Lévy-Khintchin Theorem for best simultaneous Diophantine approximations

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1 Introduction

In 1936, Aleksandr Khintchin showed that there exists a constant γ such that the denominators $(q_n)_{n\geq 0}$ of the convergents of the continued fraction expansions of almost all real numbers θ satisfy

$$\lim_{n\to\infty} q_n^{1/n} = \gamma$$

(see [12]). Soon afterward, in [21], in the footnote page 289, Paul Lévy gave the explicit value of the constant,

$$\gamma = \exp\frac{\pi^2}{12\ln 2}.$$

In 1983, Wieb Bosma, Hendrik Jager and Freek Wiedijk, proved the following conjecture due to Hendrik Lenstra: for almost all real numbers θ,

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{card}\{0\leq k<n : q_k\,\mathrm{d}(q_k\theta,\mathbb{Z})\leq t\} = g(t)$$

for all $t\in[0,1]$, where

$$g(t) = \int_0^t \frac{1}{2\ln 2}\frac{1-|1-2s|}{s}\,ds.$$

Later, Jager proved variants of this result in particular with the quantity $q_{k+1}\,\mathrm{d}(q_k\theta,\mathbb{Z})$ instead of $q_k\,\mathrm{d}(q_k\theta,\mathbb{Z})$.

The aim of the paper is to extend to best simultaneous Diophantine approximations, both Lévy-Khintchin's result and Bosma, Jager and Wiedijk's result.

Let d and c be two positive integers. Suppose $\mathbb{R}^d$ and $\mathbb{R}^c$ are endowed with the standard Euclidean norms $\|.\|_{\mathbb{R}^d}$ and $\|.\|_{\mathbb{R}^c}$. We prove

Theorem 1. *There exists a constant $L_{d,c}$ such that for almost all matrices $\theta\in\mathrm{M}_{d,c}(\mathbb{R})$,*

$$\lim_{n\to\infty}\frac{1}{n}\ln\|Q_n(\theta)\|_{\mathbb{R}^c} = L_{d,c}\,,$$
$$\lim_{n\to\infty}\frac{1}{n}\ln(\mathrm{d}(\theta Q_n(\theta),\mathbb{Z}^d)) = -\frac{c}{d}L_{d,c}$$

where $Q_n(\theta)\in\mathbb{Z}^c$, $n\geq 0$, is the sequence of best Diophantine approximation denominators of θ associated with the norms $\|.\|_{\mathbb{R}^d}$ and $\|.\|_{\mathbb{R}^c}$.

(See section 2.1, the definition of best Diophantine approximation denominators).

For a matrix θ in $\mathrm{M}_{d,c}(\mathbb{R})$, let denote $\beta_n(\theta) = \|Q_{n+1}(\theta)\|_{\mathbb{R}^c}^c \,\mathrm{d}(\theta Q_n(\theta), \mathbb{Z}^d)^d$.

Theorem 2. *1. There exists a probability measure $\nu_{d,c}$ on $\mathbb{R}$ such that for almost all matrices $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$, $\nu_{d,c}$ is the weak limit of the sequence of probability measures*

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{\beta_k(\theta)}$$

where δ_a is the Dirac measure at a.

2. The support of the measure $\nu_{d,c}$ is included in a bounded interval, and contains 0 provided that $c + d \geq 3$.

Lévy-Khintchin's result has already been extended to multi-dimensional settings. For instance, for almost all θ in $\mathbb{R}^d$, the denominators $(J_n(\theta))_{n\geq 0}$ of the Jacobi-Perron expansion of θ satisfy $\lim_{n\to\infty} \frac{1}{n} \ln J_n(\theta) = c_d$ for some constant c_d (see [4]). The common proofs of such results use ergodic theory. The one-dimensional Lévy-Khintchin's result can be proved with Birkhoff ergodic Theorem, while the growth rate of the Jacobi-Perron denominators can be derived from Oseledec multiplicative ergodic Theorem. In both cases, the proof depends on the existence of an underlying dynamical system: the Gauss map or the Jacobi-Perron map (see [25] for many examples of these kinds of maps). However, no such map associated with best Diophantine approximations is known when $d + c \geq 3$. One classical way to circumvent this problem is to use the action of the diagonal flow

$$g_t = \begin{pmatrix} e^{ct} I_d & 0 \\ 0 & e^{-dt} I_c \end{pmatrix} \in \mathrm{SL}(d+c, \mathbb{R})$$

on the space of unimodular lattices $\mathcal{L}_{d+c} = \mathrm{SL}(d+c,\mathbb{R})/\mathrm{SL}(d+c,\mathbb{Z})$. For instance, in [10] this flow is used to prove that the sequence of best Diophantine approximation denominators of almost all θ in $\mathrm{M}_{d,c}(\mathbb{R})$ satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|Q_n(\theta)\|_{\mathbb{R}^c} \leq K_{d,c}$$

for some constant $K_{d,c}$. When $c = 1$, it is also possible to derive this inequality from a Theorem of W. M. Schmidt (see [9]).

In this work, as in [10], the flow (g_t) is the main tool. Together with the flow, an important ingredient is a surface S of co–dimension 1 transverse to the flow and the first return map associated with the flow. Such transversals have been widely used and we only mention two closely related works.

Firstly, P. Arnoux and A. Nogueira in [1], have used transversals to naturally obtain invariant measures associated with multidimensional continued fraction algorithms. Furthermore, in the case of the one dimensional continued fraction algorithm, their approach leads to an interpretation of the Lévy's constant as the average return time of the flow on the transversal.

Secondly, in some cases, the transformation induced on a sub-interval by an interval exchange transformation T_1 is an interval exchange transformation T_2 of the same kind as T_1. In such situations, the map $T_1 \to T_2$ can be seen as the Gauss map of a "multidimensional continued fraction algorithm". In [27], W. Veech used a transversal to prove that this Gauss map admits an unique absolutely continuous invariant measure up to a scalar multiple.

In our case, the transversal is the set of unimodular lattices the first two minimums of which are equal (actually, the definition is slightly more restrictive, see section 3.1 for the exact definition of the transversal). It is crucial to observe that the visiting times of the transversal are given by a formula involving best simultaneous Diophantine approximations, see Lemma 15. Making use of Birkhoff Theorem, this observation leads to a Lévy-Khintchin result in the space of lattices and to a formula close to the Arnoux-Nogueira interpretation of the Lévy's constant:

$$L_{d,c} = \frac{d}{\mu_S(S)} \int_S \tau \, d\mu_S = \frac{d \times \mu(\mathcal{L}_{d+c})}{\mu_S(S)} \tag{1}$$

where μ is the invariant measure in the space of lattices, μ_S the invariant measure induced by the flow on the transversal S and τ the return time to S, see Theorem 20 and Corollary 22.

The second step of the proof of Theorem 1 consists in converting an almost all result in the space of lattices $\mathcal{L}_{d+c}$ into an almost all result in $\mathrm{M}_{d,c}(\mathbb{R})$. To achieve this goal, we prove a general result, Theorem 24, which might be of independent interest. At first sight, this result might appear as an easy consequence of the following standard fact: the set of lattices associated with the matrices θ in $\mathrm{M}_{d,c}(\mathbb{R})$, is the expanding direction of the flow g_t. However, an example shows that Theorem 24 depends on some properties of the transversal, see section 8.

When $d = 1$ or 2 and $c = 1$, the submanifold S and the measure μ_S can be entirely calculated (see section 7). When $d = c = 1$, thanks to Siegel formula giving the volume of the modular space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$, computing Lévy's constant $L_{1,1} = \ln \gamma$ is easy; it is even possible to determine the first return map to the transversal. It turns out that this first return map is a 2-fold extension of the natural extension of the Gauss map (see subsection 7.3). However, when $d = 2$ and $c = 1$, the calculation of $L_{2,1}$ leads to a seven-tuple integral and we only succeed in reducing it to a triple integral that can be evaluated numerically (see subsection 7.4).

When $d = c = 1$, the double inequality $\frac{1}{2} \leq q_{n+1} \, \mathrm{d}(q_n\theta, \mathbb{Z}) \leq 1$ shows that the behaviors of the two sequences $(\frac{1}{n} \ln q_n)_n$ and $(\frac{-1}{n} \ln \mathrm{d}(q_n\theta, \mathbb{Z}))_n$ are the same and each of the limits in Theorem 1 implies the other. When d is larger or equal than two, no such double inequality exists. Indeed, it has been proved in [9] that when $c = 1$ and $d \geq 2$,

$$\liminf_{n\to\infty} q_{n+1} \, \mathrm{d}(q_n\theta, \mathbb{Z}^d)^d = 0$$

for almost all θ in $\mathbb{R}^d$. Observe that Theorem 2 implies this latter result; it is an immediate consequence of the fact that 0 is in the support of the measure $\nu_{d,c}$. Hopefully for the proof of Theorem 1, the weaker inequality

$$\mathrm{d}(\theta Q_n(\theta), \mathbb{Z}^d) \geq \frac{1}{\|Q_n(\theta)\|_{\mathbb{R}^c}^{c/d} \ln \|Q_n(\theta)\|}$$

which holds almost surely by the convergence part of the Khintchin-Groshev Theorem, is enough to link both of the limits in Theorem 1.

In the last section, we extend the aforementioned result of [9] to best simultaneous approximations of matrices. Our proof leads to the stronger result

$$\liminf_{n\to\infty} \|Q_{n+k}(\theta)\|_{\mathbb{R}^c}^{c} \, \mathrm{d}(\theta Q_n(\theta), \mathbb{Z}^d)^d = 0$$

for almost all θ in $\mathrm{M}_{d,c}(\mathbb{R})$ and all $k \in \mathbb{N}$.

Obviously, if the matrix θ is badly approximable, then

$$\liminf_{n\to\infty} \|Q_{n+k}(\theta)\|_{\mathbb{R}^c}^c \,\mathrm{d}(\theta Q_n(\theta), \mathbb{Z}^d)^d > 0$$

because by definition $\liminf_{n\to\infty} \|Q_n(\theta)\|_{\mathbb{R}^c}^c \,\mathrm{d}(\theta Q_n(\theta), \mathbb{Z}^d)^d > 0$. When $d = 2$ and $c = 1$, we prove that the set of θ with $\liminf_{n\to\infty} q_{n+1} \,\mathrm{d}(q_n\theta, \mathbb{Z})^2 > 0$ is not reduced to the set of badly approximable vectors.

Historical Note: Lévy's proof does not rely on the Ergodic Theorem which was not known for non-invertible maps at that time. A proof of the Birkhoff Theorem for non-invertible maps was given by Frédéric Riesz in 1945 (see [23]) and then a proof of Lévy's Theorem using the Ergodic Theorem was given in [24]. The authors would like to thank Vitaly Bergelson for bringing [24] to their attention.

2 Notation

Let d and c be two positive integers.

2.1 Vectors and distances

Let $\|\cdot\|_{\mathbb{R}^n}$ denote the usual Euclidean norm on $\mathbb{R}^n$.

We assume $\mathbb{R}^d$ and $\mathbb{R}^c$ are equipped with the usual Euclidean norms and $\mathbb{R}^{d+c}$ is equipped with the norm $\|(u,h)\|_{\mathbb{R}^{d+c}} = \max\{\|u\|_{\mathbb{R}^d}, \|h\|_{\mathbb{R}^c}\}$.

For $X = (u,h)$ in $\mathbb{R}^{d+c}$, let $|X|_- = \|h\|_{\mathbb{R}^c}$ denote the *height* of the vector X and $|X|_+ = \|u\|_{\mathbb{R}^d}$ denote the norm of the projection of X in the horizontal space. We also denote $X_+ = u$ and $X_- = h$ the vertical and horizontal components of X.

For a vector X in $\mathbb{R}^{d+c}$, let $C(X)$ denote the closed cylinder

$$C(X) = B_{\mathbb{R}^d}(0, |X|_+) \times B_{\mathbb{R}^c}(0, |X|_-),$$

and if Y is another vector, let $C(X,Y)$ denote the closed cylinder

$$C(X,Y) = B_{\mathbb{R}^d}(0, |X|_+) \times B_{\mathbb{R}^c}(0, |Y|_-).$$

In all situations, let $\mathrm{d}(x,y)$ denote the distance associated with the underlying norm between the two points x and y and $\mathrm{d}(x,A)$ the distance between the point x and the set A.

2.2 Matrices

Let I_n denote the identity matrix in $\mathrm{M}_n(\mathbb{R})$.

We fix once and for all a norm on $\mathrm{M}_{d+c}(\mathbb{R})$. All the distances and the balls in the space of matrices are associated with this norm. When E is a subset of $\mathrm{M}_{d+c}(\mathbb{R})$, let $B_E(x,r)$ denote the set of matrices in E within a distance from x smaller than r.

Let $\mathcal{L} = \mathcal{L}_{d+c}$ denote the space of $(d+c)$-dimensional unimodular lattices in $\mathbb{R}^{d+c}$ which we identify with $\mathrm{SL}(d+c,\mathbb{R})/\mathrm{SL}(d+c,\mathbb{Z})$.

For $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$, let M_θ denote the matrix

$$\begin{pmatrix} I_d & -\theta \\ 0 & I_c \end{pmatrix} \in \mathrm{SL}(d+c,\mathbb{R})$$

and $\Lambda_\theta = M_\theta \mathbb{Z}^{d+c}$ the lattice associated with M_θ.

Let $\mathcal{H}_{>0}$ denote the subgroup of all matrices M_θ, $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$, and let $\mathbb{T}_{d,c}$ denote its image in $\mathcal{L}_{d+c}$.

In the same manner, let $\mathcal{H}_{<}$ denote the subgroup of $\mathrm{SL}(d+c,\mathbb{R})$ of matrices of the form

$$\begin{pmatrix} I_d & 0 \\ B & I_c \end{pmatrix}$$

and let $\mathcal{H}_{\leq}$ denote the subgroup of $\mathrm{SL}(d+c,\mathbb{R})$ of matrices of the form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where $A \in \mathrm{GL}(d,\mathbb{R})$, $B \in M_{c,d}(\mathbb{R})$ and $C \in \mathrm{GL}(c,\mathbb{R})$.

Let

$$g_t = \begin{pmatrix} e^{ct} I_d & 0 \\ 0 & e^{-dt} I_c \end{pmatrix} \in \mathrm{SL}(d+c,\mathbb{R}),$$

$t \in \mathbb{R}$, denote the standard diagonal flow, $E_- = \{0\} \times \mathbb{R}^c$ denote the contracting direction of the flow and $E_+ = \mathbb{R}^d \times \{0\}$ denote the expanding direction of the flow. We also refer to E_+ as the horizontal subspace and to E_- as the vertical subspace.

2.3 Lattices

Suppose $\mathbb{R}^n$ is equipped with a norm $\|.\|$. For a lattice Λ and an integer $i \in \{1, ..., n\}$, let $\lambda_i(\Lambda)$ denote the i-th minimum of the lattice Λ with respect to the norm $\|.\|$, i.e.,

$$\lambda_i(\Lambda, \|.\|) = \min\{\lambda > 0 : B(0,\lambda) \cap \Lambda \text{ contains at least } i \text{ independent vectors}\}.$$

Observe that $\lambda_1(\Lambda)$ is the length of the shortest nonzero vector in Λ. When there is no ambiguity about the norm we write $\lambda_i(\Lambda)$ instead of $\lambda_i(\Lambda, \|.\|)$.

