected components of U_n. Denote by $\mathbf{U}$ the set of connected components of U_n

The set $\bigcup_{i=1}^k \partial\mathcal{W}_i$ is a finite union of segments. Denote by n_0 the number of such segments. Now, the boundary of U_n is a finite union of segments U_n, $S_1 \cup ... \cup S_N$ with

$$N \leq n_0 n \qquad (*)$$

(there could exist a segment of $\bigcup_{i=1}^k T^{-j}\partial\mathcal{W}_i$ included in $\bigcup_{i=1}^k T^{-l}\partial\mathcal{W}_i$). We can assume that the segments S_i are non overlapping. Thus for each pair (i,j) of distinct integers in $\{1,...,N\}$, the intersection $S_i \cap S_j$ is finite (remember that we are working in $\mathbf{T}^2$ and not in $\mathbf{R}^2$). Set $I = \bigcap_{i\neq j} S_i \cap S_j$ and $\mathcal{S} = \bigcup_{i=1}^N S_i \backslash I$. Denote by $\mathbf{S}$ the set of connected components of $\mathcal{S}$. The cardinality of $\mathbf{S}$ will give us the upper bound. Indeed, consider the set $\mathbf{P}$ of pairs (s,U) in $\mathbf{S}\times\mathbf{U}$ such that s is in the boundary of U. We have

$$\operatorname{card}\mathbf{U} \leq \operatorname{card}\mathbf{P} \leq 2\operatorname{card}\mathbf{S}.$$

Let L be the maximum of the length of the segments in the boundaries of the $\mathcal{W}_i$. Since the intersection of two segments of $\mathbf{T}^2$ of length L meet themselves in $L+1$ points at most, the cardinal number of I is $(L+2)N^2$ at most. It follows that

$$\operatorname{card}\mathbf{S} \leq (L+2)N^2 + N \leq Cn^2$$

where $C = 2n_0^2(L+2)$, in view of $(*)$.

It remains to find a lower bound. We can assume that $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2\}$. Making use of the hypothesis, $\overline{\mathcal{W}_i^o} = \overline{\mathcal{W}_i}$, it is obvious that for $i = 1, 2$,

$\partial\overline{\mathcal{W}_i}$ contains no isolated point and $\partial\overline{\mathcal{W}_i} = \partial\mathcal{W}_i = \partial\mathcal{W}_i^o$. It is also important to notice that $\partial\mathcal{W}_1 = \partial\mathcal{W}_2$. Making use of the aperiodicity, we shall prove that the boundary of $\mathcal{W}_1$ and $\mathcal{W}_2$ contains at least two segments say S_1 and S_2 which are not parallel. Indeed, suppose on the contrary that the boundary of $\mathcal{W}_1$ consists of one segment or two parallel segments. In the former case, there exist $a, \overrightarrow{u}$ in $\mathbf{R}^2$ and λ in $\mathbf{R}$ such that $\partial\overline{\mathcal{W}_1} = \mathbf{p}([a, a+\lambda\overrightarrow{u}])$, $|\overrightarrow{u}| < 1$ and $\lambda > 1$. If $x \in \mathcal{W}_i^o$, $i = 1$ or 2, and $y = x + \mathbf{p}(\overrightarrow{u})$ then y must be in $\mathcal{W}_i^o$ or in $\partial\overline{\mathcal{W}_i}$, therefore y is in $\overline{\mathcal{W}_i}$. Since the partition $\{\mathcal{W}_1, \mathcal{W}_2\}$ is aperiodic, $\overrightarrow{u}$ is a lattice vector and since $|\overrightarrow{u}| < 1$, $\overrightarrow{u} = 0$. It follows that $\partial\overline{\mathcal{W}_i}$ is reduced to a single point which is impossible. In the latter case there exists $a_1, a_2, \overrightarrow{u}$ in $\mathbf{R}^2$ and λ_1, λ_2 in $\mathbf{R}$ such that $\partial\overline{\mathcal{W}_1} = \mathbf{p}([a_1, a_1 + \lambda_1\overrightarrow{u}]) \cup \mathbf{p}([a_2, a_2 + \lambda_2\overrightarrow{u}])$, $|\overrightarrow{u}| < 1$ and $\lambda_1, \lambda_2 > 1$. As in former case we see that $\overrightarrow{u} = 0$ which leads to a contradiction.

We have $\partial\mathcal{W}_1 = \partial\mathcal{W}_2$, and these boundaries contain the two segments S_1 and S_2 which are not reduced to single points, therefore there exist two open balls B_1 and B_2 such that for $i = 1, 2$,

$$B_i \cap (\partial\mathcal{W}_1 = \partial\mathcal{W}_2) = B_i \cap S_i,$$

is a diameter of the ball B_i. Hence, each ball B_i is the disjoint union of the two open half disks $B_i \cap \mathcal{W}_1^o$, and $B_i \cap \mathcal{W}_2^o$ and of the diameter $B_i \cap S_i$. This means that for $i = 1, 2$, there exist a point A_i in $\mathbf{R}^2$ and two vectors $\overrightarrow{u_i}$ and $\overrightarrow{v_i}$ in $\mathbf{R}^2 \backslash \{0\}$ such that ,

$$\begin{gathered}
\mathbf{p}([A_i - \overrightarrow{u_i}, A_i + \overrightarrow{u_i}]) \subset S_i, \ \overrightarrow{u_i} \perp \overrightarrow{v_i}, \\
\mathbf{p}(A_i + \mathcal{R}_i^+) \subset \mathcal{W}_2, \\
\mathbf{p}(A_i + \mathcal{R}_i^-) \subset \mathcal{W}_1
\end{gathered}$$

where

$$\begin{aligned}
\mathcal{R}_i^+ &= \{t\overrightarrow{u_i} + s\overrightarrow{v_i} : t \in]-1, 1[, \ s \in]0, 1[\}, \\
\mathcal{R}_i^- &= \{t\overrightarrow{u_i} + s\overrightarrow{v_i} : t \in]-1, 1[, \ s \in]-1, 0[\},
\end{aligned}$$

$i = 1, 2$. Since the segments S_1 and S_2 are not parallel, the vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are not collinear. Choosing some shorter vectors we can assume that

$$\begin{aligned}
\mathcal{R}_1^+ &= \{t\overrightarrow{u_1} + s\overrightarrow{u_2} : t \in]0, 1[, \ s \in]0, 1[\}, \\
\mathcal{R}_1^- &= \{t\overrightarrow{u_1} + s\overrightarrow{u_2} : t \in]0, 1[, \ s \in]-1, 0[\}
\end{aligned}$$

and that

$$\begin{aligned}
\mathcal{R}_2^+ &= \{t\overrightarrow{u_2} + s\overrightarrow{u_1} : t \in]0, 1[, \ s \in]0, 1[\} = \mathcal{R}_1^+, \\
\mathcal{R}_2^- &= \{t\overrightarrow{u_2} + s\overrightarrow{u_1} : t \in]0, 1[, \ s \in]-1, 0[\}.
\end{aligned}$$

Consider the parallelograms $\mathbf{P}$, $\mathbf{P}_1$ and $\mathbf{P}_2$ defined by

$$\begin{aligned}
\mathbf{P} &= \{t\overrightarrow{u_1} + s\overrightarrow{u_2} : t \in]0, 1/2[, \ s \in]0, 1/2[\}, \\
\mathbf{P}_1 &= \{t\overrightarrow{u_1} + s\overrightarrow{u_2} : t \in]-1/2, 0[, \ s \in]0, 1/2[\}, \\
\mathbf{P}_2 &= \{t\overrightarrow{u_1} + s\overrightarrow{u_2} : t \in]0, 1/2[, \ s \in]-1/2, 0[\},
\end{aligned}$$

For all integer n, set

$$J_{i,n} = \{k \in \{0, ..., n\} : T^{-k} A_i \in \mathbf{p}(\mathbf{P}_i)\},\ i = 1, 2.$$

By uniform distribution of the sequence $(T^{-k}(A_i))_{k \in \mathbf{N}}$, there exist a positive constant c and an integer N, such that for all $n \geq N$,

$$\text{card } J_{i,n} \geq cn,\ i = 1, 2.$$

By the lemma 12 (below), there is a constant K and two subsets $J'_{1,n}$ and $J'_{2,n}$ of $J_{1,n}$ and $J_{2,n}$ such that

$$n_i = \text{card } J'_{i,n} \geq \frac{c}{K} n,\ i = 1, 2$$

and for all $p, q \in J'_{i,n}$,

$$(T^{-p} A_i + \mathbf{p}(\{t\overrightarrow{u_i} : t \in [0, 1]\})) \cap (T^{-q} A_i + \mathbf{p}(\{t\overrightarrow{u_i} : t \in [0, 1]\}) = \emptyset,\ i = 1, 2.$$

Now, for all $p \in J'_{i,n}$, $\mathbf{P}$ is included in the union of $T^{-p} A_i + \mathcal{R}_i^+$, $T^{-p} A_i + \mathcal{R}_i^-$ and the segment $T^{-p} S_i$, $i = 1, 2$. Furthermore,

$$\mathbf{P} \cap (T^{-p} A_i + \mathcal{R}_i^+) \neq \emptyset,\ \mathbf{P} \cap (T^{-p} A_i + \mathcal{R}_i^-) \neq \emptyset,\ i = 1, 2.$$

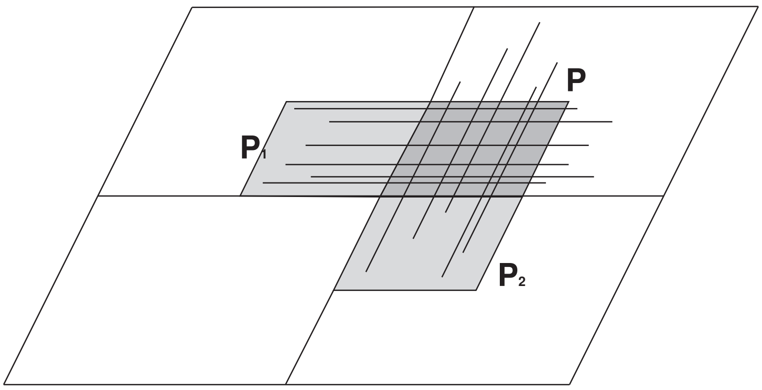

Figure 1:

Hence for each $p \in J'_{1,n}$, the horizontal segment $T^{-p} A_1 + \mathbf{p}(\{t\overrightarrow{u_1} : t \in [0, 1]\})$ crosses the parallelogram $\mathbf{P}$ and splits it into two parts (see figure 1). The lower one is in $T^{-p}(\mathcal{W}_1)$ and the above one is in $T^{-p}(\mathcal{W}_2)$. The same holds for each inclined segment $T^{-p} A_2 + \mathbf{p}(\{t\overrightarrow{u_2} : t \in [0, 1]\})$ with $p \in J'_{2,n}$.