3 Best approximations

3.1 Best Diophantine approximations

Multidimensional extensions of the classical continued fraction expansion cannot conciliate all the properties of the one dimensional expansion. For instance, it is not possible to conciliate the unimodularity and the best approximation property (see [17] and [22]). The "best simultaneous Diophantine approximations" is the multidimensional extension based solely on the best approximation property. It has been studied by many Authors, see for instance [14, 15, 16, 17, 18, 7, 9, 11, 22].

Definition 3. *Let $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$.*
1. A nonzero vector $Q \in \mathbb{Z}^c$ is a best simultaneous Diophantine approximation denominator of θ if for all nonzero U in $\mathbb{Z}^c$,

$$\|U\|_{\mathbb{R}^c} < \|Q\|_{\mathbb{R}^c} \Rightarrow \mathrm{d}(\theta Q, \mathbb{Z}^d) < \mathrm{d}(\theta U, \mathbb{Z}^d)$$
$$\|U\|_{\mathbb{R}^c} \leq \|Q\|_{\mathbb{R}^c} \Rightarrow \mathrm{d}(\theta Q, \mathbb{Z}^d) \leq \mathrm{d}(\theta U, \mathbb{Z}^d).$$

2. An element (P, Q) in $\mathbb{Z}^d \times \mathbb{Z}^c$ is a best Diophantine approximation vector of θ if Q is a best simultaneous Diophantine approximation denominator of θ and if

$$\|\theta Q - P\|_{\mathbb{R}^d} = \mathrm{d}(\theta Q, \mathbb{Z}^d).$$

If the equation $\theta Q = 0 \mod \mathbb{Z}^d$ has no nontrivial solution $Q \in \mathbb{Z}^d$, the set of best Diophantine approximation denominators of θ is infinite. Numbering the set of best approximation denominators in ascending order of the norm $q = \|Q\|_c$, we obtain two sequences

$$q_0 = q_0(\theta) = \lambda_1(\mathbb{Z}^c) < q_1 = q_1(\theta) = \|Q_1(\theta)\|_c < ... < q_n = q_n(\theta) = \|Q_n(\theta)\|_c < ....$$

and

$$r_0 = r_0(\theta) = \mathrm{d}(\theta Q_0, \mathbb{Z}^d) > r_1 = r_1(\theta) = \mathrm{d}(\theta Q_1, \mathbb{Z}^d) > ... > r_n = r_n(\theta) = \mathrm{d}(\theta Q_n, \mathbb{Z}^d) > ....$$

When $d = c = 1$, by the best approximation property, the integers $q_0, q_1, ..., q_n, ...$ are the denominators of the ordinary continued fraction expansion of θ. The only slight difference is that in the ordinary continued fraction expansion, it can happen that $q_0 = q_1 = 1$. In this case, the indices are shifted by one.

3.2 Minimal vectors in lattices

The notion of minimal vector goes back to Voronoï. He used minimal vectors to find units in cubic fields (see [28] and also [5, 6]). It allows to convert statements about best simultaneous Diophantine approximations of vectors or of matrices into statements about lattices (see [7, 11]).

Definition 4. *Let $M \in \mathrm{SL}(d+c, \mathbb{R})$ and let $\Lambda = M\mathbb{Z}^{d+c} \in \mathcal{L}_{d+c}$. A nonzero vector $X \in \Lambda$ is a minimal vector of Λ (with respect to the norms $\|\cdot\|_{\mathbb{R}^d}$ and $\|\cdot\|_{\mathbb{R}^c}$) if the only nonzero vectors $Y \in \Lambda$ in the cylinder $C(X)$ are such that $C(X) = C(Y)$, i.e.,*

$$|X|_+ = |Y|_+ \,,\ |X|_- = |Y|_- \,.$$

If two minimal vectors X and Y define the same cylinder we say that they are equivalent.

Observe that for each lattice Λ, there exists a minimal vector X that is a shortest vector of Λ with respect to the norm $\|.\|_{\mathbb{R}^{d+c}}$. There might exist other shortest vectors and even other shortest vector that are minimal. The set of minimal vectors is generally infinite but might be finite. For $d, c \geq 1$, it is easily shown that there are at least two linearly independent minimal vectors in any lattice. This lower bound may be achieved, for instance with $\Lambda = \mathbb{Z}^{1+1}$.

Given a lattice Λ in $\mathcal{L}_{d+c}$, we select one minimal vector in each equivalent class of minimal vectors. We number these vectors in ascending order of heights. Such a numbering exists because 0 is the only possible limit point of the set of heights of minimal vectors (see the proof of Lemma 7) We get a sequence

$$...X_{-n}(\Lambda), ..., \ X_{-1}(\Lambda)\,, \ X_0(\Lambda), \ X_1(\Lambda), ...$$

This sequence might be finite, infinite one-sided or two-sided. The sequence $(|X_n(\Lambda)|_+)_n$ is decreasing while the sequence $(|X_n(\Lambda)|_-)_n$ is increasing. The numbering with increasing heights is unique up to a shift on the indices. Though, this shift is not really relevant, we will fix later a convenient way of choosing $X_0(\Lambda)$ (see the section about return times).

We shall always use the following notations

$$q_n(\Lambda) = |X_n(\Lambda)|_- \quad \text{ and } \quad r_n(\Lambda) = |X_n(\Lambda)|_+\,.$$

The following Lemma is easy and very important. It shows that for $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$, the sequences $(q_n(\Lambda_\theta))_n$ and $(q_n(\theta))_n$ are deduced one another by a shift. Therefore, if one of the two limits

$$\lim_{n\to\infty} \frac{1}{n} \ln q_n(\theta), \text{ and } \lim_{n\to\infty} \frac{1}{n} \ln q_n(\Lambda_\theta)$$

exists, then the other exists and have the same value. The same results holds with the sequences $(r_n(\theta))_n$ and $(r_n(\Lambda_\theta))_n$.

Lemma 5. *Let θ be in $\mathrm{M}_{d,c}(\mathbb{R})$.*
1. If $X = M_\theta Y$ is a minimal vector of the lattice Λ_θ with positive height , $|X|_- > 0$, then Y is best a approximation vector of θ.
2. Conversely if Y in $\mathbb{Z}^{d+c}$ is a best approximation vector of θ such that

$$\|Y_+ - \theta Y_-\|_{\mathbb{R}^d} < \lambda_1(\mathbb{Z}^d)$$

then $X = M_\theta Y$ is a minimal vector of Λ_θ.

Proof. 1. Set $Q = Y_-$. Since X is a minimal vector, $\mathrm{d}(\theta Q, \mathbb{Z}^d) = |X|_+$. Suppose that $U \in \mathbb{Z}^c$ and $V \in \mathbb{Z}^d$ are such that

$$\|\theta U - V\|_{\mathbb{R}^d} = \mathrm{d}(\theta U, \mathbb{Z}^d) \le \mathrm{d}(\theta Q, \mathbb{Z}^d) = |X|_+$$

and

$$\|U\|_{\mathbb{R}^c} \le \|Q\|_{\mathbb{R}^c} = |Y|_-\,,$$

then the vector

$$M_\theta \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} V - \theta U \\ V \end{pmatrix}$$

is in the cylinder $C(X)$, and therefore by definition of minimal vectors, we have

$$\|\theta U - V\|_{\mathbb{R}^d} = \mathrm{d}(\theta U, \mathbb{Z}^d) = \mathrm{d}(\theta Q, \mathbb{Z}^d)$$

and

$$\|U\|_{\mathbb{R}^c} = \|Q\|_{\mathbb{R}^c}\,.$$

It follows that Q is a best approximation denominator.
2. Conversely suppose that $Y=(P,Q)$ is a best approximation vector such that

$$\|P-\theta Q\|_{\mathbb{R}^d} < \lambda_1(\mathbb{Z}^d).$$

If a nonzero vector

$$Z = M_\theta \begin{pmatrix} V \\ U \end{pmatrix} \in C(X),$$

then we have

$$\|V-\theta U\|_{\mathbb{R}^d} \le \|P-\theta Q\|_{\mathbb{R}^d} < \lambda_1(\mathbb{Z}^d),$$

hence U is not zero. We also have

$$\|U\|_{\mathbb{R}^c} \le \|Q\|_{\mathbb{R}^c}$$

hence by definition of best approximation denominator, we have $|Z|_- = \|U\|_{\mathbb{R}^c} = \|Q\|_{\mathbb{R}^c} = |X|_-$ and $|Z|_+ = \|V-\theta U\|_{\mathbb{R}^d} = \|P-\theta Q\|_{\mathbb{R}^d} = |X|_+$, hence X is minimal. □

The classical inequality

$$q_{n+1} r_n \le 1$$

which holds for the one-dimensional continued fraction expansion of a real number can be extended to minimal vectors of lattices or to best approximation vectors. This fact is well known but it is worth stating it.

Lemma 6. *There is a constant $C_{d,c}$ depending only on c and d such that for all lattice $\Lambda \in \mathcal{L}_{d+c}$ or all matrices $\theta \in \mathrm{M}_{d,c}$ and all integers n, we have*

$$q_{n+1}^c(\Lambda) r_n^d(\Lambda) \le C_{d,c},$$
$$q_{n+1}^c(\theta) r_n^d(\theta) \le C_{d,c}.$$

Proof. Just use the first Minkowski Theorem with the cylinder defined by two consecutive minimal vectors or best approximation vectors. □

The classical inequality

$$q_{n+2} \ge 2q_n$$

which holds for the denominators of the one-dimensional continued fraction expansion of a real number can be extended to minimal vector of lattices. This inequality has already been extended to best simultaneous Diophantine approximations, see [16], [17] and [11]. The extension to minimal vectors of lattices is straightforward.

Lemma 7. *There is a positive integer constant $A = A(d,c)$ such that for any Λ in $\mathcal{L}_{d+c}$ and any $n \in \mathbb{Z}$, if $X_n(\Lambda),\ X_{n+1}(\Lambda), ..., X_{n+A}(\Lambda)$ exist, then*

$$q_{n+A}(\Lambda) \ge 2q_n(\Lambda),$$
$$r_{n+A}(\Lambda) \le \frac{1}{2} r_n(\Lambda)$$

Proof. Let A be an integer constant such that if A points $(x_1, y_1), ..., (x_A, y_A)$ are in the product of balls $B_{\mathbb{R}^d}(0, r_1) \times B_{\mathbb{R}^c}(0, r_2)$ with $r_1, r_2 \geq 0$ then there exist two indices $i \neq j$ such that $\|x_i - x_j\|_{\mathbb{R}^d} \leq \frac{1}{2} r_1$ and $\|y_i - y_j\|_{\mathbb{R}^c} \leq \frac{1}{2} r_2$. With this choice of the constant A, if $k \geq A$ is a positive integer such that $q_{n+k}(\Lambda) \leq 2q_n(\Lambda)$, then there are two integers $0 \leq i < j \leq k$ such that the vector $X_{n+j}(\Lambda) - X_{n+i}(\Lambda)$ satisfies both conditions

$$\begin{cases} |X_{n+j}(\Lambda) - X_{n+i}(\Lambda)|_+ \leq \frac{1}{2} r_n(\Lambda) \\ |X_{n+j}(\Lambda) - X_{n+i}(\Lambda)|_- \leq \frac{1}{2} 2q_n(\Lambda) \end{cases}$$

which contradicts the definition of $X_n(\Lambda)$. The same way of reasoning leads to the other inequality. □

We shall use several times the following very simple Lemma which is a consequence of the following observation. For any minimal vector X of a lattice Λ in $\mathbb{R}^{d+c}$ and any $t \in \mathbb{R}$, $g_t X$ is a minimal vector of the lattice $g_t \Lambda$. It follows that

Lemma 8. *Let Λ be in $\mathcal{L}_{d+c}$ and let t be in $\mathbb{R}$. The sequence of minimal vectors of the lattice $g_t \Lambda$ is $(g_t(X_n(\Lambda)))_n$.*

4 The surfaces S and S'

We assume that $\mathbb{R}^{d+c}$ is endowed with the norm

$$\|(x, y)\|_{\mathbb{R}^{d+c}} = \max\{\|x\|_{\mathbb{R}^d}, \|y\|_{\mathbb{R}^c}\}.$$

The main idea of the proofs of Theorems 1 and 2 is to induce the flow g_t on the surface

$$\{\Lambda \in \mathcal{L}_{d+c} : \lambda_2(\Lambda) = \lambda_1(\Lambda)\}$$

where $\lambda_1(\Lambda)$ and $\lambda_2(\Lambda)$ are the first two minima of the lattice Λ associated with the above norm. And then, to use Birkhoff ergodic Theorem with the first return map associated with the flow g_t. For technical reason it is better to slightly change the surface. For instance the above set is not a submanifold of $\mathcal{L}_{d+c}$. It could have some “branching points” while a slightly smaller set is clearly a submanifold, see Lemma 11. It will be convenient to use two surfaces S and S' for the proof of Theorem 1. These two surfaces are very similar; we state all the results we need for both surfaces but we only perform the proofs for the first surface S.

4.1 Definition of S

The surface S is the set of lattices Λ in $\mathcal{L}_{d+c}$ such that there exist two independent vectors $v_0^S(\Lambda)$ and $v_1^S(\Lambda)$ in Λ such that:

- $|v_1^S(\Lambda)|_+$ and $|v_0^S(\Lambda)|_-$ are $< |v_1^S(\Lambda)|_- = |v_0^S(\Lambda)|_+$,
- the only nonzero points of Λ in the ball $B_{\mathbb{R}^{d+c}}(0, \lambda_1(\Lambda))$ are $\pm v_0^S(\Lambda)$ and $\pm v_1^S(\Lambda)$.

Observe that for Λ in S, $v_0^S(\Lambda)$ and $v_1^S(\Lambda)$ are unique up two sign and are consecutive minimal vectors of Λ.

Since $\pm v_0^S(\Lambda)$ and $\pm v_1^S(\Lambda)$ are the only nonzero points of Λ in the ball $B_{\mathbb{R}^{d+c}}(0, \lambda_1(\Lambda))$, S is included in the set

$$\{\Lambda \in \mathcal{L}_{d+c} : \lambda_1(\Lambda) = \lambda_2(\Lambda)\}.$$

4.2 Definition of S'

The surface S' is the set of lattices in $\mathcal{L}_{d+c}$ such that there exists a vectors $w_0^{S'}(\Lambda)$ in Λ such that:

- the only nonzero points of Λ in the ball $B_{\mathbb{R}^{d+c}}(0, \lambda_1(\Lambda))$ are $\pm w_0^{S'}(\Lambda)$,
- the ball $B_{\mathbb{R}^{d+c}}(0, \lambda_1(\Lambda))$ is equal to the cylinder $C(w_0^{S'}(\Lambda))$.

Observe that $w_0^{S'}(\Lambda)$ is unique up to sign and is a minimal vector of Λ.

4.3 Lattices bases and minima

We shall need the following results about lattices.