Therefore, $\mathbf{P}$ is divided in $m = (n_1 + 1)(n_2 + 1)$ parallelograms $\mathcal{Q}_1, ..., \mathcal{Q}_m$ (see figure 1) such that if $x \in \mathcal{Q}_i^o$ and $y \in \mathcal{Q}_j^o$ with $i \neq j$ there exists p_1 in $J'_{1,n}$ or p_2 in $J'_{2,n}$, with

$$\text{card}(\{x, y\} \cap (T^{-p_1} A_1 + \mathcal{R}_1^-)) = 1$$

or

$$\text{card}(\{x, y\} \cap (T^{-p_2} A_2 + \mathcal{R}_1^-)) = 1.$$

It follows that the p_1-th or the p_2-th letter of the words $\pi_n(x)$ and $\pi_n(y)$ are different. The number of such words is therefore at least m. By minimality, it follows that

$$p_n(\pi(x_0)) \geq m \geq \frac{c^2}{K^2} n^2. \square$$

Lemma 12 *Let $\alpha \in \mathbf{R}^2$ and $T : \mathbf{T}^2 \to \mathbf{T}^2$ be the translation defined by $T(x) = x + \alpha$. If T is ergodic and $S = [s_0, s_1]$ is a segment of $\mathbf{R}^2$, then there exists a constant K such that for any x in $\mathbf{T}^2$,*

$$\operatorname{card} \mathbf{p}(S) \cap \{T^n x : n \in \mathbf{N}\} \leq K.$$

Proof. If the slope of S is rational then for all x in $\mathbf{T}^2$ there is at most one integer n with $T^n x \in \mathbf{p}(S)$ for the translation T is ergodic. Therefore we can assume that the slope of S is irrational. Consider the line $\mathcal{D}$ parallel to S containing 0. The projection $\mathbf{p}$ is one to one on $\mathcal{D}$. Set $q_0 = \min\{q \in \mathbf{N}\backslash\{0\} : q\alpha \in \mathcal{D} + \mathbf{Z}^2\}$. There is a point P_0 in $\mathbf{Z}^2$ such that $q_0\alpha + P_0 \in \mathcal{D}$. Set $r = d(0, q_0\alpha + P_0)$ and $\lambda = \frac{l}{r}$ where l is the length of the segment S.

Let us show that if A is in $S + \mathbf{Z}^2$ and if $k > \lambda$ then $A + kq_0\alpha$ is not in $S + \mathbf{Z}^2$. Suppose on the contrary that the point $A + kq_0\alpha \in S + \mathbf{Z}^2$. It follows that $kq_0\alpha$ is in $S - S + \mathbf{Z}^2$ and that there exists P in $\mathbf{Z}^2$ such that $kq_0\alpha + P \in S - S$. Since $S - S \subset \mathcal{D}$, the point $P - kP_0 = kq_0\alpha + P - k(q_0\alpha + P_0)$ is in $\mathcal{D}$ which implies that $P = kP_0$ and that

$$|kq_0\alpha + P| = k\,|q_0\alpha + P_0| > l.$$

But $kq_0\alpha + P$ belongs to the segment $S - S$ which is included in $\{x : |x| < l\}$. This contradicts the above inequality.

Fix an integer $k_0 > \lambda$ and set $Q = k_0 q_0$. For each $r = 0, ..., Q - 1$ there is at most one integer a such that $(aQ + r)\alpha \in S + \mathbf{Z}^2$ and therefore there is at most Q integers q such that $q\alpha \in S + \mathbf{Z}^2$. $\square$

6 Proof of proposition 8

6.1 Pieces exchange

The proof of proposition 8 needs several easy auxiliary results. Among them, proposition 13 and lemma 15 may be of independent interest.

Let $\mathcal{W}_1, ..., \mathcal{W}_k$, and $\mathcal{A}$ be a measurable subsets of $\mathbf{R}^d$. Recall that $\{\mathcal{W}_1, ..., \mathcal{W}_p\}$ is a measurable partition of $\mathcal{A}$ modulo 0 if
- $\mathcal{A} = \mathcal{W}_1 \cup ... \cup \mathcal{W}_p \mod 0$,
- the set $\mathcal{W}_1, ..., \mathcal{W}_p$ are disjoint modulo 0.

Proposition 13 *Let $\alpha = (\alpha_1, ..., \alpha_d)$ be in $\mathbf{R}^d$ and let $\mathcal{D}$ be a measurable subset of $\mathbf{R}^d$. Assume that $1, \alpha_1, ..., \alpha_d$ are linearly independent over $\mathbf{Q}$.*

If there exist a measurable partition modulo 0, $\{\mathcal{W}_1, ..., \mathcal{W}_p\}$ *of* $\mathcal{D}$ *and* $e_1, ..., e_p$ *in* $\mathbf{Z}^d$ *such that*

$$\mathcal{D} = \bigcup_{i=1}^{p} (\mathcal{W}_i + \alpha - e_i) \mod 0,$$

then the map $X \in \mathbf{T}^d \mapsto \mathrm{card}(X \cap \mathcal{D}) \in \mathbf{N}$ *is a.e. constant (an element* X *in* $\mathbf{T}^d$ *is a subset of* $\mathbf{R}^d$*).*

Notation: Let $\mu(\mathcal{A})$ denote the Lebesgue measure of a measurable subset $\mathcal{A}$ of $\mathbf{R}^d$ of $\mathbf{T}^d$.

Proof of proposition 13.

The sets $\mathcal{W}_i + \alpha - e_i$ have the same Lebesgue measure as the sets $\mathcal{W}_i$, $i = 1, ..., p$, which are disjoint, moreover

$$\bigcup_{i=1}^{p} (\mathcal{W}_i + \alpha - e_i) = \mathcal{D} = \bigcup_{i=1}^{p} \mathcal{W}_i \mod 0,$$

therefore $\mu((\mathcal{W}_i + \alpha_i - e_i) \cap (\mathcal{W}_j + \alpha_j - e_j)) = 0$, for all $i \neq j$. Removing a subset of zero measure to each $\mathcal{W}_i$, we can assume that the sets $\mathcal{W}_i$ are disjoint as well as the sets $\mathcal{W}_i + \alpha - e_i$, and that $\bigcup_{i=1}^{p} (\mathcal{W}_i + \alpha - e_i)$ is included in $\mathcal{D}$.

Consider the map $nb : X \in \mathbf{T}^d \mapsto \mathrm{card}(X \cap \mathcal{D})$ and for each integer k, the set $E_k = \{X \in \mathbf{T}^d : nb(X) \geq k\}$. We wish to prove that the measure of E_k is 0 or 1 for all integers k. Since the translation $T : X \in \mathbf{T}^2 \to X + \alpha$ is ergodic and since the Lebesgue measure is T-invariant, it is enough to prove that for all integer k, $T(E_k) \subseteq E_k \mod 0$. Fix an integer k.

Denote by $\mathcal{N}$ the set of X in $\mathbf{T}^d$ which have an empty intersection with $\mathcal{D} \backslash \bigcup_{i=1}^{p} \mathcal{W}_i$. The set $\mathcal{N}$ is of zero measure. Let X be in $E_k \backslash \mathcal{N}$ and let $x \neq y$ be two elements of $X \cap \mathcal{D}$. There is two integers $i(x)$ and $i(y)$ such that $x \in \mathcal{W}_{i(x)}$ and $y \in \mathcal{W}_{i(y)}$. If $i(x) = i(y) = i$ then $x + \alpha - e_i$ and $y + \alpha - e_i$ are distinct. If $i(x) \neq i(y)$ then $x + \alpha - e_i$ and $y + \alpha - e_i$ belong to $\mathcal{W}_{i(x)} + \alpha - e_i$ and to $\mathcal{W}_{i(y)} + \alpha - e_i$ which are disjoint sets, therefore $x + \alpha - e_{i(x)}$ and $y + \alpha - e_{i(y)}$ are distinct in all cases. It follows that the cardinality of the set $T(X) \cap \mathcal{D}$ is greater or equal to k, thus $T(X) \in E_k$.
$\square$

Fix $\alpha = (\alpha_1, ..., \alpha_d)$ in $\mathbf{R}^d$ such that 1,α_1,...,α_d are independent over $\mathbf{Q}$.

Corollary 14 *Let* $\mathcal{W}_1, ..., \mathcal{W}_p$*, and* $\mathcal{K}$ *be compact subsets of* $\mathbf{R}^d$ *and let* $e_1, ..., e_p$ *be in* $\mathbf{Z}^d$*. Assume that* $\{\mathcal{W}_1, ..., \mathcal{W}_p\}$ *is a measurable partition of* $\mathcal{K}$ *modulo* 0*. Consider the map* F *defined over the subsets of* $\mathbf{R}^d$ *by*

$$F : \mathcal{A} \subseteq \mathbf{R}^d \mapsto \bigcup_{i=1}^{p} (\mathcal{A} \cap \mathcal{W}_i + \alpha - e_i) \subseteq \mathbf{R}^d.$$

Set

$$\mathcal{D} = \bigcap_{n \in \mathbb{N}} F^n(\mathcal{K}).$$

If $F(\mathcal{K}) \subseteq \mathcal{K}$, then the map $X \in \mathbf{T}^2 \mapsto \mathrm{card}(X \cap \mathcal{D})$ is a.e. constant on $\mathbf{T}^d$.