Lemma 9. *Suppose $\mathbb{R}^n$ is equipped with any norm $\|.\|$. Let Λ be a lattice in $\mathbb{R}^n$ and let v_1, v_2 be two independent vectors of Λ such that $\|v_1\| = \lambda_1(\Lambda)$ and $\|v_2\| = \lambda_2(\Lambda)$. Then $\mathbb{Z}v_1 + \mathbb{Z}v_2$ is a primitive sub-lattice of Λ unless $\frac{1}{2}(v_1 + v_2) \in \Lambda$ and $\|v_1\| = \|v_2\| = \left\|\frac{1}{2}(v_1 + v_2)\right\|$.*

Proof. Consider the parallelogram $\mathcal{P}$ defined by the vectors v_1 and v_2. Let v be an element of Λ that belongs to the interior of $\mathcal{P}$. If v is not in the segment joining v_1 and v_2 then the distance from v to either 0 or $v_1 + v_2$ is of the form $\|t_1v_1 + t_2v_2\|$ for some positive real numbers t_1 and t_2 with $t_1 + t_2 < 1$. Hence this distance is $\leq t_1 \|v_1\| + t_2 \|v_2\| < \lambda_2(\Lambda)$ which contradicts the definition of $\lambda_2(\Lambda)$. If v is in the the segment joining v_1 and v_2 but is not the point $\frac{1}{2}(v_1+v_2)$ then the distance from v to either v_1 or v_2 is of the form $\|t(v_1 - v_2)\|$ with $t < \frac{1}{2}$ which implies that this distance is $< \lambda_2(\Lambda)$, again a contradiction. If $v = \frac{1}{2}(v_1 + v_2)$, we have $\|v\| \leq \frac{1}{2}(\|v_1\| + \|v_2\|)$ which is $< \lambda_2(\Lambda)$ unless $\|v_1\| = \|v_2\| = \|v\|$. □

It follows that when the norm is strictly convex, the sub-lattice $\mathbb{Z}v_1+\mathbb{Z}v_2$ is always primitive. In our setting despite that the norm is not strictly convex it is possible to use the above Lemma. With our choice of the norm on $\mathbb{R}^{d+c}$, the triangle inequality is strict for two vectors one inside the "top" of the cylinder $B_{\mathbb{R}^{d+c}}(0, r)$ and one inside the lateral side of $B_{\mathbb{R}^{d+c}}(0, r)$. Therefore,

Corollary 10. *Let Λ be in S. Then the vectors $v_0^S(\Lambda)$ and $v_1^S(\Lambda)$ associated with Λ are the first two vectors of a basis of Λ.*

4.4 Geometric properties of S and S'

Lemma 11. *S and S' are a submanifolds of $\mathcal{L}_{d+c}$ of dimension $(d + c)^2 - 2$, transverse to the diagonal flow g_t.*

Proof. Let Λ_0 be in S and call $v_0^S(\Lambda_0)$ and $v_1^S(\Lambda_0)$ the two vectors provided by the definition of S. By Corollary 10, $v_0^S(\Lambda_0)$ and $v_1^S(\Lambda_0)$ are the first two vectors of a basis $(b_1, ..., b_{d+c})$ of Λ_0. We can find a small enough positive real number ε such that for any $(v_1, ..., v_{d+c})$ in the open set

$$W = B_{\mathbb{R}^{d+c}}(b_1, \varepsilon) \times ... \times B_{\mathbb{R}^{d+c}}(b_{d+c}, \varepsilon),$$

- the matrix $M = M(v_1, ..., v_{d+c})$ the columns of which are the v_i, is in $\mathrm{GL}(d + c, \mathbb{R})$ and the sets WP, $P \in \mathrm{SL}(d + c, \mathbb{Z})$ are disjoint,

- the vectors $\pm v_1$ and $\pm v_2$ are the only nonzero vectors of the lattice $\Lambda = M\mathbb{Z}^{d+c}$ in the cylinder $C(v_1, v_2)$,
- $|v_1|_+ > 0$,
- $\|v\| > \|v_1\|$ and $\|v_2\|$ for all v in $\Lambda \setminus \{0, \pm v_1, \pm v_2\}$.

Consider the map

$$\begin{aligned} f &: W \to \mathbb{R}^2 \\ &: M = (v_1, ..., v_{d+c}) \to (f_1(M) = \det M, f_2(M) = |v_1|_+^2 - |v_2|_-^2). \end{aligned}$$

Then a lattice $\Lambda = M\mathbb{Z}^{d+1}$ with $M \in W$, is in S iff $f(M) = (1, 0)$. To prove that S is a submanifold, it is enough to show that the differential $Df(M)$ is onto at every point M in W. The differential of f_2 is given by

$$Df_2(M).(w_1, ..., w_{d+c}) = 2v_1^+.w_1^+ - 2v_2^-.w_2^-.$$

The linear map $Df_2(M)$ depends only on w_1 and w_2 and since $v_1^+ \neq 0$ for all M in W, $Df_2(M)$ is never the zero map. The differential of f_1 is given by

$$Df_1(M).(w_1, ..., w_{d+c}) = \sum_{i,j=1}^{d+c} (-1)^{i+j} \Delta_{i,j} w_{i,j}$$

where $w_j = (w_{i,j})_{i=1,...,d+c}$, $j = 1, ..., d+c$ and $\Delta_{i,j}$ is the (i, j)-minor of the matrix M. Since $\det M \neq 0$, one at least one of the minors $\Delta_{i,3}$, $i \leq d+c$, is not zero. Therefore the linear $Df_1(M)$ is not zero and depends on w_3. It follows that the two linear maps $Df_1(M)$ and $Df_2(M)$ are linearly independent for all M in W which implies that S is a submanifold of $\mathcal{L}_{d+c}$. To show that the flow is transverse to S, we have to check that for a matrix $M = M(v_1, \ldots, v_{d+c})$ in W such that $f(M) = 0$ we have $Df(M).(w_1, ..., w_{d+c}) \neq 0$ when $w_i = (cv_i^+, -dv_i^-)$. Now, for such w_i, $Df_2(M).(w_1, ..., w_{d+c}) = 2c\,|v_1|_+^2 + 2d\,|v_2|_-^2 > 0$, hence $Df(M).(w_1, ..., w_{d+c})$ is not zero. □

4.5 Negligible sets

A important ingredient of the proof of Theorem 1 is that, for a given lattice Λ, the visiting times t, i.e. the times t such that $g_t\Lambda \in S$, can be read from the sequence $(X_n(\Lambda))_n$ of minimal vectors. However, this reading is straightforward only for generic lattices, a small subset of lattices has to be avoided.

4.5.1 A negligible set $\mathcal{N}$ in the space lattices

Let $\mathcal{N} = \mathcal{N}_{d+c}$ be the set of lattices Λ in $\mathcal{L}_{d+c}$ such that either

- there exist two vectors v_1, and v_2, such that $v_1 \neq \pm v_2$ and $|v_1|_+ = |v_2|_+ > 0$ or $|v_1|_- = |v_2|_- > 0$,

- or there exists a nonzero vector in Λ lying in the vertical space $\{0\} \times \mathbb{R}^c$, or in the horizontal subspace $\mathbb{R}^d \times \{0\}$.

Remark 1. All the lattices Λ_θ are in $\mathcal{N}$.

Lemma 12. *$\mathcal{N}$ is negligible and g_t invariant.*

Proof. Clearly $\mathcal{N}$ is g_t invariant and the set of lattices with a nonzero vector in the vertical subspace or in the horizontal subspace is negligible. So we are reduced to prove that if $X \neq \pm Y$ are two nonzero vectors in $\mathbb{Z}^{d+c}$ the set of matrices M in $\mathrm{SL}(d+c, \mathbb{R})$ satisfying one of the equations

$$|MX|_+^2 - |MY|_+^2 = 0$$

or

$$|MX|_-^2 - |MY|_-^2 = 0$$

is of zero measure. Firstly, by symmetry, it is enough to deal with one of the equations, say the first. Secondly, by homogeneity it is equivalent to prove that the set of matrices in $M_{d+c}(\mathbb{R})$ that satisfy this equation is of zero measure. Since this is an algebraic equation, it is enough to prove that there exists at least one matrix M such that $|MX|_+^2 - |MY|_+^2 \neq 0$. If X and Y are proportional just choose a matrix M such that $|MX|_+ \neq 0$. Otherwise, first choose a vector Z in the subspace spanned by X and Y that is orthogonal to X. Observe that $Z.Y \neq 0$. Next choose a d-dimensional subspace V of $\mathbb{R}^{d+c}$ containing Z and orthogonal to X. A matrix M the first d rows of which are a basis of V, is such that $|MX|_+ = 0$ and $|MY|_+ \neq 0$. □

Remark 2. A lattice Λ that is not in $\mathcal{N}$ has a bi-infinite sequence of minimal vectors and is in S iff $\lambda_1(\Lambda) = \lambda_2(\Lambda)$.

4.5.2 A negligible set $\mathcal{M}$ in the space of matrices $\mathrm{M}_{d,c}(\mathbb{R})$

Let C be a positive real constant and let $\mathcal{M} = \mathcal{M}_{d,c} = \mathcal{M}_{d,c}(C)$ be the set of matrices $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$ such that either

- there exist two nonzero vectors $X \neq \pm Y$ in $\mathbb{Z}^{d+c}$ with nonzero heights such that $|M_\theta X|_+ = |M_\theta Y|_+$
- or there exist infinitely many pairs $X \neq \pm Y$ in $\mathbb{Z}^{d+c}$ such that $|X|_- = |Y|_- \neq 0$ and $|M_\theta X|_+, |M_\theta Y|_+ \leq C\,|X|_-^{-\frac{c}{d}}$.

The set $\mathcal{M}$ depends on the constant C. Actually, we will only use the value $C = C_{d,c}$ where $C_{d,c}$ is given by Lemma 6.

Lemma 13. *$\mathcal{M}$ is negligible.*

Proof. We prove that $\mathcal{M}$ is included in a countable union of negligible sets.

Given $X \neq \pm Y$ two nonzero vectors in $\mathbb{Z}^{d+c}$ with nonzero heights, consider the set $\mathcal{M}(X,Y)$ of matrices $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$ such that $|M_\theta X|_+ = |M_\theta Y|_+$. In order to show that $\mathcal{M}(X,Y)$ has zero measure it is enough to show that the polynomial

$$\begin{aligned} f(\theta) &= |M_\theta X|_+^2 - |M_\theta Y|_+^2 \\ &= \|X_+\|_{\mathbb{R}^d}^2 - \|Y_+\|_{\mathbb{R}^d}^2 - 2(X_+.\theta X_- - Y_+.\theta Y_-) + \|\theta X_-\|_{\mathbb{R}^d}^2 - \|\theta Y_-\|_{\mathbb{R}^d}^2 \end{aligned}$$

is not the zero polynomial.

If $X_- \neq \pm Y_-$, we can choose θ_0 such that $\|\theta_0 X_-\|^2_{\mathbb{R}^d} - \|\theta_0 Y_-\|^2_{\mathbb{R}^2} \neq 0$. With this choice, the one variable polynomial $P(t) = f(t\theta_0)$ has a nonzero degree two monomial which implies that the polynomial f is not the zero polynomial.

If $X_- = Y_-$ (the case $X_- = -Y_-$ is similar), $f(\theta) = |X|_+^2 - |Y|_+^2 - 2(X_+ - Y_+).\theta X_-$. Since $X_- \neq 0$, the map $\varphi : \theta \in \mathrm{M}_{d,c}(\mathbb{R}) \to \theta X_- \in \mathbb{R}^d$ is onto. It follows that we can choose θ such that $\theta X_- = X_+ - Y_+$. With this value of θ, we obtain $f(\theta) - f(-\theta) = -4\,|X_+ - Y_+|^2 \neq 0$ which implies that the polynomial f is not the zero polynomial. It follows that $\mathcal{M}(X, Y)$ is negligible.

Consider now, for a positive integer n, the set $\mathcal{M}_n$ of matrices $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$ such that there a pair of linearly independent vectors (X, Y) in $\mathbb{Z}^{d+c} \times \mathbb{Z}^{d+c}$ such that $n \leq |X|_-\,, |Y|_- < n+1$ and

$$|M_\theta X|_+\,, \, |M_\theta Y|_+ \leq C\,|X|_-^{-\frac{c}{d}}\,.$$

We want to prove that the set of matrices θ that are in infinitely many $\mathcal{M}_n$ is negligible. We can move in the space $\mathrm{M}_{d,c}(\mathbb{R}/\mathbb{Z})$ and consider instead the set $\mathcal{T}_n$ of $\theta \in \mathrm{M}_{d,c}(\mathbb{R}/\mathbb{Z})$ such that there exist $q, q' \in \mathbb{Z}^c$ linearly independent with $n \leq \|q\|_{\mathbb{R}^c}\,, \|q'\|_{\mathbb{R}^c} < n+1$ and

$$d(\theta q, \mathbb{Z}^d),\, d(\theta q', \mathbb{Z}^d) \leq C n^{-\frac{c}{d}}.$$

For q fixed, the measure of the set of $\theta \in \mathrm{M}_{d,c}(\mathbb{R}/\mathbb{Z})$ such that $d(\theta q, \mathbb{Z}^c) \leq C n^{-\frac{c}{d}}$ is $a_{d,c} n^{-c}$ where the constant $a_{d,c}$ depends only on C and the dimensions. When the inequality holds simultaneously for two linearly independent integer vectors q and q', the measure is the square of $a_{d,c} n^{-c}$. It follows that the measure of $\mathcal{T}_n$ is bounded above by

$$u_n = \mathrm{card}\{(q, q') \in \mathbb{Z}^c \times \mathbb{Z}^c : n \leq \|q\|_{\mathbb{R}^c}\,, \|q'\|_{\mathbb{R}^c} < n+1\} \times a_{d,c}^2 n^{-2c}.$$

By Borel-Cantelli, it is enough to prove that the $\Sigma_n u_n < \infty$. Now $\mathrm{card}\{q \in \mathbb{Z}^c : n \leq \|q\|_{\mathbb{R}^c} < n+1\} \ll n^{c-1}$, hence

$$u_n \ll n^{-2}$$

and we are done.

□

4.6 Visiting and return times

Let Λ be in $\mathcal{L}_{d+c}$. By definition of S, when $g_t\Lambda$ is in S, $v_0^S(g_t\Lambda)$ and $v_1^S(g_t\Lambda)$ are two consecutive minimal vectors of $g_t\Lambda$. Therefore,

$$\left\{ \begin{array}{l} v_0^S(g_t\Lambda) = g_t X_k(\Lambda) \\ v_1^S(g_t\Lambda) = g_t X_{k+1}(\Lambda) \end{array} \right. .$$

for an integer k. Hence $e^{ct}\,|X_k(\Lambda)|_+ = e^{-dt}\,|X_{k+1}(\Lambda)|_-$ which implies

$$t = \frac{1}{d+c} \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)}.$$

It follows that the set of real numbers t such that $g_t\Lambda \in S$ is included in the set

$$V_\Lambda(S) = \{t_k = \frac{1}{d+c} \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)} \;:\; k \in \mathbb{Z}\}.$$

It can happen that some values t_k are skipped, but in that case, Λ must be in $\mathcal{N}$. So, when Λ is not in $\mathcal{N}$, $g_t\Lambda \in S$ iff $t \in V_\Lambda(S)$. For the surface S', the same results hold with

$$V_\Lambda(S') = \{t'_k = \frac{1}{d+c} \ln \frac{q_k(\Lambda)}{r_k(\Lambda)} \;:\; k \in \mathbb{Z}\}.$$

It follows that for almost all Λ, both the backward trajectory $(g_t\Lambda)_{t\le 0}$ and the forward trajectory $(g_t\Lambda)_{t\ge 0}$ visit the two surfaces S and S' infinitely often. Therefore the first return/entrance times in S and S',

$$\tau(\Lambda) = \inf\{t > 0 : g_t(\Lambda) \in S\} \in \mathbb{R}_{>0} \cup \{\infty\},$$
$$\tau'(\Lambda) = \inf\{t > 0 : g_t(\Lambda) \in S'\} \in \mathbb{R}_{>0} \cup \{\infty\},$$

are finite almost everywhere and the first return/entrance maps

$$R(\Lambda) = g_{\tau(\Lambda)}\Lambda,$$
$$R'(\Lambda) = g_{\tau'(\Lambda)}\Lambda$$

are defined for all Λ that are not in $\mathcal{N}$.