This corollary is a simple consequence of the proposition 13 and of the following lemma which is a slight extension of the well known fact:

If $(\mathcal{K}_n)$ is a sequence of nested compact sets and if $f : \mathcal{K}_0 \to \mathcal{K}_0$ is a continuous map, then $f(\cap_{n\geq 0}\mathcal{K}_n) = \cap_{n\geq 0} f(\mathcal{K}_n)$.

Lemma 15 *Let $\mathcal{W}_1$,..., $\mathcal{W}_p$, and $\mathcal{K}$ be compact subsets of $\mathbf{R}^d$ such that $\mathcal{K} = \mathcal{W}_1 \cup ... \cup \mathcal{W}_p$. Assume that $f_i : \mathcal{W}_i \to \mathcal{K}$, $i = 1, ..., p$, are continuous maps. Consider the map F defined over the subsets of $\mathcal{K}$ by*

$$F : \mathcal{A} \subseteq \mathcal{K} \to \bigcup_{i=1}^{p} f_i(\mathcal{A} \cap \mathcal{W}_i) \subseteq \mathcal{K}.$$

Set $\mathcal{D} = \bigcap_{n\in\mathbf{N}} F^n(\mathcal{K})$. Then $F(\mathcal{D}) = \mathcal{D}$.

The proof of lemma 15 is easy and we leave it to the reader.

6.2 End of proof of proposition 8

We have to show that the set $\mathcal{D}_\alpha = \overline{\{F^n(\{x_0\}) : n \in \mathbf{N}\}}$ is a fundamental domain. We first prove that the set $\mathcal{A} = \cap_{n\geq 0}\overline{\cup_{k\geq n} F^k(\{x_0\})}$ is a fundamental domain. Next, we prove that $\mathcal{D}_\alpha$ is equal to the set $\cup_{n\geq 0} F^n(\{x_0\}) \cup \mathcal{A}$. Proposition 8 will follow for $\cup_{n\geq 0} F^n(\{x_0\})$ is countable.

Since $x_0 \in \mathcal{K}$, $\mathcal{A}$ is included in $\bigcap_{n\geq 0} F^n(\mathcal{K}) = \mathcal{D}$. By the previous corollary, the map $X \in \mathbf{T}^d \mapsto \mathrm{card}(X \cap \mathcal{D})$ is a.e. constant on $\mathbf{T}^d$. By hypothesis $\mu(\mathcal{K}) < 2$ and since $\mathcal{D}$ is included in $\mathcal{K}$, we have $\mu(\mathcal{D}) < 2$. Hence, $\mathrm{card}(X \cap \mathcal{D}) = 0$ for almost every $X \in \mathbf{T}^d$ or $\mathrm{card}(X \cap \mathcal{D}) = 1$ for almost every $X \in \mathbf{T}^d$. Thus the only thing to show is that $\mathcal{A}$ contains a fundamental domain. Recall that $\mathbf{p} : \mathbf{R}^d \to \mathbf{T}^d$ denote the canonical projection. For each integer $k > 0$, choose a point x_k in $F^k(x_0)$. Since $\mathbf{p}(x_k) = \mathbf{p}(x_0 + k\alpha)$, the set $S = \{\mathbf{p}(x_k) : k \in \mathbf{N}\}$ is everywhere dense in $\mathbf{T}^d$. Let Y be an element of $\mathbf{T}^d$ and $(Y_p = \mathbf{p}(x_{k_p}))_{p\in\mathbb{N}}$ be a sequence in S which converges to Y. The subsequence $(x_{k_p})_p$ being bounded, we can assume, that $(x_{k_p})_p$ converge to $y \in \mathbf{R}^d$. Since $\mathcal{A}$ is closed, $y \in \mathcal{A}$, hence $Y \in \mathbf{p}(\mathcal{A})$. Therefore $\mathcal{A}$ contains a fundamental domain.

Let x be a point in $\mathcal{D}_\alpha$ not in $\cup_{n\geq 0} F^n(\{x_0\})$. There exists a sequence $(k_p)_{p\in\mathbf{N}}$ of integers and a sequence $(y_p)_{p\in\mathbf{N}}$ such that $\lim_{p\to\infty} y_p = x$ and for all p, $y_p \in F^{k_p}(\{x_0\})$. If the sequence $(k_p)_{p\in\mathbf{N}}$ is bounded by an integer N, then for each p, y_p belongs to $\cup_{n\leq N} F^n(\{x_0\})$ which is a finite set. Therefore, $x = \lim_{p\to\infty} y_p$ must belongs to this finite set which is included in $\cup_{n\geq 0} F^k(\{x_0\})$. It follows that the sequence $(k_p)_{p\in\mathbf{N}}$ is unbounded. Extracting a subsequence we may suppose that $k_p \to \infty$ as p goes to infinity. It follows that $x = \lim_{p\to\infty} y_p$ is in $\overline{\cup_{k\geq n} F^k(\{x_0\})}$ for all integers n which shows that x is in $\mathcal{A}$. □

7 Proof of proposition 9

1. Since $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_p\}$ is an α-stable set of compact subsets, we have $F(\mathcal{K} = \cup_{i=1}^p \mathcal{W}_i) \subset \mathcal{K}$, where the map $F : \mathcal{A} \subset \mathbf{R}^d \to F(\mathcal{A}) \subset \mathbf{R}^d$ is defined by $F(\mathcal{A}) = \bigcup_{i=1}^p (\mathcal{A} \cap \mathcal{W}_i + \alpha + e_i)$.

Step 1. *For all integer n, $F^n(\{x_0\})$ consists of a single point x_n and x_0 is a limit point of the sequence $(x_n)_{n\in\mathbf{N}}$.*
Since the set $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_p\}$ is a regular set of compact subsets, the interiors of the $\mathcal{W}_i$, $i = 1, ..., p$, are disjoint and since $x_0 + \mathbf{N}\alpha + \mathbf{Z}^d$ does not meet the boundaries of $\mathcal{W}_i$, $i = 1, ..., p$, for all integer n, $F^n(\{x_0\})$ consists of a single point which we denote by x_n. Hence

$$\mathcal{D}_\alpha = \overline{\{x_n : n \in \mathbf{N}\}}.$$

By proposition 8, $\mathcal{D}_\alpha$ is a fundamental domain. The set

$$\mathcal{A} = \cap_{n\geq 0}\overline{\{x_k : k \geq n\}}$$

of limit point of the sequence $(x_n)_{n\in\mathbf{N}}$ is a closed subset of $\mathcal{D}_\alpha$ and $\mathcal{D}_\alpha \backslash \mathcal{A}$ is countable, therefore $\mathcal{A}$ is fundamental domain. It follows that there exists a sequence of lattice points $(P_n)_{n\in\mathbf{N}}$ in $\mathbf{Z}^d$ such that x_0 is a limit point of the sequence $(x_n + P_n)_{n\in\mathbf{N}}$. Since the sequence $(x_n)_{n\in\mathbf{N}}$ is bounded, $P_n = P$ for infinitely many integers n. Therefore, $x_0 - P$ belongs to $F^k(\mathcal{K})$ for all integer k. But by hypothesis, there is an integer k such that $F^k(\mathcal{K}) \cap (x_0 + \mathbf{Z}^d) = \{x_0\}$, thus $P = 0$ and x_0 is a limit point of the sequence $(x_n)_{n\in\mathbf{N}}$.

Step 2. $F(\mathcal{D}_\alpha) = \mathcal{D}_\alpha$.
It follows from step 1 that x_0 is in $\cap_{n\in\mathbf{N}} F^n(\mathcal{K}) = \mathcal{D}$ and that

$$\mathcal{D}_\alpha = \overline{\{x_n : n \in \mathbf{N}\}} \subset \mathcal{D}.$$

Since

$$F(\{x_n : n \in \mathbf{N}\}) \subset \{x_n : n \in \mathbf{N}\},$$

$$\begin{aligned} F(\mathcal{D}_\alpha) &= F(\overline{\{x_n : n \in \mathbf{N}\}}) \subset \overline{F(\{x_n : n \in \mathbf{N}\})} \\ &\subset \overline{\{x_n : n \in \mathbf{N}\}} \\ &= \mathcal{D}_\alpha. \end{aligned}$$

It follows that the sequence $(F^n(\mathcal{D}_\alpha))_{n\in\mathbf{N}}$ is nested. Since x_0 is a limit point of the sequence $(x_n)_{n\in\mathbf{N}}$, x_0 belongs to $\cap_{n\in\mathbf{N}} F^n(\mathcal{D}_\alpha)$. But $F(\cap_{n\in\mathbf{N}} F^n(\mathcal{D}_\alpha)) \subset \cap_{n\in\mathbf{N}} F^n(\mathcal{D}_\alpha)$, therefore the whole sequence $(x_n)_{n\in\mathbf{N}}$ is in $\cap_{n\in\mathbf{N}} F^n(\mathcal{D}_\alpha)$. Finally, this shows that $F(\mathcal{D}_\alpha) = \mathcal{D}_\alpha$.

Step 3. $\overline{\mathcal{D}_\alpha^o} = \mathcal{D}_\alpha$.
By Bair's theorem, the interior of a compact set which is a fundamental domain can't be empty. Therefore the interior $\mathcal{D}_\alpha^o$ of $\mathcal{D}_\alpha$ is nonempty. Let x be in $\mathcal{D}_\alpha^o \backslash \mathcal{N}$ where $\mathcal{N} = \cup_{i=1}^p \partial\mathcal{W}_i$. There is a positive real number r such that $B(x, r) \subset \mathcal{D}_\alpha^o \backslash \mathcal{N}$. This ball $B(x, r)$ is included in the interior of one of the $\mathcal{W}_i$, therefore $F(B(x, r))$ is a ball $B(y, r)$ where $\{y\} = F(\{x\})$.

Moreover $F(\mathcal{D}_\alpha) \subset \mathcal{D}_\alpha$, thus y is in $\mathcal{D}_\alpha^o$.
Since the translation $T : u \in \mathbf{T}^2 \to u + \alpha \in \mathbf{T}^2$ is minimal, the set $\cup_{n\in\mathbf{N}} F^n(\mathcal{D}_\alpha^o)$ contains a fundamental domain. But this set is included in

$$\begin{aligned}\cup_{n\in\mathbf{N}} F^n(\mathcal{D}_\alpha) &= \mathcal{D}_\alpha \\ &\subset \mathcal{D} = \cap_{n\in\mathbf{N}} F^n(\mathcal{K})\end{aligned}$$

and by hypothesis, $(x_0 + \mathbf{Z}^2) \cap F^k(\mathcal{K}) = \{x_0\}$, therefore, there exists an integer $k_0 \geq 0$ and an element z in the interior of $\mathcal{D}_\alpha^o$ such that $x_0 \in F^{k_0}(\{z\})$. The point z is in $x_0 - k_0\alpha + \mathbf{Z}^2$ which does not intersect $\mathcal{N}$, thus $F(\{z\})$ is a single point which is in $\mathcal{D}_\alpha^o$. Iterating this process, we see that

$$F(\{z\}), F^2(\{z\}), ..., F^k(\{z\}) = x_0, x_1, ..., x_n, ... \in \mathcal{D}_\alpha^o.$$

It follows that $\overline{\mathcal{D}_\alpha^o} = \mathcal{D}_\alpha$. This shows that for each point x in $\mathcal{D}_\alpha^o$ there is no lattice point $P \neq 0$ such that $x + P \in \mathcal{D}_\alpha$, for, $x \in \mathcal{D}_\alpha^o$ and $x + P \in \overline{\mathcal{D}_\alpha^o}$ contradict the fact that $\mathcal{D}_\alpha$ is fundamental domain.

Step 4. *Set* $\mathcal{R}_i = \overline{\{x_n : x_n \in \mathcal{W}_i\}}$ *and* $\mathcal{U}_i = \mathcal{W}_i \cap \mathcal{D}_\alpha$. *We have* $\mathcal{R}_i \subset \mathcal{U}_i$, $\overline{\mathcal{R}_i^o} = \mathcal{R}_i$ *and* $\mathcal{R}_i^o = \mathcal{D}_\alpha^o \cap \mathcal{W}_i^o$, $i = 1, ..., p$.
Clearly

$$\mathcal{R}_i = \overline{\{x_n : x_n \in \mathcal{W}_i\}} \subset \mathcal{W}_i \cap \mathcal{D}_\alpha = \mathcal{U}_i,$$

and

$$\begin{aligned}\mathcal{D}_\alpha &= \overline{\{x_n : n \in \mathbf{N}\}} \\ &= \cup_{i=1}^p \overline{\{x_n : x_n \in \mathcal{W}_i\}} \\ &= \cup_{i=1}^p \mathcal{R}_i.\end{aligned}$$

Let x be in $\mathcal{U}_i^o$. Since the set $\mathcal{W} = \{\mathcal{W}_1, ..., \mathcal{W}_p\}$ is a regular (see definition 7), the point x which is in $\mathcal{W}_i^o$ is not in $\cup_{j\neq i}\mathcal{W}_j$. Therefore x is not in $\cup_{j\neq i}\mathcal{R}_j$ and since $\mathcal{U}_i^o = \mathcal{W}_i^o \cap \mathcal{D}_\alpha^o$, we get

$$\mathcal{U}_i^o \subset \mathcal{D}_\alpha^o \setminus \cup_{j\neq i} \mathcal{R}_j.$$

It follows that $\mathcal{U}_i^o \subset \mathcal{R}_i$ and that $\mathcal{R}_i^o = \mathcal{U}_i^o = \mathcal{D}_\alpha^o \cap \mathcal{W}_i^o$. But, for all integer n, x_n is never on the boundary of the $\mathcal{W}_i$, $i = 1, ..., p$, thus

$$\{x_n : x_n \in \mathcal{W}_i\} \subset \mathcal{W}_i^o \cap \mathcal{D}_\alpha^o$$

and therefore $\mathcal{R}_i \subset \overline{\mathcal{W}_i^o \cap \mathcal{D}_\alpha^o}$ and

$$\overline{\mathcal{R}_i^o} = \mathcal{R}_i.$$

2. Suppose now that the boundary of each $\mathcal{W}_i$ is a finite union of segments and let us show that the boundary of each $\mathcal{R}_i$ is a finite union of segments. Choose a closed square $\mathcal{C}$ of nonempty interior included in $\mathcal{D}_\alpha^o$. Since the translation $T : x \in \mathbf{T}^2 \to x + \alpha \in \mathbf{T}^2$ is minimal and since

$\mathbf{T}^2$ is compact, there exists n_0 in $\mathbf{N}$ such that $\cup_{n=\mathbf{0}}^{n_0} T^n(\mathbf{p}(\mathcal{C}^o)) = \mathbf{T}^2$. It follows that $\cup_{n=0}^{n_0} F^n(\mathcal{C})$ is a fundamental domain which is included in $\mathcal{D}_\alpha$. Since $\mathcal{D}_\alpha^o \cap (\mathcal{D}_\alpha^o + P) = \emptyset$ for all non zero P in $\mathbf{Z}^d$, $\mathcal{D}_\alpha^o$ must be included in $\cup_{n=0}^{n_0} F^n(\mathcal{C})$, this shows that $\mathcal{D}_\alpha = \cup_{n=0}^{n_0} F^n(\mathcal{C})$.

Finally, by our hypothesis, the boundary of each $\mathcal{W}_i$ is a finite union of segments, therefore the boundary of $\mathcal{D}_\alpha = \cup_{n=0}^{n_0} F^n(\mathcal{C})$ is also a finite union of segments. It follows that the same property holds for $\mathcal{U}_i$ and thus for $\mathcal{U}_i^o$ (see lemma below whose proof is left to the reader). But $\mathcal{U}_i^o = \mathcal{R}_i^o$ and $\partial \mathcal{R}_i = \partial \mathcal{R}_i^o$ therefore the boundary of $\mathcal{R}_i$ is a finite union of segments. $\square$

Lemma 16 *Let F be closed subset of $\mathbf{R}^2$. If its boundary is a finite union of segments then the boundary of the interior of F is finite union of segments (the boundary of the interior may be strictly included in the boundary of F).*

8 Proof of proposition 10

Let us show that the partition of the torus

$$\mathcal{R} = \{\mathbf{p}(\mathcal{R}_1), \mathbf{p}(\mathcal{R}_2)\backslash \mathbf{p}(\mathcal{R}_1), ..., \mathbf{p}(\mathcal{R}_p)\backslash \cup_{i=1}^{p} \mathbf{p}(\mathcal{R}_i)\}$$

is aperiodic.

Let a be in $\mathbf{R}^d \backslash \mathbf{Z}^d$. By assumption, there exist an x in $\mathcal{T}$ and i in $\{1, ..., p\}$ such that $x \in \mathcal{T} \cap \mathcal{W}_i^o$ and $x + a \notin \mathcal{W}_i + \mathbf{Z}^d$. Since $(\mathcal{T} + \mathbf{Z}^d \backslash \{0\}) \cap \mathcal{K} = \emptyset$ and since $\mathcal{D}_\alpha$ is a fundamental domain included in $\mathcal{K}$, $\mathcal{T}$ must be included in $\mathcal{D}_\alpha$. Therefore $x \in \mathcal{D}_\alpha^o \cap \mathcal{W}_i^o = \mathcal{R}_i^o$ (see the proof of proposition 9, step 4). It follows that $y = \mathbf{p}(x)$ is in $\mathbf{p}(\mathcal{R}_i^o)$ and that $y + a = \mathbf{p}(x + a)$ is in $\mathbf{T}^d \backslash \mathbf{p}(\mathcal{W}_i)$ which is included in $\mathbf{T}^2 \backslash \mathbf{p}(\mathcal{R}_i)$**.** It follows that the partition $\mathcal{R}$ is aperiodic.

For all i, $\overline{\mathbf{p}(\mathcal{R}_i^o)}$ contains $\mathbf{p}(\overline{\mathcal{R}_i^o}) = \mathbf{p}(\mathcal{R}_i)$, thus the partition $\mathcal{R}$ is regular. The last thing to prove is that $\partial \mathbf{p}(\mathcal{R}_i^o) = \mathbf{p}(\partial \mathcal{R}_i^o)$ for it will show that the boundary of $\mathbf{p}(\mathcal{R}_i^o)$ is a finite union of segments. On the one hand,

$$\begin{aligned}\partial \mathbf{p}(\mathcal{R}_i^o) = \overline{\mathbf{p}(\mathcal{R}_i^o)} \backslash \mathbf{p}(\mathcal{R}_i^o) &\subset \mathbf{p}(\mathcal{R}_i) \backslash \mathbf{p}(\mathcal{R}_i^o) \\ &\subset \mathbf{p}(\mathcal{R}_i \backslash \mathcal{R}_i^o) = \mathbf{p}(\partial \mathcal{R}_i^o).\end{aligned}$$

On the other hand, $\mathbf{p}(\mathcal{D}_\alpha \backslash \mathcal{R}_i^o)$ and $\mathbf{p}(\mathcal{R}_i^o)$ have empty intersection for $\mathcal{D}_\alpha = \overline{\mathcal{D}_\alpha^o}$; therefore $\mathbf{p}(\mathcal{R}_i \backslash \mathcal{R}_i^o)$ and $\mathbf{p}(\mathcal{R}_i^o)$ have empty intersection. Since $\mathbf{p}(\mathcal{R}_i \backslash \mathcal{R}_i^o)$ is included in $\mathbf{p}(\mathcal{R}_i)$, it is also included in $\mathbf{p}(\mathcal{R}_i) \backslash \mathbf{p}(\mathcal{R}_i^o)$. $\square$

9 Proof of Theorem 1

9.1 The compact $\mathcal{K}$

Recall that $e_0 = (0, 0), e_1 = (1, 0)$ and $e_2 = (0, 1)$. We wish to use propositions 8, 9 and 10. This is why we want to find an α-stable and regular set $\mathcal{W} = \{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2\}$ of compact subsets. The set $\mathcal{W}$ will be linked to the sequence $(P_n)_{n\in\mathbf{N}}$ whose definition use the the Voronoï's cells $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2$ of the points e_0, e_1 and e_2 (i.e. $\mathcal{V}_i = \{x \in \mathbf{R}^2 : d(x, e_i) \leq d(x, e_j),\ j \neq i\}$). We have

$$\begin{aligned}
\mathcal{V}_0 &= \{(x_1, x_2) \in \mathbf{R}^2 : x_1, x_2 \leq \frac{1}{2}\},\\
\mathcal{V}_1 &= \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq \frac{1}{2} \text{ and } x_1 \geq x_2\},\\
\mathcal{V}_2 &= \{(x_1, x_2) \in \mathbf{R}^2 : x_2 \geq \frac{1}{2} \text{ and } x_1 \leq x_2\}.
\end{aligned}$$

Luckily, it is not hard to find a compact set $\mathcal{K}$ such that $\{\mathcal{W}_0 = \mathcal{K} \cap (\mathcal{V}_0 - \alpha),\ \mathcal{W}_1 = \mathcal{K} \cap (\mathcal{V}_1 - \alpha),\ \mathcal{W}_2 = \mathcal{K} \cap (\mathcal{V}_2 - \alpha)\}$ is an α-stable set of compact subsets. Set (see figure 2)

$$\begin{aligned}
\mathcal{K} &= \text{conv}(\{(-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (1, 0), (0, 1), (-\frac{1}{2}, \frac{1}{2})\},\\
\mathcal{T}_0 &= \text{conv}\{(0, 0), (1, 0), (0, 1)\}
\end{aligned}$$

where $\text{conv}(\mathcal{A})$ denote the convex hull of a subset $\mathcal{A}$ of $\mathbf{R}^2$.