For an integer $n \ge 1$, denote τ_n the n-th return (or entrance) time in S, i.e.

$$\tau_n(\Lambda) = \sum_{k=0}^{n-1} \tau(R^k(\Lambda))$$

($R^0(\Lambda) = \Lambda$ for all Λ in $\mathcal{L}_{d+1}$). It will be convenient to choose the numbering of the sequence of minimal vectors $(X_n(\Lambda))$ in order to have simple formulas for the return time and the return map.

Numbering convention: *For a lattice $\Lambda \in \mathcal{L}_{d+c}$, $n = 0$ is the smallest integer $n \in \mathbb{Z}$ such that*

$$|X_{n+1}(\Lambda)|_- \ge |X_n(\Lambda)|_+$$

when the set of such integers is non empty.

With this numbering convention, for all Λ is in S, we have

$$X_0(\Lambda) = v_0^S(\Lambda)$$

and for all Λ is in S', we have

$$X_0(\Lambda) = w_0^{S'}(\Lambda).$$

Moreover when $\Lambda \notin S$ is not in $\mathcal{N}$,

$$|X_1(\Lambda)|_- - |X_0(\Lambda)|_+ > 0,$$
$$\tau(\Lambda) = \frac{1}{d+c} \ln \left(\frac{|X_1(\Lambda)|_-}{|X_0(\Lambda)|_+} \right),$$

and

$$g_{\tau(\Lambda)}X_0(\Lambda) = \pm v_0^S(R(\Lambda)) = \pm X_0(R(\Lambda)).$$

Let us summarize the above.

Lemma 14. *Let Λ be a lattice in $\mathcal{L}_{d+c}$.*
1. The set of visiting times in S is included in $V_\Lambda(S)$ and the set of visiting times in S' is included in $V_\Lambda(S')$.
2. Suppose Λ is not in $\mathcal{N}$. The set of visiting times in S is equals to $V_\Lambda(S)$ and the set of visiting times in S' is equals to $V_\Lambda(S')$.

For θ in $\mathrm{M}_{d,c}(\mathbb{R})$, we also need to connect the visiting times of the surface S with the best approximation vectors of θ.

Lemma 15. *Let θ be in $\mathrm{M}_{d,c}(\mathbb{R}) \setminus \mathcal{M}$. Then for all large enough integers n,*

$$t_n(\theta) = \frac{1}{d+c}\ln\frac{q_{n+1}(\theta)}{r_n(\theta)}$$

and

$$t'_n(\theta) = \frac{1}{d+c}\ln\frac{q_n(\theta)}{r_n(\theta)}$$

are visiting times for the surfaces S and S' respectively.

Proof. Let θ be in $\mathrm{M}_{d,c}(\mathbb{R}) \setminus \mathcal{M}$. Consider the sequence of all best approximation vectors $(Y_n(\theta))_{n\in\mathbb{N}}$ of θ. By Lemma 5, there are integers n_1 and k such that $X_{n+k}(\Lambda_\theta) = M_\theta Y_n(\theta)$ for all $n \geq n_1$. Since θ is not in $\mathcal{M}$, by Lemma 6 there is another integer n_2 such that for all $n \geq n_2$, the only nonzero vector of Λ_θ in the box $\mathcal{C}(X_{n+k}(\theta), X_{n+k+1}(\theta))$ are $\pm X_{n+k}(\theta)$ and $\pm X_{n+k+1}$. This means that for all n large enough, the times

$$t_n(\theta) = \frac{1}{d+c}\ln\frac{q_{n+1}(\theta)}{r_n(\theta)} \text{ and } t'_n(\theta)) = \frac{1}{d+c}\ln\frac{q_n(\theta)}{r_n(\theta)}$$

are visting times for the surfaces S and S'. ☐

4.7 Functions defined on S

Let Λ be in S. By definition of S, the functions

$$\rho, \rho^* : \Lambda \in S \to \ln\frac{|v_1^S(\Lambda)|_-}{|v_0^S(\Lambda)|_-},\ \ln\frac{|v_0^S(\Lambda)|_+}{|v_1^S(\Lambda)|_+} \in \mathbb{R}_{>0} \cup \{+\infty\}$$

are well defined on S. The next Lemma is easy, its proof is close to beginning of the proof of Lemma 11 and is omitted.

Lemma 16. *The functions $\Lambda \in \mathcal{L}_{d,c} \to |v_0^S(\Lambda)|_-$, $|v_0^S(\Lambda)|_+$, $|v_1^S(\Lambda)|_-$, $|v_1^S(\Lambda)|_+$ are continuous and thus the functions ρ and ρ^* are continuous.*

The following Lemma is important. On the one hand, it will imply that the functions ρ and ρ^* are integrable. On the other hand, it will explain the connection between the Lévy's constant $L_{d,c}$ and the average return times on S.

Lemma 17. *Let Λ be a lattice in $S \setminus \mathcal{N}$. Then*

$$\tau(\Lambda) = \frac{1}{d+c}(\rho(R(\Lambda)) + \rho^*(\Lambda)).$$

Proof. Let Λ be in $S \setminus \mathcal{N}$. By definition of S, $q_1(\Lambda) = r_0(\Lambda)$. Hence, by Lemma 14,

$$\begin{aligned}(d+c)\tau(\Lambda) &= \ln \frac{q_2(\Lambda)}{r_1(\Lambda)} \times \frac{r_0(\Lambda)}{q_1(\Lambda)} \\ &= \ln \frac{q_2(\Lambda)}{q_1(\Lambda)} + \ln \frac{r_0(\Lambda)}{r_1(\Lambda)} \\ &= \rho(R(\Lambda)) + \rho^*(\Lambda).\end{aligned}$$

□

5 Finiteness of the induced measure on S and S'

Fix a measure μ on $\mathcal{L}_{d+c}$ invariant by the $\mathrm{SL}(d+c,\mathbb{R})$ action. Recall that μ is unique up to a multiplicative constant. Since S is a submanifold of $\mathcal{L}_{d+c}$ transverse to the flow $(g_t)_{t\in\mathbb{R}}$ there exists a unique measure μ_S defined on S by the following property:

For all Λ in S, there exists a neighborhood W of Λ in S and $\varepsilon_\Lambda > 0$ such that for all Borel subsets $V \subset W$ and all $0 \le \varepsilon \le \varepsilon_\Lambda$,

$$\mu(\cup_{t\in[0,\varepsilon]} g_t V) = \varepsilon \mu_S(V).$$

The measure μ_S is the measure induced by the flow. It is well known that the measure μ_S is R-invariant.

The flow induces a measure $\mu_{S'}$ on S' as well. Let us prove that these measures are finite. This is a simple consequence of the next Lemma which will be very important in the proof of Theorem 24 about the almost sur convergence in $M_{d,c}(\mathbb{R})$.

Lemma 18. *1. There exists an integer constant A such that $\tau_A(\Lambda) \ge 1$ for all Λ in S and $\tau'_A(\Lambda) \ge 1$ for all Λ in S'.*

Proof. Let A be the constant given by Lemma **??** about the growth rate of the sequences $(q_n(\Lambda))_n$ and $(r_n(\Lambda))_n$. For all integers k, we have

$$\begin{aligned}\frac{1}{d+c}\left(\ln \frac{q_{k+A+1}(\Lambda)}{r_{k+A}(\Lambda)} - \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)}\right) &\ge \frac{1}{d+c} \ln \frac{q_{k+A+1}(\Lambda)}{q_{k+1}(\Lambda)} \\ &\ge \frac{1}{d+c}\ln 2.\end{aligned}$$

Since by Lemma 14, the set of visiting times of Λ is included in

$$V_\Lambda(S) = \{t_k = \frac{1}{d+c} \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)} \;:\; k \in \mathbb{Z}\},$$

$\tau_A(\Lambda) \ge \frac{1}{d+c}\ln 2$. Multiplying A by the smallest integer larger thna $\frac{d+c}{\ln 2}$ we are done. □

Proposition 19. *$\mu_S(S)$ and $\mu_{S'}(S')$ are finite and nonzero.*

Proof. Since S is nonempty and transverse to the flow, $\mu_S(S) > 0$.

Since μ_S is R-invariant, for all k

$$\int_S \tau(R^k\Lambda) d\mu_S(\Lambda) = \int_S \tau(\Lambda) d\mu_S(\Lambda),$$

and by Kac's return time Theorem,

$$\int_S \tau(\Lambda) d\mu_S(\Lambda) = \mu(\mathcal{L}_{d+1}),$$

therefore,

$$\int_S \tau_A(\Lambda) d\mu_S(\Lambda) = \int_S \sum_{k=0}^{A-1} \tau(R^k\Lambda) d\mu_S(\Lambda) = A\mu(\mathcal{L}_{d+c}).$$

By the above Lemma, $\tau_A \geq 1$ on S, hence $\mu_S(S) \leq A\,\mu(\mathcal{L}_{d+c})$ which is finite by Siegel's Theorem. □

6 Almost sure convergence in the space of lattices

6.1 Consequence of the Birkhoff Theorem

Theorem 20. *There exist two positive constants $L_{d,c}$ and $L^*_{d,c}$ such that for almost all lattices Λ in $\mathcal{L}_{d+c}$,*

$$\lim_{n\to\infty} \frac{1}{n} \ln q_n(\Lambda) = \frac{1}{\mu_S(S)} \int_S \rho d\mu_S = L_{d,c} > 0,$$

$$\lim_{n\to\infty} \frac{-1}{n} \ln r_n(\Lambda) = \frac{1}{\mu_S(S)} \int_S \rho^* d\mu_S = L^*_{d,c} > 0,$$

$$\lim_{n\to\infty} \frac{1}{n} \tau_n(\Lambda) = \frac{1}{d+c}(L_{d,c} + L^*_{d,c}) = \frac{\mu(\mathcal{L}_{d+c})}{\mu_S(S)}.$$

Moreover, these two constants do not depend on the particular choice of the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{R}^c$.

Proof. Let Λ be in $S \setminus \mathcal{N}$. By Lemma 17, $\tau(\Lambda) = \frac{1}{d+c}(\rho(R(\Lambda)) + \rho^*(\Lambda))$. Because the spaces of lattices has finite measure, the return time τ is in $\mathcal{L}^1(S)$ and therefore the non negative functions $\rho \circ R$ and ρ^* are also in $\mathcal{L}^1(S)$. Making use of the Birkhoff's Theorem with the functions ρ and ρ^*, we obtain the almost everywhere convergence of the sums

$$\frac{1}{N} \sum_{n=0}^{N-1} \rho \circ R^n, \ \frac{1}{N} \sum_{n=0}^{N-1} \rho^* \circ R^n$$

on S to R-invariant functions. Now the ergodicity of the flow g_t implies the ergodicity of the return map R. Therefore $\frac{1}{N}\sum_{k=0}^{N-1} \rho \circ R^n$ and $\frac{1}{N}\sum_{n=0}^{N-1} \rho^* \circ R^n$ converge almost everywhere on S to the constants

$$L_{d,c} = \frac{1}{\mu_S(S)} \int_S \rho d\mu_S \text{ and } L^*_{d,c} = \frac{1}{\mu_S(S)} \int_S \rho^* d\mu_S.$$

We would like to see that the Birkhoff sums converge to the same limits almost everywhere in the whole space of lattices. Let Λ be in $\mathcal{L}_{d+c} \setminus (S \cup \mathcal{N})$ and let n be a positive integer. By Lemma 14 and the numbering convention, for all $n \geq 1$,

$$\begin{aligned}\rho \circ R^n(\Lambda) &= \ln \frac{|X_1(R^n(\Lambda))|_-}{|X_0(R^n(\Lambda))|_-} \\ &= \ln \frac{\left|g_{\tau_n(\Lambda)}(X_n(\Lambda))\right|_-}{\left|g_{\tau_n(\Lambda)}(X_{n-1}(\Lambda))\right|_-} \\ &= \ln \frac{|X_n(\Lambda)|_-}{|X_{n-1}(\Lambda)|_-}\end{aligned}$$

and

$$\begin{aligned}\rho^* \circ R^n(\Lambda) &= \ln \frac{|X_0(R^n(\Lambda))|_+}{|X_1(R^n(\Lambda))|_+} \\ &= \ln \frac{|X_{n-1}(\Lambda)|_+}{|X_n(\Lambda)|_+}.\end{aligned}$$

as well. It follows that if the Birkhoff sums $\frac{1}{N}\sum_{n=1}^{N} \rho \circ R^n(\Lambda)$ and $\frac{1}{N}\sum_{n=1}^{N} \rho^* \circ R^n(\Lambda)$ converge to $L_{d,c}$ and $L^*_{d,c}$, then

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \ln q_N(\Lambda) &= L_{d,c} \\ \lim_{N\to\infty} \frac{-1}{N} \ln r_N(\Lambda) &= L^*_{d,c}.\end{aligned}$$

Now the image by the map $R : \mathcal{L}_{d+c} \to S$ of a subset of nonzero measure in $\mathcal{L}_{d+c}$ is a set of nonzero measure in S, therefore the sums $\frac{1}{N}\sum_{n=1}^{N} \rho \circ R^n$ and $\frac{1}{N}\sum_{n=1}^{N} \rho^* \circ R^n$ converge almost everywhere in $\mathcal{L}_{d+c}$ to $L_{d,c}$ and $L^*_{d,c}$.

By Lemma 7, we know that the sequences $(q_n(\Lambda))_n$ and $(r_n(\Lambda)^{-1})_n$ have at least exponential growth rate; therefore, the constants $L_{d,c}$ and $L^*_{d,c}$ are > 0.

By Lemma 17, for all Λ in $S \setminus \mathcal{N}$ and $k \in \mathbb{N}$,

$$\tau_{k+1}(\Lambda) - \tau_k(\Lambda) = \frac{1}{d+c}(\rho(R^{k+1}(\Lambda)) + \rho^*(R^k(\Lambda))),$$

hence

$$\lim_{n\to\infty} \frac{1}{n}\tau_n(\Lambda) = \frac{1}{d+c}(L_{d,c} + L^*_{d,c})$$

almost everywhere.