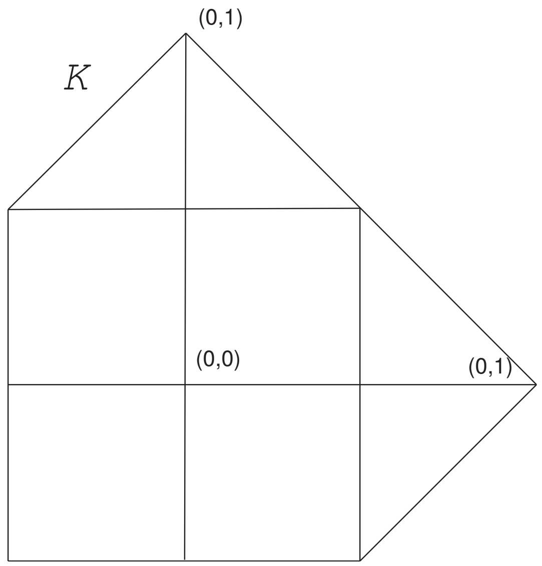

Figure 2: The domain $\mathcal{K}$. The distance between vertical or horizontal lines is $\frac{1}{2}$.

Lemma 17 *We have*

$$\begin{aligned}
(i)\quad & (\mathcal{K} + \mathcal{T}_0) \cap \mathcal{V}_0 \subseteq \mathcal{K},\\
(ii)\quad & (\mathcal{K} + \mathcal{T}_0) \cap \mathcal{V}_1 - e_1 \subseteq \mathcal{K},\\
(iii)\quad & (\mathcal{K} + \mathcal{T}_0) \cap \mathcal{V}_2 - e_2 \subseteq \mathcal{K}.
\end{aligned}$$

We leave the proof to the reader, it reduces to straightforward computations.

Corollary 18 *Let α be in $\mathbf{R}^2$. Set $\mathcal{A}_i = \{x \in \mathbf{R}^2 : x + \alpha \in \mathcal{V}_i\}$ and $\mathcal{W}_i = \mathcal{K} \cap \mathcal{A}_i$, $i = 0, 1, 2$. Consider the map F defined on the subsets of $\mathbf{R}^2$ by*

$$F : \mathcal{A} \subseteq \mathbf{R}^2 \to \bigcup_{i=0}^{2} (\mathcal{A} \cap \mathcal{W}_i + \alpha - e_i) \subseteq \mathbf{R}^2.$$

If $\alpha \in \mathcal{T}_0$ then $F(\mathcal{K}) \subset \mathcal{K}$.

Proof. Let x be in $\mathcal{K} \cap \mathcal{W}_i$. Since x is in $\mathcal{W}_i$, the point $x + \alpha$ is in $\mathcal{V}_i$. Since α is in $\mathcal{T}_0$, $x + \alpha$ is in $\mathcal{K} + \mathcal{T}_0$. Therefore $x + \alpha$ is in $(\mathcal{K} + \mathcal{T}_0) \cap \mathcal{V}_i$. By the previous lemma, it follows that the point $x + \alpha - e_i$ is in $\mathcal{K}$. $\square$

9.2 Proof of Theorem 1: 1, 2, 3 and 4.

In order to use propositions 8, 9 and 10 with the set of compact subsets $\{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2\}$ defined in corollary 18, we have to check all the assumptions these propositions.
- The above corollary asserts that the collection $\{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2\}$ of compact subsets is α-stable and it is easily seen that $\{\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2\}$ is regular.
- Let $x_0 = 0$. The boundaries of the $\mathcal{W}_i$ are are made of segments which are either included in rational line or included in a rational line translated by $-\alpha$. Since these boundaries do not contains 0, the full orbit $x_0 + \mathbf{Z}\alpha + \mathbf{Z}^2$ never meet them. Moreover the points $(0,1)$ and $(1,0)$ are not in $F(\mathcal{K})$, thus $F(\mathcal{K}) \cap (x_0 + \mathbf{Z}^2) = \{x_0\}$.
- Let $\mathcal{T}$ be the interior of the triangle $\Delta = \mathrm{conv}\{(-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\}$. Clearly

$$(\mathcal{T} + \mathbf{Z}^2 \backslash \{0\}) \cap \mathcal{K} = \emptyset.$$

Denote by $\mathcal{A}$ the set of elements a in $\mathbf{R}^2 \backslash \mathbf{Z}^2$ such that there is an integer i in $\{0, 1, 2\}$ and a point x in $\mathcal{T} \cap \mathcal{W}_i^o$ such that $x + a \notin \mathcal{W}_i + \mathbf{Z}^2$. The last assumption of proposition 10 which remains to be checked is: $\mathcal{A} = \mathbf{R}^2 \backslash \mathbf{Z}^2$, or equivalently, $\mathcal{A}$ contains $]-1, 0] \times [0, 1[\backslash \mathbf{Z}^2$.

This depends on elementary but rather tedious computations that we present below.

We can assume that $\alpha_1 < \alpha_2$. Set $a_1 = \frac{1}{2} - \alpha_1$ and $a_2 = \frac{1}{2} - \alpha_2$. Note that the conditions $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 < 1$ and $\alpha_1 < \alpha_2$ imply that $0 < a_1 < \frac{1}{2}$ and $-a_1 < a_2 < a_1$ for $\alpha_1, \alpha_2 \notin \mathbf{Q}$.

Set $\mathcal{S}_i = \mathcal{T} \cap \mathcal{W}_i^o$, $i = 0, 1, 2$,

$$O = (0, 0),\ A = (\frac{1}{2} - \alpha_1, \frac{1}{2} - \alpha_2) = (a_1, a_2),$$
$$B = (a_1, -a_1),\ C = (\frac{1}{2}, -\frac{1}{2}),\ D = (\frac{1}{2}, a_2 - a_1 + \frac{1}{2})$$

and

$$E = (\frac{1}{2}, \frac{1}{2}),\ F = (-\frac{1}{2}, \frac{1}{2}),\ G = (-a_2, a_2),$$
$$H = (-\frac{1}{2}, -\frac{1}{2}),\ I = (a_1, -\frac{1}{2}),\ K = (-\frac{1}{2}, a_2)$$

(see figure 3).

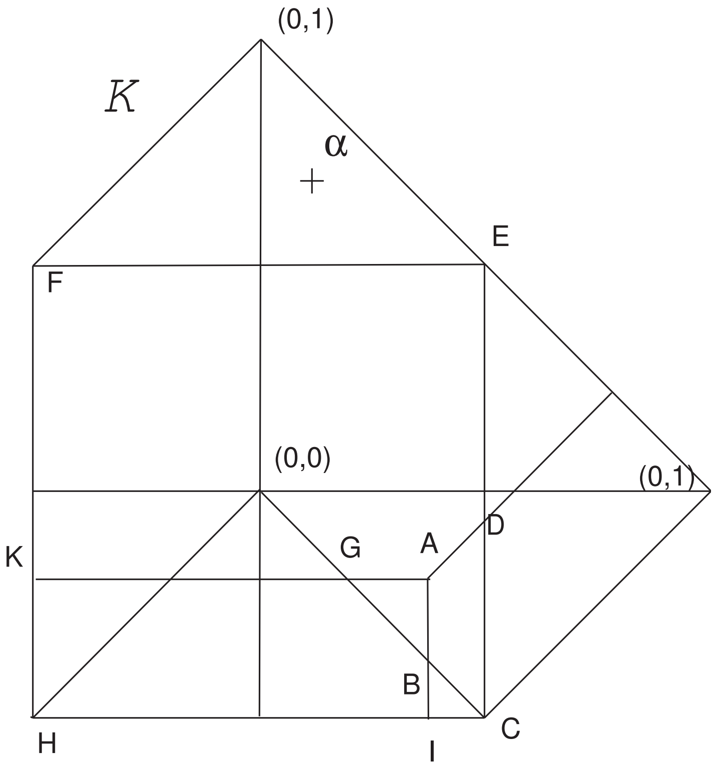

Figure 3:

It is easy to check that

$$\mathcal{S}_0 = \operatorname{conv}\{A, G, B\}^o,$$
$$\mathcal{S}_1 = \operatorname{conv}\{A, B, C, D\}^o,$$
$$\mathcal{S}_2 = \operatorname{conv}\{A, D, E, F, G\}^o.$$

Since $\mathcal{W}_0 = \operatorname{conv}\{A, I, H, K\}$, $\mathcal{W}_0 + \mathbf{Z}^2$ does not meet the strip $\mathcal{B}_0 = \{(\alpha, \beta) \in \mathbf{R}^2 : a_2 < \beta < \frac{1}{2}\}$. Thus, for all a in the strip $\mathcal{B}_1 = \{(\alpha, \beta) : 0 < \beta < \frac{1}{2} - a_2\}$ there is an x in $\mathcal{S}_0$ such that $x + a$ is in $\mathcal{B}_0$ and therefore is not in $\mathcal{W}_0 + \mathbf{Z}^2$.