Finally, let us proof that the constants $L_{d,c}$ and $L^*_{d,c}$ do not depend on the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{R}^c$. For a matrix A_d in $\mathrm{SL}(d,\mathbb{R})$ and a matrix A_c in $\mathrm{SL}(c,\mathbb{R})$, let denote A the matrix

$$A = \begin{pmatrix} A_d & 0 \\ 0 & A_c \end{pmatrix} \in \mathrm{SL}(d+c,\mathbb{R}).$$

Since the action of A on $\mathcal{L}_{d+c}$ is measure preserving,

$$\lim_{n\to\infty}\frac{1}{n}\ln q_n(A\Lambda)=L_{d,c}$$
$$\lim_{n\to\infty}\frac{-1}{n}\ln r_n(A\Lambda)=L^*_{d,c}$$

almost everywhere $\mathcal{L}_{d+1}$. Now a vectors AX in the lattice $A\Lambda$ is minimal iff X is a minimal vector of the of the lattice Λ with respect to the new Euclidean norms $\|.\|_{A_d,\mathbb{R}^d}$ and $\|.\|_{A_c,\mathbb{R}^c}$ where

$$\|u\|_{A,\mathbb{R}^d}=\|Au\|_{\mathbb{R}^d}\,,\text{ and }\|v\|_{A,\mathbb{R}^c}=\|Av\|_{\mathbb{R}^c}\,.$$

Since up to multiplicative constants, all the Euclidean norms are of the above form, the constants $L_{d,c}$ and $L^*_{d,c}$ do not depend on the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{R}^c$. □

6.2 A consequence of Borel-Cantelli Lemma

Proposition 21. *With the notation of Theorem 20 we have*

$$cL_{d,c}=dL^*_{d,c}.$$

Proof. The inequality $cL_{d,c}\le dL^*_{d,c}$ is easy to prove. By Lemma 6, for all lattices Λ in $\mathcal{L}_{d+c}$ and all n, we have $q^c_{n+1}(\Lambda)r^d_n(\Lambda)\le C_{d,c}$. Hence for a lattice Λ such that

$$\lim_{n\to\infty}\frac{1}{n}\ln q_n(\Lambda)=L_{d,c}$$
$$\lim_{n\to\infty}\frac{-1}{n}\ln r_n(\Lambda)=L^*_{d,c},$$

we have,

$$cL_{d,c}-dL^*_{d,c}=c\lim_{n\to\infty}\frac{\ln q_n(\Lambda)}{n}+d\lim_{n\to\infty}\frac{\ln r_n(\Lambda)}{n}=\lim_{n\to\infty}\frac{\ln q^c_n(\Lambda)r^d_n(\Lambda)}{n}\le 0.$$

The converse inequality uses Borel-Cantelli Lemma. Let $\varphi\,:]0,\infty[\to]0,\infty[$ be a decreasing function such that $\sum_{n\ge1}\varphi(n)<\infty$, for instance $\varphi(t)=\frac{1}{t^\alpha}$ with $\alpha>1$. Since for such a function φ, $\liminf_{n\to\infty}\frac{1}{n}\ln\varphi(n)=0$, the inequality $cL_d-dL^*_d\ge0$ holds provided that for almost all lattices Λ, we have $q^c_n(\Lambda)r^d_n(\Lambda)\ge\varphi(n)^{d+c}$ for n large enough. Let K be a constant that will be chosen later. For each integer $n\ge1$, consider the set A_n of lattices Λ in $\mathcal{L}_{d+c}$ such that

$$\lambda_1(\Lambda)\le K\varphi(n)$$

and the set $B_n=g_{t_n}A_n$ where $t_n=\frac{1}{d}(\ln\varphi(n)-n)$. It is well known that the function $\lambda_1^{-1}:\mathcal{L}_{d+c}\to\mathbb{R}$ is integrable, see for instance [3] p. 27 (actually, the only important fact is that a positive power of λ_1^{-1} is integrable). Making use of the Markov inequality, we obtain

$$\mu(B_n)=\mu(A_n)\le\frac{\left\|\lambda_1^{-1}\right\|_1}{\frac{1}{K\varphi(n)}}\ll\varphi(n).$$

Therefore, by Borel-Cantelli Lemma, the set $\mathcal{B}$ of lattices Λ in $\mathcal{L}_{d+c}$ such that $\Lambda \in B_n$ for infinitely many integers n, is negligible. Suppose now that Λ is a lattice such that $q_n^c(\Lambda) r_n^d(\Lambda) \leq \varphi(n)^{d+c}$ for infinitely many n. For each integer $n \geq 1$, set $k_n = k_n(\Lambda) = \lfloor \ln q_n(\Lambda) \rfloor$. By Theorem 20, for almost all lattices we have $k_n \leq (L_{d,c}+1)n$ for n large enough. Therefore for almost all lattices, for n large enough, if $q_n^c(\Lambda) r_n^d(\Lambda) \leq \varphi(n)^{d+c}$ then the vector $g_{t_{k_n}}^{-1}(X_n(\Lambda))$ satisfies both

$$\begin{aligned}
\left| g_{t_{k_n}}^{-1}(X_n(\Lambda)) \right|_+ &= r_n(\Lambda) e^{\frac{c}{d}(k_n - \ln \varphi(k_n))} \\
&\leq r_n(\Lambda) q_n^{\frac{c}{d}}(\Lambda) \varphi(k_n)^{-\frac{c}{d}} \\
&\leq \varphi(n)^{\frac{d+c}{d}} \varphi(k_n)^{-\frac{c}{d}} \\
&= \varphi(k_n) \times \left(\frac{\varphi(n)}{\varphi(k_n)} \right)^{\frac{d+c}{d}} \\
&\leq \varphi(k_n) \times \left(\frac{\varphi(n)}{\varphi((L_{d,c}+1)n)} \right)^{\frac{d+c}{d}} \leq K\varphi(k_n)
\end{aligned}$$

for some constant K depending only on φ (we use that $\frac{\varphi(t)}{\varphi((L_{d,c}+1)t)}$ is bounded above which is obviously true when $\varphi(t) = \frac{1}{t^\alpha}$) and

$$\left| g_{t_{k_n}}^{-1}(X_n(\Lambda)) \right|_- = q_n(\Lambda) e^{-k_n + \ln \varphi(k_n)} \leq e\varphi(k_n) \leq K\varphi(k_n).$$

Thus there are infinitely many n such that $\Lambda \in B_{k_n}$. Since the sequence $(k_n)_n$ goes to infinity, $\Lambda \in \mathcal{B}$. It follows that for almost all lattices Λ, $q_n^c(\Lambda) r_n^d(\Lambda) \geq \varphi(n)$ for n large enough and we are done. □

As an immediate consequence of the previous Proposition and of Theorem 20 we have:

Corollary 22.

$$L_{d,c} = \frac{d}{\mu_S(S)} \int_S \tau \, d\mu_S = \frac{d \times \mu(\mathcal{L}_{d+1})}{\mu_S(S)}.$$

7 Parametrization of S when $c = 1$

This section is not necessary neither for the proofs of Theorems 1 and 2, nor for sections 8 and 9.

The aim is to show that the computation of the constant $L_{d,c}$ is theoretically feasible in the case $c = 1$. However if the case $d = 1$ is easy (see below), the case $d = 2$ is already difficult. It is possible to give an integral formula for $L_{2,1}$. However, we are not able to compute the integral, only a numerical estimation of the integral has been carried out. An exact description of S when $d \geq 3$ seems to be rather difficult.

In this section we assume $c = 1$.

7.1 rkN decomposition

In this subsection we give a parametrization of a set of lattices that contains S.

Let Λ be a lattice in S and let $v_0^S(\Lambda) = u_1$ and $v_1^S(\Lambda) = u_2$ be the two vectors associated with Λ by the definition of S (see the definition of S section 4.1). When $d \geq 2$, we suppose these two vectors have non negative heights and when $d = 1$, we only suppose that u_2 has a nonnegative height. Since u_1 and u_2 are independent shortest vectors, by Corollary 10, they are the first two vectors of a basis of Λ. Thus, there is a matrix $M \in \mathrm{SL}(d+1, \mathbb{R})$ defining Λ the first two columns of which are the vectors u_1 and u_2.

When $d = 1$, using the scaling factor $r = |u_1|_+ = |u_2|_- > 0$, we can write

$$M = rN,$$

where N is in the set U_1 of 2×2 matrices such that

$$\det N > 0,$$
$$n_{1,1} = n_{2,2} = 1,$$
$$|n_{2,1}|, |n_{1,2}| < 1.$$

When $d \geq 2$, let denote $(e_1, e_2, ..., e_{d+1})$ the standard basis of $\mathbb{R}^{d+1}$. Using the same scaling factor $r = |u_1|_+ = |u_2|_- > 0$ and an orthogonal matrix k that fixes e_{d+1} and sends e_1 onto $\frac{1}{r}u_{1,+}$, we can find a matrix $N = (n_{i,j})_{1\leq i,j \leq d+1}$ such that

$$M = rkN,$$
$$\det N > 0,$$

$$n_{1,1} = n_{d+1,2} = 1 > n_{d+1,1} = |u_1|_- \geq 0, \tag{2}$$
$$\|(n_{1,2}, ..., n_{d,2})\|_{\mathbb{R}^d} < 1, \tag{3}$$
$$n_{2,1} = ... = n_{d,1} = 0. \tag{4}$$

When $d \geq 2$, k is chosen in the group

$$K_d = \{k \in \mathrm{SO}(d+1) : ke_{d+1} = e_{d+1}\}.$$

and using the decomposition of a $d \times d$ matrix in a product of an orthogonal matrix with positive determinant and of an upper triangular matrix we can even suppose that

$$n_{i,j} = 0, \text{ for all } 1 \leq j < i \leq d. \tag{5}$$

For $d \geq 2$, let denote U_d the set of $(d+1) \times (d+1)$-matrices such that (2), (3), and (5) hold ((5 implies (4)).

Since $\det M = 1$, the scaling factor r must be equal to $(\det N)^{-\frac{1}{d+1}}$. Puting $K_1 = \{I_2\}$, for all $d \geq 1$, the map

$$(k, N) \in K_d \times U_d \to (\det N)^{-\frac{1}{d+1}} kN$$

provides a natural parametrization of a subset Σ in $\mathrm{SL}(d+1, \mathbb{R})$ the projection in $\mathcal{L}_{d+1}$ of which contains S. The main problem is now to find which of these couples (k, N) are such that $rkN\mathbb{Z}^{d+1} \in S$ and to select a fundamental domain in this set of couples. This problem reduces to finding the set of matrices N such that

- $N \in U_d$,
- The first two columns u_1 and u_2 of N are in the unit ball $B_{\mathbb{R}^{d+1}}(0,1)$ and are the only nonzero vectors of the lattice $N\mathbb{Z}^{d+1}$ in this ball,

then select a fundamental domain in this set of matrices N.

This is easy when $d = 1$ and doable when $d = 2$. When $d = 1$, it is even possible to find the first return map R.

Another issue is to find the measure μ_S on S induced by the flow g_t and the invariant measure μ of $\mathcal{L}_{d+1}$. This comparatively easier issue can be performed for all d without knowing explicitly S.

7.2 The induced measure μ_S

Consider the manifold $V_d = \mathbb{R}_{>0} \times \mathbb{R}\times K_d \times U_d$ and the submanifold

$$W = \{(\Delta, t, k, N) \in V_d : t = 0,\ \Delta = 1\} = \{1\} \times \{0\} \times K_d \times U_d.$$

together with the maps

$$\begin{aligned} F &: V_d \to \mathrm{GL}(d+1,\mathbb{R}) \\ &: (\Delta, t, k, N) \to \left(\frac{\Delta}{\det N}\right)^{\frac{1}{d+1}} g_t k N \end{aligned}$$

and $\overline{F} : V_d \to \mathrm{GL}(d+1,\mathbb{R})/\mathrm{SL}(d+1,\mathbb{Z})$ defined by

$$\overline{F}(\Delta, t, k, N) = F(\Delta, t, k, N)\mathbb{Z}^{d+1}.$$

By the discussion of the previous subsection, $\overline{F}$ provides a parametrization of S:

$$S \subset \overline{F}(W).$$

We would like to compute the measure μ_S in the coordinates $(1, 0, k, N)$. The submanifold W is equipped with the reference measure

$$\mu_{K_d} \otimes \lambda_{U_d}$$

where λ_{U_d} is the Lebesgue measure on U_d and μ_{K_d} is the invariant measure on K_d associated with the invariant volume form γ on K_d that is dual to the exterior product of the invariant vector fields generated by the standard skew symmetric matrices $(A_{i,j} = E_{ij} - E_{ji})_{1\le j<i\le d}$. The induce measure μ_S can be expressed with the parametrization $\overline{F}$, we give without proof an explicit formula in next Proposition.

Proposition 23. *Assume $d \ge 2$. Suppose that $\mathcal{D}$ is an open subset of W such that $\overline{F}(\mathcal{D}) \subset S$ and the restriction of $\overline{F}$ to $\mathcal{D}$ is one to one. Then the image by $\overline{F}$ of the measure*

$$1_{\mathcal{D}}\,(\frac{1}{\det N})^{d+1}(\prod_{j=1}^{d-1} n_{j,j}^{d-j})\,\mu_{K_d} \otimes \lambda_{U_d}$$

is the restriction of μ_S to $\overline{F}(\mathcal{D})$.

Remark 3. Recall that $n_{1,1} = 1$ in the above formula.

Remark 4. When $d = 1$, the above measure has the density

$$f(N) = f(n_{2,1}, n_{1,2}) = \frac{1}{(1 - n_{2,1} n_{1,2})^2}.$$

with respect to the Lebesgue measure on the two dimensional set U_1 of matrices N.

7.3 Determination of the surface S, $c = 1$, $d = 1$

We already have a map

$$\begin{aligned} U_1 &\to \mathcal{L}_2 \\ N &\to (\det N)^{-\frac{1}{2}} N\mathbb{Z}^2 \end{aligned}$$

that sends U_1 onto a set that contains S. By definition of S, the image of a matrix

$$N = \begin{pmatrix} 1 & n_{1,2} \\ n_{2,1} & 1 \end{pmatrix} \in U_1$$

is in S iff the only nonzero vectors of the lattice $\Lambda = N\mathbb{Z}^2$ in the ball $B_{\mathbb{R}^{1+1}}(0, 1)$ are the two columns of N up to sign. We obtain that $\Lambda \in S$ iff

- $0 < |n_{2,1}|, |n_{1,2}| < 1$,
- the signs of $n_{1,2}$ and $n_{2,1}$ are opposite.

So the map F defined on $]0, 1[^2 \times \{-1, 1\}$ defined by

$$(x, y, \epsilon) \to \frac{1}{(1 + xy)^{1/2}} \begin{pmatrix} 1 & -\epsilon x \\ \epsilon y & 1 \end{pmatrix} \mathbb{Z}^2$$

provide a parametrization of S and it is easy to see that F is a bijection. By Remark 4, with these coordinates, the function

$$f(x, y, \epsilon) = \frac{1}{(1 + xy)^2}.$$

is density the measure μ_S with respect to the Lebesgue measure. Therefore $\mu_S(S) = 2 \ln 2$. With the Siegel formula ([26]) and Corollary 22, we obtain the Lévy's constant

$$L_{1,1} = \frac{\mu(\mathcal{L}_{d+c})}{\mu_S(S)} = \frac{\zeta(2)}{2 \ln 2} = \frac{\pi^2}{12 \ln 2}.$$

7.3.1 Determination of the first return map, $c = 1,\ d = 1$

Let
$$\Lambda = F(x, y, \epsilon) = \frac{1}{(1+xy)^{1/2}} \begin{pmatrix} 1 & -\epsilon x \\ \epsilon y & 1 \end{pmatrix} \mathbb{Z}^2$$
be in S. By Lemma 14, to find the first return $R(\Lambda)$ in S, it is enough to find the minimal vector $X_2(\Lambda)$. Then $R(\Lambda)$ is given by $g_{\tau(\lambda)}(\Lambda)$ with
$$\tau(\Lambda) = \frac{1}{2} \ln \left(\frac{|X_2(\Lambda)|_-}{|X_1(\Lambda)|_+} \right).$$
By corollary 10, the first minimal vectors $X_0(\Lambda)$ and $X_1(\Lambda)$ form a basis of Λ. The minimal vector $X_2(\Lambda)$ is the vector of the form $X = aX_0(\Lambda) + bX_1(\Lambda)$ in the strip $|X|_+ < |X_1(\Lambda)|_+ = x$ with $a, b \in \mathbb{Z}$, and with the smallest height. It is not difficult to see that
$$X_2(\Lambda) = \epsilon X_0(\Lambda) + \lfloor \frac{1}{x} \rfloor X_1(\Lambda).$$
So we obtain $R(\Lambda) = F(x', y', \epsilon')$ where
$$\begin{aligned} \epsilon' &= -\epsilon \\ x' &= \{\frac{1}{x}\} \\ y' &= \frac{1}{y + \lfloor \frac{1}{x} \rfloor} \end{aligned}$$
and we see that return map R is a two-fold extension of the natural extension of the Gauss map.