For all x in $\mathcal{T}$, the equivalence class $x + \mathbf{Z}^2$ intersects $\mathcal{K}$ at exactly one point which is x, therefore the set $\mathcal{S}_2$ does not intersect $\mathcal{W}_1 + \mathbf{Z}^2$. It follows that for all a in $\mathcal{S}_2 - \mathcal{S}_1$, there exists x in $\mathcal{T} \cap \mathcal{W}_1^o$ such that $x + a \notin \mathcal{W}_i + \mathbf{Z}^2$. The set $\mathcal{S}_2 - \mathcal{S}_1$ contains

$$\mathcal{C}_1 = \operatorname{conv}\{D - A, A - D, G - D, F - D, F - C, E - C\}^o$$

and $D-A=(\frac{1}{2}-a_1,\frac{1}{2}-a_1)$, $G-D=(-a_2-\frac{1}{2},a_1-\frac{1}{2})$, $F-D=(-1,a_1-a_2)$, $F-C=(-1,1)$, and $E-C=(0,1)$. Since the point $F-D$ is in the strip $\mathcal{B}_1$, the set $\mathcal{A}$ contains the square $]-1,0]\times]0,1[$. So it remains to prove that $\mathcal{A}$ contains the "segment" $\mathbf{S}=]-1,0[\times\{0\}$.

It is easy to see that $\mathcal{E}_1=\mathrm{conv}(O,H,I,B)^o$ and $\mathcal{W}_1+\mathbf{Z}^2$ have an empty intersection. It follows that $\mathcal{E}_1-\mathcal{S}_1$ is included in $\mathcal{A}$. The set $\mathcal{E}_1-\mathcal{S}_1$ contains

$$\begin{aligned}\mathcal{C}_2&=\mathrm{conv}\{O=C-C,-C,H-C,H-D\}\\&=\mathrm{conv}\{(0,0),(-\frac{1}{2},\frac{1}{2}),(-1,0),(-1,-1-a_2+a_1)\}^o.\end{aligned}$$

Since $-1-a_2+a_1<0$, the segment $\mathbf{S}$ is included in $\mathcal{C}_2$ and therefore in $\mathcal{A}$.

Finally, we have proved that all the assumptions of propositions 8, 9 and 10 hold with $\mathcal{W}=\{\mathcal{W}_0,\mathcal{W}_1,\mathcal{W}_2\}$ defined in corollary 18, $\mathcal{K}$, $\mathcal{T}=\mathrm{conv}\{(-\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},-\frac{1}{2})\}^o$, and $x_0=0$. It follows that:

- $\mathcal{D}_\alpha=\overline{\{n\alpha-P_n:n\in\mathbf{N}\}}$ is a regular fundamental domain,
- $\mathcal{R}_i=\overline{\{n\alpha-P_n:P_{n+1}-P_n=e_i\}}=\mathcal{W}_i\cap\mathcal{D}_\alpha$, $i=0,1,2$,
- $\{\mathbf{p}(\mathcal{R}_0),\mathbf{p}(\mathcal{R}_1)\backslash\mathbf{p}(\mathcal{R}_0),\mathbf{p}(\mathcal{R}_2)\backslash(\mathbf{p}(\mathcal{R}_0)\cup\mathbf{p}(\mathcal{R}_1))\}$ is a finite aperiodic and regular partition of $\mathbf{T}^2$, and the boundaries of the elements of this partition are finite union of segments.

Note that $T^n x_0\in\mathcal{R}_i$ if and only if $P_{n+1}-P_n=e_i$. Now by Theorem A, (Y,S) is minimal, by Theorem B, (Y,S) is uniquely ergodic, and the two dynamical systems $(\mathbf{T}^2,T)$ and (Y,S) endowed with their unique invariant probability measures are metrically isomorphic. Furthermore, by Theorem 2, the complexity of the coding sequences is of quadratic growth. Finally the sequence $\omega_0=(P_{n+1}-P_n)_{n\in\mathbf{N}}$ is a natural coding, for

$$\mathcal{R}_i+\alpha-e_i\subset\overline{\{n\alpha-P_n+\alpha-e_i:P_{n+1}-P_n\}}\subset\mathcal{D}_\alpha.\ \square$$

9.3 Proof of Theorem 1: 5.

Let $w=(P_{n+1}-P_n)_{n=p,...,p+l-1}$ and $w'=(P_{n+1}-P_n)_{n=q,...,p+l-1}$ be two subwords of the same length of the sequence $(P_{n+1}-P_n)_{n\in\mathbf{N}}$. Set for $i=1,2$,

$$\begin{aligned}a_i&=\mathrm{card}\{n\in\{p,...,p+l-1\}:P_{n+1}-P_n=e_i\},\\a_i'&=\mathrm{card}\{n\in\{q,...,q+l-1\}:P_{n+1}-P_n=e_i\}.\end{aligned}$$

By lemma 17, the sequence $(n\alpha-P_n)_{n\in\mathbf{N}}$ lies in $\mathcal{K}$. Since $1,\alpha_1,\alpha_2$ are independent over the rational numbers, $n\alpha-P_n$ never belongs to the boundary of $\mathcal{K}$ which is composed of segments with rational endpoints. Since the distance associated with the supremum norm between two interior points of $\mathcal{K}$ is $<\frac{3}{2}$, it follows that for all integers n and m,

$$\|n\alpha-P_n-(m\alpha-P_m)\|_\infty<\frac{3}{2}$$

where $\|.\|_\infty$ denotes the supremum norm. Thus

$$\|l\alpha - (P_{p+l} - P_p)\|_\infty = \|((p+l)\alpha - P_{p+l}) - (p\alpha - P_p)\|_\infty < \frac{3}{2},$$
$$\|l\alpha - (P_{q+l} - P_q)\|_\infty = \|((q+l)\alpha - P_{q+l}) - (p\alpha - P_q)\|_\infty < \frac{3}{2},$$

and

$$\|(P_p - P_{p+l}) - (P_q - P_{q+l})\|_\infty < 3,$$

but this last norm is an integer, therefore

$$\|(P_p - P_{p+l}) - (P_q - P_{q+l})\|_\infty \leq 2.$$

Finally,

$$P_{p+l} - P_p = l\alpha - a_1e_1 - a_2e_2,$$
$$P_{q+l} - P_q = l\alpha - a_1'e_1 - a_2'e_2,$$

thus

$$\|(a_1 - a_1')e_1 + (a_2 - a_2')e_2\|_\infty \leq 2. \ \square$$

9.4 Proof of corollary 5

We have just seen that the sequence $\omega_0 = (P_{n+1} - P_n)_{n\in\mathbf{N}}$ is 2-balanced over the letters e_1 and e_2. Since the alphabet of the sequence ω_0 has only three letters, the sequence ω_0 is 4-balanced over the three letters. This means that the coding sequence of $x_0 = 0$ with respect of the partition

$$\{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2\} = \{\mathbf{p}(\mathcal{R}_0), \mathbf{p}(\mathcal{R}_1)\backslash\mathbf{p}(\mathcal{R}_0), \mathbf{p}(\mathcal{R}_2)\backslash(\mathbf{p}(\mathcal{R}_0) \cup \mathbf{p}(\mathcal{R}_1)\}$$

is 4-balanced.

Since the dynamical system (Y, S) is minimal, all element of Y are 4-balanced sequences. Moreover the coding map π is a metric isomorphism of the two dynamical systems $(\mathbf{T}^2, T)$ and (Y, S), thus, for almost all x in $\mathbf{T}^2$, the coding sequence $\pi(x)$ is 4-balanced. By the Birkhoff ergodic theorem, for almost all x in $\mathbf{T}^2$ and for $i = 0, 1, 2$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\mathcal{P}_i}(T^k x) = \mu(\mathcal{P}_i).$$

Fix a positive integer q and set $N_i = N_i(x, q) = \mathrm{card}\left\{k < q{:}T^k x \in \mathcal{P}_i\right\}$. Let x be in $\mathbf{T}^2$ such that the sequence $\pi(x)$ is 4-balanced. For $i = 0, 1, 2$ and for all n in $\mathbf{N}$,

$$N_i - 4 \leq \mathrm{card}\{nq \leq k < (n+1)q : T^k x \in \mathcal{P}_i\} \leq N_i + 4.$$

Hence, for all n in $\mathbf{N}$,

$$\frac{n(N_i - 4)}{nq} \leq \frac{1}{nq} \sum_{k=0}^{qn-1} 1_{\mathcal{P}_i}(T^k x) \leq \frac{n(N_i + 4)}{nq},$$

thus, letting n go to infinity,

$$N_i - 4 \leq q\mu(\mathcal{P}_i) \leq N_i + 4$$

which shows that $\mathcal{P}_i$ is a bounded remainder set. The rest of the proposition 1 follows immediately from proposition 10. $\square$

10 Description of the fundamental domain $\mathcal{D}_\alpha$

It is possible to compute explicitly the fundamental domain $\mathcal{D}_\alpha$ in Theorem 1. The next Theorem explains it. We shall not prove this Theorem whose proof is elementary but rather tedious. We need some notations. The point

$$S = (\frac{1}{2}, \frac{1}{2})$$

plays a particular role as well as the square

$$\mathcal{C} = [-\frac{1}{2}, \frac{1}{2}]^2,$$

and the triangles

$$\begin{aligned}
\mathcal{T} &= \mathrm{conv}\{(\frac{1}{2}, -\frac{1}{2}),\ (\frac{1}{2}, \frac{1}{2}),\ (-\frac{1}{2}, \frac{1}{2})\},\\
\mathcal{T}_1 &= \mathrm{conv}\{(\frac{1}{2}, -\frac{1}{2}),\ (\frac{1}{2}, \frac{1}{2}),\ (1, 0)\}\backslash\mathcal{C},\\
\mathcal{T}_2 &= \mathrm{conv}\{(\frac{1}{2}, \frac{1}{2}),\ (-\frac{1}{2}, \frac{1}{2}),\ (0, 1)\}\backslash\mathcal{C}.
\end{aligned}$$

Observe that $\mathcal{K} = \mathcal{C} \cup \mathcal{T}_1 \cup \mathcal{T}_2$.

Theorem 3 *Let n_0 be the smallest integer $n \geq 1$ such that $F^n(S) \in \mathcal{C}$. Set*

$$\begin{aligned}
\mathcal{D}_1 &= \cup_{n=0}^{n_0-1} F^n(\mathcal{T}) \cap \mathcal{T}_1,\\
\mathcal{D}_2 &= \cup_{n=0}^{n_0-1} F^n(\mathcal{T}) \cap \mathcal{T}_2.
\end{aligned}$$

Then

$$\mathcal{D}_\alpha = \mathcal{T} \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup (\overline{\mathcal{T}_1\backslash\mathcal{D}_1} - e_1) \cup (\overline{\mathcal{T}_2\backslash\mathcal{D}_2} - e_2).$$

In figure 4, the two regions are translates of the fundamental domain $\mathcal{D}_\alpha$ associated to $\alpha = (0.263361...,\ 0.715678...)$.

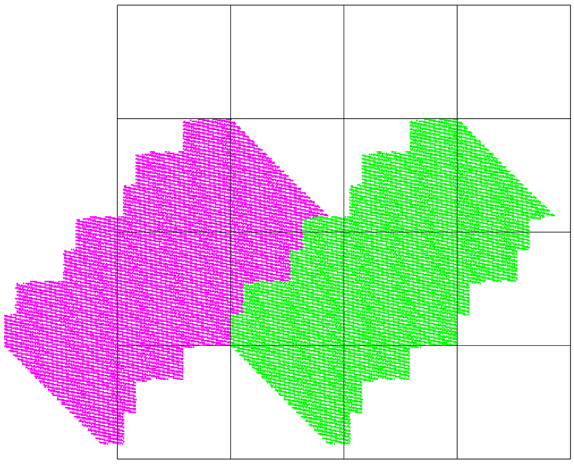

Figure 4: The y-axis is downward and the distance between vertical or horizontal lines is $\frac{1}{2}$

11 Miscellaneous remarks

1. Both Sturmian's and Rauzy's coding are closely related to Diophantine approximations. Sturmian sequences are linked to the continued fraction algorithm ([2], [3]) and the Tribonacci sequence is linked to the best Diophantine approximations of (ξ, ξ^2) with respect to a well chosen norm ([4], [5], [11]). We do not know whether the coding described in Theorem 1 enjoys any properties of that kind.

2. Let $0 < \alpha < \frac{1}{2}$ be a real number and $T : x \in \mathbf{T}^1 \to x + \alpha$ the associated translation. Consider the Sturmian partition $I_0 = [0, 1 - \alpha[$, $I_1 = [1 - \alpha, 1[$. A key fact to understand the link between the continued fraction expansion of α and the coding with respect to the partition I_0, I_1, is that the map T_0 induced by T on I_0:
$T_0 x = T^{n(x)} x$ *where* $n(x) = \min\{n \geq 1 : T^n x \in I_0\}$ *(i.e. the first return map),*
is an exchange of two intervals (see [Bé, Fe, Za]). Therefore, this map is itself a translation of a one-dimensional torus which allows a recursion process. In the case of $\alpha \in \mathbf{R}^2$, a natural question arises:
Call T_i the induced map (the first return map) on the set

$$\mathcal{R}_i = \overline{\{n\alpha - P_n : P_{n+1} - P_n = e_i\}},$$

$i = 0, 1$, and 2. Is there an i and a lattice Λ_i of $\mathbf{R}^2$ such that $\mathcal{R}_i$ is a fundamental domain of $\mathbf{R}^2$ for the action of Λ_i and such that T_i induces a minimal translation on $\mathbf{R}^2/\Lambda_i$?

A necessary condition is that $\mathcal{R}_i$ is a bounded remainder set for α, which was shown by Rauzy ([14]). By Proposition 5, this is actually the case. In the one-dimensional case it is not difficult to see that $[0, 2\alpha]$ is a bounded remainder set for which the induced transformation is an exchange of three intervals rather than two (see [7]), so Rauzy's condition is not sufficient. Nevertheless, Ferenczi [7] has found a necessary and sufficient condition for a subset $\mathcal{A}$ of $\mathbf{T}^2$ to be bounded remainder set. Whereas Ferenczi's condition does not give a direct answer to the previous question, it might help.

3. One can try to characterize the sequences ω_0 obtained in Theorem 1. They have two combinatorial properties, the complexity p_{ω_0} is of quadratic growth and the sequence ω_0 is 2-balanced over the letters 1 and 2. But it is very likely that there exist sequences $\omega \in \{0, 1, 2\}^{\mathbf{N}}$ for which these properties both hold and which are not associated with an α by Theorem 1. Nevertheless, on can observe that the property of being balanced is crucial. Indeed, if $\omega \in \{e_0, e_1, e_2\}^{\mathbf{N}}$ is C-balanced over the letters 1 and 2, one can prove that there exits a unique α in $\mathbf{R}^2$ such that $(n\alpha - \sum_{k=0}^{n-1} \omega(k))_{n\geq 1}$ is a bounded sequence.

4. Although the extension of Theorem 1 in the tree-dimensional case could be possible, it may be better to break the symmetry between the coordinates. Indeed, if want to keep exactly the same process which defines the sequence $(P_n)_{n\in\mathbf{N}}$ of lattice points in Theorem 1, we have to consider $\alpha = (\alpha_1, \alpha_2, \alpha_2)$ in the tetrahedron $\mathcal{T} = \{\alpha = (\alpha_1, \alpha_2, \alpha_2) : \alpha_1, \alpha_2, \alpha_2 \geq 0 \text{ and } \alpha_1 + \alpha_2 + \alpha_2 \leq 1\}$. But unlike in the two-dimensional case, there are some $\beta = (\beta_1, \beta_2, \beta_3)$ in $\mathbf{R}^3$ for which there are no $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathcal{T}$ such that $\beta_i = \pm\alpha_i \bmod \mathbf{Z}$. To overcome this drawback, one can try to extend to the three-dimensional case the following two-dimensional process: take α in the box $\mathcal{B} = \{\alpha = (\alpha_1, \alpha_2) : 0 \leq \alpha_1, \alpha_2 \leq \frac{1}{2}\}$ instead of the triangle $\mathcal{T}$ and define the sequence $(P_n = (x_n, y_n))_{n\in\mathbf{N}}$ by $P_0 = (0, 0)$ and
- $P_{n+1} = P_n + e_2$ if $(n+1)\alpha_2 \geq y_n + \frac{1}{2}$,
- $P_{n+1} = P_n + e_1$ if $(n+1)\alpha_2 < y_n + \frac{1}{2}$ and $(n+1)\alpha_1 \geq x_n + \frac{1}{2}$,
- $P_{n+1} = P_n + e_0$ if $(n+1)\alpha_2 < y_n + \frac{1}{2}$ and $(n+1)\alpha_1 < x_n + \frac{1}{2}$.

It is easy to see that the sequence $(n\alpha - P_n)_{n\in\mathbf{N}}$ stays in the compact set $\mathcal{K} = [-\frac{1}{2}, \frac{1}{2}]^2 \cup [\frac{1}{2}, 1] \times [-\frac{1}{2}, 0]$, which is the key argument in the proof of Theorem 1 (see section 8.1).

12 Appendix

12.1 Proof of Theorem A

Notation. For all finite sequences $\mathcal{W}_{i_0}, ..., \mathcal{W}_{i_n}$ of elements of $\mathcal{W}$, we denote by $[\mathcal{W}_{i_1}, ..., \mathcal{W}_{i_n}]$ the set of all $\omega = (\omega(n))_{n\in\mathbf{N}}$ in Ω such that

$\omega(0) = \mathcal{W}_{i_0}, ..., \omega(n) = \mathcal{W}_{i_n}$. These sets are called *cylinders* and they generate the topology of Ω.

Let Z be the set of ω in Ω such that $\mathcal{W}_\omega = \cap_{n\geq 0} T^{-n}\overline{\omega(n)}$ is nonempty. We wish to show that $\mathcal{W}_\omega$ consists of exactly one point for all ω in Z. Suppose on the contrary that two distinct points x and y are in $\mathcal{W}_\omega$. Since the partition $\mathcal{W}$ is aperiodic, there exists a point x' in the interior of some $\mathcal{W}_i$ such that $y' = x' + (y - x)$ is not in the closure of $\mathcal{W}_i$. Thus, there exists a positive number ε such that the ball $B(x', \varepsilon)$ is inside the interior of $\mathcal{W}_i$ and the ball $B(y', \varepsilon)$ is outside of the closure of $\mathcal{W}_i$. Since the map T is minimal, we can find an integer $n \geq 0$ with $T^n x \in B(x', \varepsilon)$. The map T is a translation, therefore $T^n y = y - x + T^n x$ and the point $T^n y$ is in the ball $B(y', \varepsilon)$. It follows that $T^n x \in \mathcal{W}_i^o$ and $T^n y \notin \overline{\mathcal{W}_i}$ which shows that $\omega(n) = \mathcal{W}_i$ and $\omega(n) \neq \mathcal{W}_i$, which is impossible.

Since for every ω in Z, $\mathcal{W}_\omega$ consists of exactly one point, we can define a map $\psi : Z \to \mathbf{T}^2$ by $\psi(\omega) = x$ where x is the unique point of $\mathcal{W}_\omega$. Obviously, for all x in $\mathbf{T}^2$, $x \in \mathcal{W}_{\pi(x)}$, therefore, $\pi(x) \in Z$, $\psi(\pi(x)) = x$ and $\psi \circ \pi = Id_{\mathbf{T}^2}$.

Let us show the continuity of the map ψ. Let ω_0 be in Z and x_0 be the image of ω_0. Again, by definition of ψ, for all ω in Z, $\psi(\omega) \in \mathcal{W}_\omega = \cap_{n\geq 0} T^{-n}\overline{\omega(n)}$, thus $T^n\psi(\omega) \in \overline{\omega(n)}$ for all integer $n \geq 0$. By definition of ψ, the image of the cylinder $[\omega_0(0), ..., \omega_0(n)]$ is included in the closed set $F_n = \cap_{0\leq p\leq n} T^{-p}\overline{\omega_0(p)}$. The intersection of the closed sets F_n, $n \in \mathbf{N}$, is $\{x_0\}$, therefore for all $\varepsilon > 0$, there exists an integer n_ε such that $F_{n_\varepsilon} \subset B(x_0, \varepsilon)$. It follows that $\psi([\omega_0(0), ..., \omega_0(n_\varepsilon)] \subset B(x_0, \varepsilon)$.