7.4 Value of Lévy's constant when $d = 2$ and $c = 1$,

An exact description of S is possible when $d = 2$ and $c = 1$. Together with the expression of the measure μ_S in Proposition 23, this lead to a closed formula for Lévy's constant as a seven-tuple integral of an algebraic function over an union of domains the boundaries of which are algebraic surfaces of degree at most two. We are not able to compute this seven-tuple integral. However using Octave, Seraphine Xieu (see [29]) has compute a numerical approximation of Levy's constant
$$L_{2,1} = 1.135256974\ldots$$
This can be compared with the one dimensional Levy's constant
$$L_{1,1} = 1.186569111\ldots$$

8 Almost sure convergence in $\mathrm{M}_{d,c}(\mathbb{R})$

8.1 A general result

Recall that $\mathcal{H}_{\leq}$ is the subgroup of $\mathrm{SL}(d+c, \mathbb{R})$ defined by
$$\mathcal{H}_{\leq} = \{h \in \mathrm{SL}(d+c, \mathbb{R}) : h = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\}$$

with $A \in \mathrm{GL}(d, \mathbb{R})$, $B \in \mathrm{M}_{c,d}(\mathbb{R})$ and $C \in \mathrm{GL}(c, \mathbb{R})$. We say that a function $f : \mathcal{L}_{d+c} \to \mathbb{R}$ is *uniformly continuous in the $\mathcal{H}_{\leq}$-direction* if for all ε there exists $\beta > 0$ such that for all $\Lambda \in \mathcal{L}_{d+c}$ and all $h \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \beta)$, $|f(h\Lambda) - f(\Lambda)| \leq \varepsilon$.

Theorem 24. *1. Let $\varphi : S \to \mathbb{R}$ be a function continuous almost everywhere on S. Suppose there exists a non negative function $f : \mathcal{L}_{d+c} \to \mathbb{R}_{\geq 0}$ that is continuous, uniformly continuous in the $\mathcal{H}_{\leq}$-direction, integrable and such that $|\varphi| \leq f$ on S. Then,*

$$\int_S f d\mu_S < +\infty$$

and for almost all θ in $\mathrm{M}_{d,c}(\mathbb{R})$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) = \frac{1}{\mu_S(S)} \int_S \varphi d\mu_S.$$

2. The same result holds for S' instead of S

We can formulate Theorem 24 for a general surface S. The assumptions about S are:

- S is a co–dimension one submanifold transverse to the flow,
- the number of visiting times in a time interval of length 1 is bounded above by a universal constant A (Lemma 18),
- Lemma 25 below holds for S.

The other assumptions and the conclusion are the same as in Theorem 24.

For a compact subset K of the submanifold S and $\delta > 0$, let denote

$$U(K, \delta) = \{g_t h\Lambda : t \in [0, 1],\ h \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \delta),\ \Lambda \in S \setminus K\}.$$

Lemma 25. *For all $\varepsilon > 0$, there exists a compact subset K in S and $\delta > 0$ such that $\mu(U(K, \delta)) \leq \varepsilon$.*

This Lemma also holds for S' and is proven below only for S. This is the key Lemma because it explains that the part of S near its "boundary" is not relevant.

Next Proposition is an important step toward Theorem 24. An example shows that without some assumptions about the boundary of S such as Lemma 25, neither Theorem 24 nor Proposition 26 hold.

Proposition 26. *Let $\varphi : S \to \mathbb{R}$ be a bounded continuous function. Then for almost all θ in $\mathrm{M}_{d,c}(\mathbb{R})$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) = \frac{1}{\mu_S(S)} \int_S \varphi d\mu_S.$$

8.1.1 Auxiliary Lemmas

We will need three Lemmas the proofs of which are omitted. The first one only use that S and S' are transverse to the flow together with the inverse mapping Theorem.

Lemma 27. *For all compact subset K in S (or in S'), there exist α and $\eta > 0$ such that*
- the map $(t, \Lambda) \to g_t\Lambda$ is one to one on $[-\alpha, \alpha] \times K$,
- for all $h \in B(I_{d+c}, \eta)$ and all Λ in K, there exists an unique $t = t(h, \Lambda) \in [-\alpha, \alpha]$ such that $g_{-t}h\Lambda \in S$.
- the maps $\sigma : (h, \Lambda) \to t = t(h, \Lambda)$ and $\pi : (h, \Lambda) \to g_{-t}h\Lambda$ are continuous on $B(I_{d+c}, \eta) \times K$ and the values of τ are in $[-\alpha/4, \alpha/4]$.

The second Lemma is a purely theoretical measure result.

Lemma 28. *Let X and Y be locally compact second countable metric spaces. Let μ_X and μ_Y be two measures on X and Y finite on compact subsets. Suppose $\psi : X \to Y$ is a continuous map such that every y in Y has at most N preimages and such that for all x in X there exists a compact neighborhood ω_x of x with the following property:*
- ψ is one to one on ω_x,
- the image by ψ of the measure $1_{\omega_x}\mu_X$ is the measure $1_{\psi(\omega_x)}\mu_Y$.
Then for all nonnegative measurable function $f : Y \to \mathbb{R}$,

$$\int_X f \circ \psi \, d\mu_X \leq N \int_Y f \, d\mu_Y.$$

The last Lemma is an easy consequence of the previous Lemma and of the definition of the induced measure μ_S.

Lemma 29. *Let U be an open subset in $\mathcal{L}_{d+c}$ such that for all Λ in U, $g_t\Lambda \in U$ for all t in a time interval I_Λ of length 1 containing 0. Then*

$$\mu_S(U \cap S) \leq 4A\mu(U)$$

where A is the maximum number of entrance times in S of a flow trajectory during a time interval of length 1 (see Lemma 18).

Remark 5. The constant $4A$ is certainly not the best one.

Remark 6. The assumption U Borel subset should be sufficient.

8.1.2 An example

We want to construct a co-dimension one submanifold V in $\mathcal{L}_{d+c}$ transverse to the flow g_t together with a bounded continuous function $\varphi : V \to \mathbb{R}$ such that for a set of positive measure of $\theta \in M_{d,c}(\mathbb{R})$, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ R_V^k(\Lambda_\theta)$ does not converge to $\frac{1}{\mu_V(V)}\int_V \varphi d\mu_V$ where R_V is the first return map in V and μ_V is the invariant measure induced by the flow. The idea is the following. Take V an open set in S. Then μ_V is the restriction of μ_S to V. Suppose that the open set V can be chosen in order that for all $\theta \in M_{d,c}(\mathbb{R})$, and all $k \geq 1$,

$$R_V^k(\Lambda_\theta) = R_S^k(\Lambda_\theta)(= R^k(\Lambda_\theta)).$$

Then if $\varphi : S \to \mathbb{R}$ is a non negative continuous function not identically zero with support included in V, the sequences

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi\circ R_V^k(\Lambda_\theta),\ \frac{1}{n}\sum_{k=0}^{n-1}\varphi\circ R_S^k(\Lambda_\theta)$$

converge to the same limit which cannot be equal to both $\frac{1}{\mu_S(S)}\int_S \varphi d\mu_S$ and $\frac{1}{\mu_V(V)}\int_V \varphi d\mu_V = \frac{1}{\mu_S(V)}\int_S \varphi d\mu_S$ provided that $\mu_S(V) < \mu_S(S)$. So we are reduced to constructing V.

Observe that Theorem 24 implies that for such a V, $\varphi = 1_V$ is not almost everywhere continuous on S which means that the boundary of V in S has positive measure. Moreover, it shows that the assumption about the continuity of the function φ in Theorem 24, cannot be dropped.

8.1.3 Construction of V

Consider the set $\mathbb{T}$ of lattices Λ_θ such that the coefficients of θ are all in $[0,1]$. It is a compact subset in $\mathcal{L}_{d+c}$ containing all the lattices Λ_θ. Denote W_ε the open ball $B(I_{d+c},\varepsilon)$ in $\mathrm{SL}(d+c,\mathbb{R})$. We consider the open sets

$$U_n(\varepsilon) = \bigcup_{t\in[n,n+1]} g_t(W_\varepsilon\mathbb{T})$$

and for a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers, we consider the open set

$$U = U((\varepsilon_n)_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} U_n(\varepsilon_n).$$

Take $V = S\cap U$. For all $t\geq$ and all θ, $g_t\Lambda_\theta = g_t I_{d+c}\Lambda_\theta$ is in U, hence for all $k\in\mathbb{N}$, we have

$$R_S^k(\Lambda_\theta) = R_V^k(\Lambda_\theta).$$

So we are reduce to show that when the sequence $(\varepsilon_n)_n$ is small enough,

$$\mu_S(V) < \mu_S(S).$$

By definition of $U_n(\varepsilon)$, if $\Lambda = g_t g\Lambda_\theta$ with $t\in[n,n+1]$, $g\in W_\varepsilon$ and $\Lambda_\theta\in\mathbb{T}$, then

$$g_s g_t g\Lambda_\theta \in U_n(\varepsilon)$$

for all s in the interval $[n-t,n+1-t]$. So U satisfies the assumption of the Lemma 29 and therefore

$$\mu_S(U\cap S) \leq 4A\sum_{n\in\mathbb{N}}\mu(U_n(\varepsilon_n)).$$

Using that $g_t g\Lambda = g_n(g_{t-n}gg_{-(t-n)})g_{t-n}\Lambda$, we see that

$$\begin{aligned} U_n(\varepsilon) &= \{g_t\Lambda : t\in[n,n+1],\ \Lambda\in W_\varepsilon\mathbb{T}\} \\ &\subset g_n W_{\varepsilon'}\{g_s\Lambda : s\in[0,1],\ \Lambda\in\mathbb{T}\} \end{aligned}$$

where ε' is such that $g_s W_\varepsilon g_{-s} \subset W_{\varepsilon'}$ for all s in $[0,1]$. Furthermore, the compact set $\{g_s\Lambda : s \in [0,1],\ \Lambda \in \mathbb{T}\}$ has zero measure because it has dimension $cd+1$ which is $< (c+d)^2 - 1$. Therefore

$$\lim_{\varepsilon' \to 0} \mu(W_{\varepsilon'}\{g_s\Lambda : s \in [0,1],\ \Lambda \in \mathbb{T}\}) = 0,$$

which implies

$$\lim_{\varepsilon \to 0} \mu(U_n(\varepsilon)) = 0.$$

So there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n\in\mathbb{N}} \mu(U_n(\varepsilon_n)) < \frac{1}{4A}\mu(S)$$

and for such a sequence, the sets $U = \cup_{n\in\mathbb{N}} U_n(\varepsilon_n)$ and $V = U \cap S$ are the ones we are looking for which ends the construction of a counter-example to Theorem 24 without assumption about the boundary of S.

8.2 Proof of Proposition 26

Let $\varphi : S \to \mathbb{R}$ be a continuous bounded function.

Let ε be a positive real number, let K and δ be associated with ε by Lemma 25, and α and η associated with K by Lemma 27.

Preliminary observations. Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of reals numbers in $]0,\eta[$ tending to zero and set $L_{a_n} = B(I_{d+c}, a_n) \times K$. Since the intersection of all the compact sets L_{a_n}, $n \in \mathbb{N}$, is $L_0 = \{I_{d+c}\} \times K$ and since the map $\psi(g,\Lambda) = \varphi(\pi(g,\Lambda)) - \varphi(\Lambda)$ is continuous, we have

$$\cap_{n\geq 0}\psi(L_{a_n}) = \psi(\cap_{n\geq 0} L_{a_n}) = \psi(L_0) = \{0\}.$$

Therefore, for n large enough $\psi(L_{a_n}) \subset]-\varepsilon,\varepsilon[$ which implies there exists $\beta > 0$ such that for all $\Lambda \in K$ and all $g \in B(I_{d+c},\beta)$,

$$|\varphi(\pi(g,\Lambda)) - \varphi(\Lambda)| \leq \varepsilon. \tag{6}$$

Finally, let $\gamma > 0$ be such that for all $s \geq 0$ and all $h \in B_{\mathcal{H}_\leq}(I_{d+c},\gamma) B^{-1}_{\mathcal{H}_\leq}(I_{d+c},\gamma)$,

$$\mathrm{d}(g_s h g_{-s}, I_{d+c}) \leq \min(\delta,\beta,\eta).$$

For $T \geq 0$, Λ a lattice, and E a subset of S, denote

$$I(T,\Lambda,E) = \{t \in [0,T] : g_t\Lambda \in E\}.$$

For almost all $\theta \in M_{d,c}(\mathbb{R})$, we can fix $h_\theta \in B_{\mathcal{H}_\leq}(I_{d+c},\gamma)$ such that the conclusion of Birkhoff Theorem holds for the flot g_t or the first return map in S and the lattice $h_\theta\Lambda_\theta$. Observe that $h_\theta = h_{\theta,\varepsilon}$ depends on ε. It is understood that we shall use Birkhoff Theorem in countably many situations. We fix a sequence ε_n going to zero and for each $\varepsilon = \varepsilon_n$ and we use three times Birkhoff Theorem and the ergodicity of the flow: for almost all θ,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_{U(K,\delta)}(g_t h_\theta \Lambda_\theta)dt = \frac{1}{\mu(\mathcal{L}_{d+c})}\mu(U(K,\delta)) \leq \varepsilon, \tag{7}$$

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{card} I(T,h_\theta\Lambda_\theta,S)=\frac{1}{\mu_S(S)}\int_S \tau_S d\mu_S=\frac{\mu(\mathcal{L}_{d+c})}{\mu_S(S)},\tag{8}$$

where τ_S is the first return time in S, and

$$\lim_{T\to\infty}\frac{1}{\operatorname{card} I(T,h_\theta\Lambda_\theta,S)}\sum_{t\in I(T,h_\theta\Lambda_\theta,S)}\varphi(g_th_\theta\Lambda_\theta)=\frac{1}{\mu_S(S)}\mu_S(\varphi).\tag{9}$$

Let T be positive and let $s_1<...<s_m$ be the elements of $I(T,\Lambda_\theta,S\setminus K)$, we have

$$g_th_\theta\Lambda_\theta=g_{t-s_i}(g_{s_i}h_\theta g_{-s_i})g_{s_i}\Lambda_\theta\in U(K,\delta)$$

for all $t\in[s_i,s_i+1]$ and we can extract a subsequence $s_{n_1},...,s_{n_p}$ defined by $n_1=1$ and $n_{i+1}=\min\{j:s_j\geq s_{n_i}+1\}$. Now by Lemma 18, there is an absolute constant A such that there are at most A elements of $I(T,\Lambda_\theta,S\setminus K)$ ($\subset I(T,\Lambda_\theta,S)$) in an interval of length 1, hence $Ap\geq m$. Therefore, by (7)

$$\frac{m}{A}\leq p\leq\int_0^{T+1}1_{U(K,\delta)}(g_th_\theta\Lambda_\theta)dt\leq 2(T+1)\varepsilon$$

and hence

$$\operatorname{card} I(T,\Lambda_\theta,S\setminus K)\leq 3AT\varepsilon\tag{10}$$

for T large enough: $T\geq T(\Lambda_\theta,\varepsilon)$. We will also need to bound above the number of elements of $I(T,h_\theta\Lambda_\theta,S\setminus K)$ and making use of (7), the same way of reasoning leads to the same result

$$I(T,h_\theta\Lambda_\theta,S\setminus K)\leq 3AT\varepsilon\tag{11}$$

for $T\geq T(\Lambda_\theta,\varepsilon)$.

Heart of the proof. We want to compare

$$\Sigma_1=\frac{1}{\operatorname{card} I(T,\Lambda_\theta,S)}\sum_{t\in I(T,\Lambda_\theta,S)}\varphi(g_t\Lambda_\theta)$$

with

$$\Sigma_2=\frac{1}{\operatorname{card} I(T,h_\theta\Lambda_\theta,S)}\sum_{t\in I(T,h_\theta\Lambda_\theta,S)}\varphi(g_th_\theta\Lambda_\theta)$$

because by (9), this latter sum tends to $\frac{1}{\mu_S(S)}\int_S\varphi d\mu_S$ when T goes to infinity. We split $\sum_{t\in I(T,\Lambda_\theta,S)}$ in two sums $\sum_{t\in I(T,\Lambda_\theta,K)}$ and $\sum_{t\in I(T,\Lambda_\theta,S\setminus K)}$. Observe that for $t\in I(T,\Lambda_\theta,K)$, $g_th_\theta\Lambda_\theta=(g_th_\theta g_{-t})g_t\Lambda_\theta$ is of the form $g\Lambda$ with $g\in B(I_{d+c},\eta)$ and $\Lambda\in K$, this allows to use Lemma 27. We use the notation of Lemma 27 and for t in $I(T,\Lambda_\theta,K)$, we denote $t'=\sigma(g_th_\theta g_{-t},g_t\Lambda_\theta)$. By (6), we have

$$\left|\sum_{t\in I(T,\Lambda_\theta,K)}\varphi(g_t\Lambda_\theta)-\sum_{t\in I(T,\Lambda_\theta,K)}\varphi(\pi(g_th_\theta g_{-t},g_t\Lambda_\theta))\right|\leq\varepsilon\operatorname{card} I(T,\Lambda_\theta,K).$$

Now,

$$\pi(g_th_\theta g_{-t},g_t\Lambda_\theta)=g_{-t'}g_th_\theta g_{-t}g_t\Lambda_\theta=g_{t-t'}h_\theta\Lambda_\theta$$

hence

$$\left|\sum_{t\in I(T,\Lambda_\theta,K)} \varphi(g_t\Lambda_\theta) - \sum_{t\in I(T,\Lambda_\theta,K)} \varphi(g_{t-t'}h_\theta\Lambda_\theta)\right| \leq \varepsilon \operatorname{card} I(T,\Lambda_\theta,K).$$

Observe that the map $t \in I(T,\Lambda_\theta,K) \to t-t'$ is one to one because $t' \in [-\frac{\alpha}{4}, \frac{\alpha}{4}]$ and the gap between two visiting times of K is $\geq \alpha$. Observe also that $t-t' \in I(T,h_\theta\Lambda_\theta,S)$ except possibly for the first and the last element of $t \in I(T,h_\theta\Lambda_\theta,S)$. On the one hand, it follows that

$$\begin{aligned}
&\left|\sum_{t\in I(T,\Lambda_\theta,S)} \varphi(g_t\Lambda_\theta) - \sum_{t\in I(T,h_\theta\Lambda_\theta,S)} \varphi(g_t h_\theta\Lambda_\theta)\right| \\
&\qquad\qquad \leq \varepsilon \operatorname{card} I(T,\Lambda_\theta,K) \\
&\qquad\qquad + \|\varphi\|_\infty \left(\operatorname{card} I(T,\Lambda_\theta,K\setminus S) + \operatorname{card} I(T,h_\theta\Lambda_\theta,S) - \operatorname{card} I(T,\Lambda_\theta,K) + 2\right).
\end{aligned}$$

On the other hand, it follows that

$$\operatorname{card} I(T,h_\theta\Lambda_\theta,S) \geq \operatorname{card} I(T,\Lambda_\theta,K) - 2$$

and the same way of reasoning leads to

$$\operatorname{card} I(T,\Lambda_\theta,S) \geq \operatorname{card} I(T,h_\theta\Lambda_\theta,K) - 2.$$

Making use of (10) and (11), we obtain

$$\begin{aligned}
-2 \leq \operatorname{card} I(T,h_\theta\Lambda_\theta,S) &- \operatorname{card} I(T,\Lambda_\theta,K) = \\
&\operatorname{card} I(T,h_\theta\Lambda_\theta,S) - \operatorname{card} I(T,h_\theta\Lambda_\theta,K) \\
&+ \operatorname{card} I(T,h_\theta\Lambda_\theta,K) - \operatorname{card} I(T,\Lambda_\theta,S) \\
&+ \operatorname{card} I(T,\Lambda_\theta,S) - \operatorname{card} I(T,\Lambda_\theta,K) \\
&\leq 3AT\varepsilon + 2 + 3AT\varepsilon = 6AT\varepsilon + 2,
\end{aligned}$$

hence (using (10) once again)

$$\left|\sum_{t\in I(T,\Lambda_\theta,S)} \varphi(g_t\Lambda_\theta) - \sum_{t\in I(T,h_\theta\Lambda_\theta,S)} \varphi(g_t h_\theta\Lambda_\theta)\right| \leq \varepsilon \operatorname{card} I(T,\Lambda_\theta,K) + (9A\varepsilon T + 2)\|\varphi\|_\infty$$

for $T \geq T(\Lambda_\theta, \varepsilon)$. We obtain

$$\begin{aligned}
|\operatorname{card} I(T,h_\theta\Lambda_\theta,S) - \operatorname{card} I(T,\Lambda_\theta,S)| &\leq |\operatorname{card} I(T,h_\theta\Lambda_\theta,S) - \operatorname{card} I(T,\Lambda_\theta,K)| \\
&+ |\operatorname{card} I(T,\Lambda_\theta,S\setminus K)| \\
&\leq 9AT\varepsilon + 2
\end{aligned}$$

as well. Relation (8) implies that $\operatorname{card} I(T,h_\theta\Lambda_\theta,S) \geq aT$ for $T \geq T(\Lambda_\theta,\varepsilon)$ where $a = \frac{1}{2}\frac{\mu(\mathcal{L})}{\mu_S(S)}$.

All together, for $T \geq T(\Lambda_\theta, \varepsilon)$, we obtain

$$
\begin{aligned}
|\Sigma_2 - \Sigma_1| &\leq \left| \Sigma_2 - \frac{\operatorname{card} I(T, \Lambda_\theta, S)}{\operatorname{card} I(T, h_\theta \Lambda_\theta, S)} \Sigma_1 \right| + \left| \frac{\operatorname{card} I(T, \Lambda_\theta, S) - \operatorname{card} I(T, h_\theta \Lambda_\theta, S)}{\operatorname{card} I(T, h_\theta \Lambda_\theta, S)} \right| |\Sigma_1| \\
&\leq \frac{1}{\operatorname{card} I(T, h_\theta \Lambda_\theta, S)} \left| \sum_{t \in I(T, h_\theta \Lambda_\theta, S)} \varphi(g_t h_\theta \Lambda_\theta) - \sum_{t \in I(T, \Lambda_\theta, S)} \varphi(g_t \Lambda_\theta) \right| \\
&+ \left| \frac{\operatorname{card} I(T, \Lambda_\theta, S) - \operatorname{card} I(T, h_\theta \Lambda_\theta, S)}{\operatorname{card} I(T, h_\theta \Lambda_\theta, S)} \right| \|\varphi\|_\infty \\
&\leq \frac{\varepsilon \operatorname{card} I(T, \Lambda_\theta, K) + (9AT\varepsilon + 2) \|\varphi\|_\infty}{\operatorname{card} I(T, h_\theta \Lambda_\theta, S)} + \frac{(9AT\varepsilon + 2) \|\varphi\|_\infty}{\operatorname{card} I(T, h_\theta \Lambda_\theta, S)} \\
&\leq \varepsilon (1 + \frac{18AT \|\varphi\|_\infty}{aT}) + \frac{4 \|\varphi\|_\infty}{aT}
\end{aligned}
$$

which is $\ll \varepsilon$ when T is large enough.

8.3 Proof of Theorem 24

Step 1. Let us show that the restriction of f to S is integrable with respect to μ_S.

We use Lemma 28 with $X =]0,1[\times S$, $Y = \mathcal{L}_{d+c}$, the map $\psi :]0,1[\times S \to \mathcal{L}$ defined by $\psi(t, \Lambda) = g_t \Lambda$, the measures $\mu_X = dt \otimes \mu_S$ and $\mu_Y = \mu$, and the function f. By definition of the induced measure, we know that the image of the restriction of $dt \otimes \mu_S$ to any small enough open subset ω is the restriction to $\psi(\omega)$ of the invariant measure μ on $\mathcal{L}_{d+c}$. Now, by Lemma 7 each element of $\mathcal{L}_{d+c}$ has at most $A+1$ ψ-preimages, therefore by Lemma 28

$$
\int_0^1 \int_S f(g_t \Lambda) d\mu_S dt \leq (A+1) \int_{\mathcal{L}_{d+c}} f d\mu.
$$

Since f is uniformly continuous in the $\mathcal{H}_\leq$ direction, there exists $\Delta > 0$ such that for all Λ, and all $t \in [0, \Delta]$, $f(g_t \Lambda) \geq f(\Lambda) - 1$. Therefore

$$
\int_0^\Delta \int_S (f(\Lambda) - 1) d\mu_S dt \leq (A+1) \int_{\mathcal{L}_{d+c}} f d\mu,
$$

which implies $\int_S f(\Lambda) d\mu_S \leq \mu_S(S) + \frac{A+1}{\Delta} \int_{\mathcal{L}_{d+c}} f d\mu < +\infty$.

Step 2: It is enough to prove the Theorem for continuous functions φ.

Indeed, since φ is continuous almost everywhere and since $|\varphi| \leq f$ with f continuous and in $L^1(\mu_S)$, for all positive integer p, there exist two continuous functions φ_p^- and φ_p^+ such that

$$
-f \leq \varphi_p^- \leq \varphi \leq \varphi_p^+ \leq f
$$

and

$$
\int_S \varphi d\mu_S - \frac{1}{p} \leq \int_S \varphi_p^- d\mu_S \leq \int_S \varphi_p^+ d\mu_S \leq \int_S \varphi d\mu_S + \frac{1}{p}.
$$

Therefore, if the convergence holds for almost every θ for all the functions φ_p^- and φ_p^+, we have

$$\begin{aligned}\int_S \varphi_p^- d\mu_S = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi_p^- \circ R^k(\Lambda_\theta) \leq \lim_{n\to\infty} \inf \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta)\\ \leq \lim_{p\to\infty} \sup \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) \leq \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi_p^- \circ R^k(\Lambda_\theta) = \int_S \varphi_p^+ d\mu_S\end{aligned}$$

which implies that for almost all θ,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) = \int_S \varphi d\mu_S.$$

So, we are reduce to prove the Theorem for φ continuous.

Step 3.

Writing $\varphi = \varphi^+ - \varphi^-$, we can suppose $\varphi \geq 0$. Using Proposition 26 with the minimum of φ and of a constant M, we obtain for almost all θ,

$$\lim_{n\to\infty} \inf \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) \geq \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \min(\varphi, M) \circ R^k(\Lambda_\theta) = \frac{1}{\mu_S(S)} \int_S \min(\varphi, M) d\mu_S,$$

hence, letting M going to infinity, we obtain

$$\lim_{n\to\infty} \inf \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) \geq \frac{1}{\mu_S(S)} \int_S \varphi d\mu_S.$$

So we have to bound above the sums $\sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta)$.

Since f is in L^1 there exists $\varepsilon' > 0$ such that for any measurable subset B in $\mathcal{L}_{d+c}$, we have

$$\mu(B) \leq \varepsilon' \Rightarrow \int_B f d\mu \leq \varepsilon.$$

This allows to strengthen Lemma 25:

Lemma 30. *For all $\varepsilon > 0$, there exists a compact subset K in S and $\delta > 0$ such that $\frac{1}{\mu(\mathcal{L}_{d+c})} \int_{U(K,\delta)} f d\mu$ and $\frac{1}{\mu(\mathcal{L}_{d+c})} \mu(U(K,\delta))$ are $\leq \varepsilon$.*

We keep all the choices and the notations of the proof of Proposition 26, and we use Birkhoff Theorem with one more function:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(g_t h_\theta \Lambda_\theta) 1_{U(K,\delta)}(g_t h_\theta \Lambda_\theta) dt = \frac{1}{\mu(\mathcal{L}_{d+c})} \int_{U(K,\delta)} f d\mu \leq \varepsilon \qquad (12)$$

so that (7), (8), (9) and (12) hold for almost all θ.

Since the function f is uniformly continuous in the $\mathcal{H}_{\leq}$-direction, there exists $\kappa > 0$ such that $f(h\Lambda) \geq f(\Lambda) - \frac{1}{2}$ for all Λ and all $h \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \kappa)$. By choosing γ small enough we can suppose that $(g_s h g_{-s}) \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \kappa)$ for all $s \geq 0$ and all $h \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \gamma)$. Furthermore, there exists a positive constant $\Delta = \Delta(\kappa)$ such that $g_t \in B_{\mathcal{H}_{\geq 0}}(I_{d+c}, \kappa)$ for all $t \in [0, \Delta]$. Therefore for all lattices Λ, all non negative real number s, all $h \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \gamma)$ and all $t \in [0, \Delta]$, we have

$$f(g_t(g_s h g_{-s}) g_s \Lambda) \geq f((g_s h g_{-s}) g_s \Lambda) - \frac{1}{2} \geq f(g_s \Lambda) - 1$$

and hence

$$f(g_{t+s} h \Lambda) = f(g_t(g_s h g_{-s}) g_s \Lambda) \geq f(g_s \Lambda) - 1. \tag{13}$$

As in the proof of Proposition 26, let $s_1 < ... < s_m$ be the elements of $I(T, \Lambda_\theta, S \setminus K)$. On the one hand $g_t h_\theta \Lambda_\theta \in U(K, \delta)$ for all $t \in [s_i, s_i + 1]$, and on the other hand, for almost all θ, (10) and (11) hold for $T \geq T(\Lambda_\theta, \varepsilon)$. We can suppose $\Delta < 1$ and since there are at most A elements of $I(T, \Lambda_\theta, S \setminus K)$ ($\subset I(T, \Lambda_\theta, S)$) in an interval of length 1, by (13) we obtain

$$\begin{aligned} \sum_{s \in I(T, \Lambda_\theta, S \setminus K)} \varphi(g_s \Lambda_\theta) &\leq \sum_{s \in I(T, \Lambda_\theta, S \setminus K)} f(g_s \Lambda_\theta) \\ &\leq \sum_{i=1}^{m} \frac{1}{\Delta} \int_{s_i}^{s_i + \Delta} (1 + 1_{U(K,,\delta)}(g_t h_\theta \Lambda_\theta) f(g_t h_\theta \Lambda_\theta)) dt \\ &\leq m + \frac{A}{\Delta} \int_0^{T+1} 1_{U(K,\delta)}(g_t h_\theta \Lambda_\theta) f(g_t h_\theta \Lambda_\theta) dt \end{aligned}$$

and with (10) and (12), this gives

$$\sum_{s \in I(T, \Lambda_\theta, S \setminus K)} f(g_s \Lambda_\theta) \leq 3AT\varepsilon + \frac{A}{\Delta} 3T\varepsilon \leq 6 \frac{A}{\Delta} T\varepsilon$$

for all $T \geq T(\Lambda_\theta, \varepsilon)$.