Let us show that Z is closed. Suppose that $(\omega_p)_{p\in\mathbf{N}}$ is a sequence in Z which converges to ω in Ω. By definition of the topology of Ω, for each integer n there is an integer p_n such that $\omega(0) = \omega_{p_n}(0)$, $\omega(1) = \omega_{p_n}(1)...,$ $\omega(n) = \omega_{p_n}(n)$. Therefore, for all integer n, $\cap_{k\leq n} T^{-k}\overline{\omega(k)} = \cap_{k\leq n} T^{-k}\overline{\omega_{p_n}(k)}$ is a nonempty closed set of $\mathbf{T}^2$, it follows that $\cap_{k\in\mathbf{N}} T^{-k}\overline{\omega(k)}$ is non empty and that ω is in Z.

Let ω be in Z. We have $\mathcal{W}_{S\omega} = \cap_{n\geq 0} T^{-n}\overline{\omega(n+1)} = T(\cap_{n\geq 1} T^{-n}\overline{\omega(n)})$ for the map T is bijective. Using again the argument of the beginning of the proof, we see that the intersection $\cap_{n\geq 1} T^{-n}\overline{\omega(n)}$ contains at most one point and since $\psi(\omega)$ is in $\cap_{n\geq 0} T^{-n}\overline{\omega(n)}$, we get $\cap_{n\geq 1} T^{-n}\overline{\omega(n)} = \{\psi(\omega)\}$. Therefore, $\mathcal{W}_{S\omega} = \{T\psi(\omega)\}$, $\mathcal{W}_{S\omega}$ is in Z, and $\psi S(\omega) = T\psi(\omega)$. Hence $S(Z) \subset Z$ and $\psi S = T\psi$ on Z.

Since the partition $\mathcal{W}$ is regular, the boundary of an element $\mathcal{W}_i$ of $\mathcal{W}$ is the same than the boundary of its interior. Therefore the boundary of each $\mathcal{W}_i$ is a closed set of empty interior. It follows by Baire's theorem that

$$\mathcal{G} = \mathbf{T}^2 \setminus \bigcup_{n\in\mathbf{N}} \bigcup_{i=1}^{k} T^{-n}\partial\mathcal{W}_i$$

is everywhere dense in $\mathbf{T}^2$. Clearly $T(\mathcal{G}) \subset \mathcal{G}$. Moreover, for all x in $\mathcal{G}$ there exists an unique ω in Z such that $\psi(\omega) = x$ because for each n there exists an unique i with $T^n x \in \overline{\mathcal{W}_i}$. This implies that the restriction of ψ to $\psi^{-1}(\mathcal{G})$ is one to one and that $\psi^{-1}(\mathcal{G}) = \pi(\mathcal{G})$. Set $L = \psi^{-1}(\mathcal{G}) = \pi(\mathcal{G})$,

$Y = \overline{L}$, and $\phi = \psi_{|Y}$. By definition, L is everywhere dense in Y and $\phi(Y)$ is a compact set containing $\phi(L) = \phi(\pi(\mathcal{G})) = \mathcal{G}$. It follows that $\phi(Y) = \mathbf{T}^2$.

Let us show that $S(Y) \subset Y$. If $\omega \in L$, then $\phi S(\omega) = T\phi(\omega) \in \mathcal{G}$ and $S(\omega)$ belongs to L. By continuity of S, it follows that $S(Y) = S(\overline{L}) \subset \overline{S(L)} \subset \overline{L} = Y$.

Let ω_0 be in L, let ω be in Z, and let $U = [\omega_0(0), ..., \omega_0(n)]$ be a neighborhood of ω_0. The open set $\mathcal{V} = \bigcap_{i=0}^{n} T^{-i}\omega_0(i)^o$ is nonempty for $x = \phi(\omega_0)$ is in $\mathcal{V}$. Since T is minimal, there exists an integer $m \geq 0$ such that $T^m\psi(\omega)$ belongs to $\mathcal{V}$. But $T^m\psi = \psi S^m$, thus $\psi S^m(\omega)$ belongs to $\mathcal{V}$. Since $\mathcal{W}$ is a partition, for each $i \leq k$, the interior of $\mathcal{W}_i$ and the closure of the other $\mathcal{W}_j$ have empty intersection; by definition of ψ, we have $\psi^{-1}(\mathcal{V}) \subset U$ and therefore $S^m(\omega) \in U$. So all the elements of L are limit points of the sequence $(S^m(\omega))_{m\geq 0}$ which shows that the restriction of S to Y is minimal. It also follows that $S(Y)$ which is compact is equal to Y. $\square$

Remark. It can happen that $Y \neq \pi(\mathbf{T}^2) \neq Z$ (obviously $\pi(\mathbf{T}^2) \subset Z$). We give an example in $\mathbf{T}^1$:
let α be in $[0, \frac{1}{3}]\backslash\mathbf{Q}$, let $T : \mathbf{T}^1 \to \mathbf{T}^1$ be the map defined by $Tx = x + \alpha$, and let $\mathcal{W} = \{\mathcal{W}_1 = [0, \alpha], \mathcal{W}_2 =]\alpha, 2\alpha], \mathcal{W}_3 =]2\alpha, 1[\}$. On the one hand, a coding of a point x in $\mathbf{T}^1\backslash\{0, \alpha, 2\alpha\}$ cannot begin by

$$\mathcal{W}_1\mathcal{W}_1,$$

but $\pi(0) = \mathcal{W}_1\mathcal{W}_1\mathcal{W}_2$ therefore $\pi(0) \notin \overline{\pi(\mathcal{G})}$. On the other hand, a coding of point can never contain the subword $\mathcal{W}_2\mathcal{W}_2$, but $\omega_\alpha = \mathcal{W}_2\mathcal{W}_2\pi(3\alpha)$ is in Z, for $\mathcal{W}_{\omega_\alpha} = \{\alpha\}$, therefore $\omega_\alpha \in Z\backslash\pi(\mathbf{T}^1)$.

13 Proof of Theorem B

We keep the notation of the previous proof. Assume that the boundaries of the $\mathcal{W}_i$ are of zero measure. This means that $\mathcal{G}$ has full measure. We claim that the coding map π is continuous on $\mathcal{G}$. Indeed, let n be an integer ≥ 0 and let x be in $\mathcal{G}$. Set $\omega_0 = \pi(x)$. The open set $\bigcap_{i=0}^{n} T^{-i}\omega_0(i)^o$ contains x and its image by π is included in the cylinder $[\omega_0(0), ..., \omega_0(n)]$.

Let $f : Y \to \mathbf{R}$ be a continuous map. The map $g = f \circ \pi : \mathcal{G} \to \mathbf{R}$ is continuous and bounded. Since $\mathcal{G}$ is everywhere dense in $\mathbf{T}^2$, we can extend g to $\mathbf{T}^2$ as a SCS map g^+ and also as SCI map g^-. These extensions are simply defined by

$$g^+(x) = \overline{\lim}_{y\to x,\, y\in\mathcal{G}} g(x) \text{ and } g^-(x) = \underline{\lim}_{y\to x,\, y\in\mathcal{G}} g(x).$$

Obviously $g^- \leq g^+$.

Since π is continuous on $\mathcal{G}$, g is continuous as well and $g = g^+ = g^-$ on $\mathcal{G}$. Moreover $\mathbf{T}^2\backslash\mathcal{G}$ is of zero measure, whence

$$\int_{\mathbf{T}^2} g\,dx = \int_{\mathbf{T}^2} g^+\,dx = \int_{\mathbf{T}^2} g^-\,dx.$$

Let $\varepsilon > 0$. There exist $h^- : \mathbf{T}^2 \to \mathbf{R}$ and $h^+ : \mathbf{T}^2 \to \mathbf{R}$ continuous such that

$$h^- \leq g^- \leq g^+ \leq h^+$$

and

$$\int_{\mathbf{T}^2} (g^- - h^-)\, dx, \ \int_{\mathbf{T}^2} (h^+ - g^+)\, dx \leq \varepsilon.$$

Since the translation T is uniquely ergodic, for all x in $\mathbf{T}^2$,

$$\int_{\mathbf{T}^2} g\, dx - \varepsilon \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h^-(T^k x) \leq \underline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g^-(T^k x)$$

and

$$\overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g^+(T^k x) \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h^+(T^k x) \leq \int_{\mathbf{T}^2} g\, dx + \varepsilon.$$

These inequalities hold for all ε, thus for all x in $\mathbf{T}^2$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g^-(T^k x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g^+(T^k x) = \int_{\mathbf{T}^2} g\, dx.$$

Call ν the image of the Lebesgue measure of $\mathbf{T}^2$ by π. Since $g^- = g^+ = g$ on $\mathcal{G}$ and since $S \circ \pi = \pi \circ T$, we get that for all $\omega = \pi(x)$ in $\pi(\mathcal{G})$ and all continuous map $f : Y \to \mathbf{R}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k \omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(T^k x)) = \int_{\mathbf{T}^2} g\, d\mu = \int_{\mathbf{Y}} f\, d\nu.$$

If λ is an other S-invariant ergodic probability on Y then by Birkhoff theorem, we have $\lambda(\pi(\mathcal{G})) = 0$. But the image of λ by ϕ (see Theorem A) is a T-invariant probability λ_ϕ with $\lambda_\phi(\mathcal{G}) = \lambda(\phi^{-1}(\mathcal{G})) = \lambda(\pi(\mathcal{G})) = 0$ which is impossible for T is uniquely ergodic. This shows that (Y, S) is uniquely ergodic.

Finally, the map $\pi : \mathcal{G} \to L$ is a metric isomorphism of the measurable dynamical systems $(\mathcal{G}, T, \mu)$ and $(\pi(\mathcal{G}), S, \nu)$. $\square$

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