We want to bound above

$$\Sigma_1(T) = \frac{1}{\operatorname{card} I(T, \Lambda_\theta, S)} \sum_{t \in I(T, \Lambda_\theta, S)} \varphi(g_t \Lambda_\theta)$$

with

$$\Sigma_2(T) = \frac{1}{\operatorname{card} I(T, h_\theta \Lambda_\theta, S)} \sum_{t \in I(T, h_\theta \Lambda_\theta, S)} \varphi(g_t h_\theta \Lambda_\theta)$$

because this last sum tends to $\frac{1}{\mu_S(S)} \int_S \varphi d\mu_S$ when T goes to infinity. We split $\sum_{t \in I(T, \Lambda_\theta, S)}$ in two sums $\sum_{t \in I(T, \Lambda_\theta, K)}$ and $\sum_{t \in I(T, \Lambda_\theta, S \setminus K)}$. As in the previous proof for T large enough, we have

$$|I(T, h_\theta \Lambda_\theta, S) - I(T, \Lambda_\theta, S)| \leq 9AT\varepsilon + 2,$$

$$I(T, h_\theta \Lambda_\theta, S) \geq aT,$$

and

$$\left| \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(g_t \Lambda_\theta) - \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(g_{t-t'} h_\theta \Lambda_\theta) \right| \leq \varepsilon \operatorname{card} I(T, \Lambda_\theta, K)$$

where $t' = \tau(g_t h_\theta g_{-t}, g_t\Lambda_\theta)$ is defined in Lemma 27. Taking into account of the first element t_{min} and of the last element in $I(T, \Lambda_\delta, K)$, the latter inequality implies that

$$\begin{aligned}\sum_{t\in I(T,\Lambda_\theta,K)} \varphi(g_t\Lambda_\theta) &\le \varphi(g_{t_{min}}\Lambda_\theta) + \sum_{t\in I(T,\Lambda_\theta,K)\setminus\{t_{min}\}} \varphi(g_{t-t'}h_\theta\Lambda_\theta) + \varepsilon\,\mathrm{card}\, I(T,\Lambda_\theta,K)\\ &\le \varphi(R\Lambda_\theta) + \sum_{t\in I(T+1,h_\theta\Lambda_\theta,S)} \varphi(g_t h_\theta\Lambda_\theta) + \varepsilon\,\mathrm{card}\, I(T,\Lambda_\theta,K).\end{aligned}$$

All together, we obtain (recall that $\varphi \ge 0$)

$$\begin{aligned}\Sigma_1(T) &\le \frac{1}{T}\varphi(R\Lambda_\theta) + \frac{\mathrm{card}\, I(T+1,h_\theta\Lambda_\theta,S)}{\mathrm{card}\, I(T,\Lambda_\theta,S)}\Sigma_2(T+1) + \varepsilon + \frac{1}{\mathrm{card}\, I(T,\Lambda_\theta,S)}\sum_{s\in I(T,\Lambda_\theta,S\setminus K)} f(g_s\Lambda_\theta)\\ &\le \frac{1}{T}\varphi(R\Lambda_\theta) + \left(1 + \left|\frac{\mathrm{card}\, I(T+1,h_\theta\Lambda_\theta,S) - \mathrm{card}\, I(T,\Lambda_\theta,S)}{\mathrm{card}\, I(T,\Lambda_\theta,S)}\right|\right)\Sigma_2(T+1) + \varepsilon\\ &+ \frac{1}{\mathrm{card}\, I(T,\Lambda_\theta,S)}6\frac{A}{\Delta}T\varepsilon\\ &\le \Sigma_2(T+1) + \frac{1}{T}\varphi(R\Lambda_\theta) + \frac{9AT\varepsilon+2+A}{aT-9AT\varepsilon-2}\Sigma_2(T+1) + \left(1 + \frac{6A}{(aT-9AT\varepsilon-2)\Delta}\right)\varepsilon\end{aligned}$$

and we are done. □

8.4 Proofs of Lemma 25

We need an auxiliary Lemma.

Lemma 31. *Let $E(\lambda,\eta)$ be the set of lattices Λ in $\mathcal{L}_{d,c}$ such that there exist two nonzero vectors $X \ne \pm X'$ of Λ in the open ball $B_{\mathbb{R}^{d+c}}(0,\lambda)$ with $\frac{1}{1+\eta} < \frac{|X|_\pm}{|X'|_\pm} < 1+\eta$ or a nonzero vector X in the open ball $B_{\mathbb{R}^{d+c}}(0,\lambda)$ with $|X|_\pm < \eta$. For all $\lambda > 0$, we have $\lim_{\eta\to 0}\mu(E(\lambda,\eta)) = 0$.*

Proof. Since $\lim_{\rho\to 0}\mu(\{\lambda_1(\Lambda)\le\rho\}) = 0$, it is enough to show that for all $\rho > 0$, $\mu(E(\lambda,\eta)\cap\{\lambda_1(\Lambda)\ge\rho\}) \to 0$ when η goes to 0. Choose a Siegel reduction domain $\mathcal{S} \subset \mathrm{SL}(d+1,\mathbb{R})$. There is a constant $c = c(\mathcal{S}) > 0$ such that for all matrices M in $\mathcal{S}$ and all vectors Y in $\mathbb{R}^{d+c}$, we have

$$\|MY\|_{\mathbb{R}^{d+c}} \ge c\lambda_1(\Lambda)\,\|Y\|_{\mathbb{R}^{d+c}}\,.$$

where $\Lambda = M\mathbb{Z}^{d+c}$ (this inequality holds for all norms with a constant c depending only on the norm, just use the norm equivalence). It follows that we can find a finite subset F_ρ of $\mathbb{Z}^{d+c}$ such that for all matrices M in $\mathcal{S}$ with $\lambda_1(M\mathbb{Z}^{d+c}) \ge \rho$, the only Y in $\mathbb{Z}^{d+c}$ such that $\|MY\|_{\mathbb{R}^{d+c}} \le \lambda$, are in F_ρ. Therefore, if a matrix M in $\mathcal{S}$ is such that $\Lambda = M\mathbb{Z}^{d+c}$ belongs to $E(\lambda,\eta)\cap\{\lambda_1(\Lambda)\ge\rho\}$ then there exist a nonzero Y in F_ρ or two nonzero vectors $Y \ne \pm Y'$ in F_ρ with

$$|MY|_\pm \le \eta$$

or

$$\frac{1}{1+\eta} < \frac{|MY|_\pm}{|MY'|_\pm} < 1+\eta.$$

For a fixed Y or a fixed pair $Y \neq \pm Y'$ of nonzero vectors in F_ρ, the measure of the set of M in $\mathcal{S}$ for which the above inequality holds, goes to 0 as η goes to 0. Since F_ρ is finite and since a Siegel domain contains a fundamental domain we are done. $\square$

Proof of Lemma 25 . Consider the set $V(\lambda, \eta, \rho) = E(\lambda, \eta) \cup \{\lambda_1(\Lambda) < \rho\}$.

Step 1: *The complementary V^C of $V(\lambda, \eta, \rho)$ is a closed subset of $\mathcal{L}_{d,c}$.*
Let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of points of V^C which converge to Γ in $\mathcal{L}_{d,c}$. First, since λ_1 is continuous, $\lambda_1(\Gamma) = \lim \lambda_1(\Lambda_n) \geq \rho$. There is a sequence of matrices $(M_n)_{n\in\mathbb{N}}$ such that $\Lambda_n = M_n\mathbb{Z}^{d+c}$ for all $n \in \mathbb{N}$ and such that $(M_n)_{n\in\mathbb{N}}$ converges to M with $\Gamma = M\mathbb{Z}^{d+c}$. We have to show that Γ is not in $E(\lambda, \eta)$. Let $X = MY$ and $X' = MY'$ be two nonzero vectors in Γ with $X \neq \pm X'$ and $\|X\|_{\mathbb{R}^{d+c}}$, $\|X'\|_{\mathbb{R}^{d+c}} < \lambda$. When n is large enough, $X_n = M_nY$ and $X'_n = M_nY'$ are in the open ball $B(0, \lambda)$ and since $\Lambda_n = M_n\mathbb{Z}^{d+c}$ is not in $E(\lambda, \eta)$ we have both

$$|M_nY|_\pm \geq \eta$$

and

$$\frac{|M_nY|_\pm}{|M_nY'|_\pm} \notin \left]\frac{1}{1+\eta}, 1+\eta\right[$$

and passing through the limit we obtain

$$\frac{|X|_\pm}{|X'|_\pm} = \frac{|MY|_\pm}{|MY'|_\pm} \notin \left]\frac{1}{1+\eta}, 1+\eta\right[$$

and

$$|X|_\pm \geq \eta.$$

Therefore Γ is not in $E(\lambda, \eta)$.

Step 2: *$F = S \setminus V(\lambda, \eta, \rho)$ is a compact subset of S when $\lambda \geq 2\max\{\lambda_1(\Lambda) : \Lambda \in \mathcal{L}_{d,c}\}$.*
Thanks to Malher compactness Theorem it is enough to prove that F is a closed subset of $\mathcal{L}_{d,c}$. Let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of points of F which converges to Γ in $\mathcal{L}_{d,c}$. We want to prove that Γ is in F. By the first step it is enough to prove that Γ is in S. Choose a Siegel domain $\mathcal{S}$. There is a sequence of matrices $M_n \in \mathcal{S}$ such that $\Lambda_n = M_n\mathbb{Z}^{d+c}$ for all $n \in \mathbb{N}$ and such that $(M_n)_{n\in\mathbb{N}}$ converges to $M \in \mathcal{S}$. For each n, there are two vectors $Y_{1,n}$ and $Y_{2,n}$ in $\mathbb{Z}^{d+c}$ such that $X_{1,n} = M_nY_{1,n}$ and $X_{2,n} = M_nY_{2,n}$ are the two vectors associated with Λ_n by the definition of S. Since the matrices M_n are all in the Siegel domain $\mathcal{S}$ and that $\|M_nY_{i,n}\|_{\mathbb{R}^{d+c}} = \lambda_1(\Lambda_n)$, $i = 1, 2$, the sequences $(Y_{i,n})_{n\in\mathbb{N}}$, $i = 1, 2$, are bounded sequence in $\mathbb{Z}^{d+c}$. Therefore extracting subsequences, we can suppose that the two sequences $(Y_{i,n})_{n\in\mathbb{N}}$ are constant: $Y_{i,n} = Y_i$ for all n, $i = 1, 2$. It follows that $\|MY_i\|_{\mathbb{R}^{d+c}} = \lim_{n\to\infty} \|M_nY_i\|_{\mathbb{R}^{d+c}} = \lim_{n\to\infty} \lambda_1(\Lambda_n) = \lambda_1(\Gamma)$. Moreover

$$|MY_1|_+ = \lim_{n\to\infty} |M_nY_1|_+ = \lim_{n\to\infty} \lambda_1(\Lambda_n) = \lambda_1(\Gamma)$$

and

$$|MY_2|_- = \lim_{n\to\infty} |M_nY_2|_- = \lim_{n\to\infty} \lambda_1(\Lambda_n) = \lambda_1(\Gamma).$$

Suppose now that $\lambda \geq 2\max\{\lambda_1(\Lambda) : \Lambda \in \mathcal{L}_{d,c}\}$, then making use of the first step we conclude that Γ is in S.

Step 3. For a neighborhood W of I_{d+c} in $\mathrm{SL}(d+c,\mathbb{R})$, set

$$U_W = \{g_t h\Lambda : t \in [0,1],\ h \in W,\ \Lambda \in V(\lambda,\eta,\rho)\}.$$

Let us show that we can choose W in order that $U_W \subset V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$. It will finish the proof of Lemma 25. Indeed, we first fix $\lambda \geq 2\max\{\lambda_1(\Lambda) : \Lambda \in \mathcal{L}_{d,c}\}$, next we take η and ρ such that $\mu(E(2e^{d+c}\lambda, 5e^{d+c}\eta)) \leq \frac{\varepsilon}{2}$ and $\mu(\{\lambda_1 < 2e^{d+c}\rho\}) \leq \frac{\varepsilon}{2}$, then we take W such that $U_W \subset V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$ and δ such that $B(I_{d+c},\delta) \subset W$. Now by the second step $K = S \setminus V(\lambda,\eta,\rho)$ is compact and since $U(K,\delta) \subset U_W \subset V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$, we have $\mu(U(K,\delta)) \leq \varepsilon$.

Let Λ be in $V(\lambda,\eta,\rho)$, h in W and $t \in [0,1]$. We explain how to successively reduce W in order to obtain the above inclusion.

Case 1. Suppose $\lambda_1(\Lambda) < \rho$. We can choose W small enough in order that $\|hX\|_{\mathbb{R}^{d+c}} \leq 2\|X\|_{\mathbb{R}^{d+c}}$ for all h in W and all X in $\mathbb{R}^{d+c}$. This implies that $\lambda_1(g_t h\Lambda) \leq 2e^{d+c}\lambda_1(\Lambda) < 2e^{d+c}\rho$, hence $g_t h\Lambda \in V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$.

Case 2. Suppose there exist a nonzero vector X in $\Lambda \cap B(0,\lambda)$ with $|X|_- < \eta$ (the case $|X|_+ < \eta$ is easier). Call $p_\pm$ the projections on the subspaces $E_\pm$ and $\|u\|$ the norm of the linear operator u associated with the norm $\|.\|_{\mathbb{R}^{d+c}}$. The vector $g_t hX$ is in the open ball $B(0, 2\lambda e^{d+c})$ and we have

$$p_- g_t hX = g_t p_- h p_- X + g_t p_- h p_+ X,$$

hence

$$|g_t hX|_- < e^{d+c}\|p_- h\|\,\eta + e^{d+c}\|p_- h p_+\|\,\lambda.$$

We can choose W in order that $\|p_- h p_+\| < \frac{\eta}{\lambda}$ and $\|p_- h\| \leq 1$. Then we have $|g_t hX|_- < 3e^{d+c}\eta$ which implies that $g_t h\Lambda \in V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$.

Case 3. Suppose there exists two distinct nonzero vectors X and X' in $\Lambda \cap B(0,\lambda)$ such that

$$|X|_-\,, |X'|_- \geq \eta \ \text{ and } \ \frac{1}{1+\eta} < \frac{|X|_-}{|X'|_-} < 1+\eta.$$

The case with $|.|_+$ is similar.
As above,

$$\begin{aligned} |hX|_- &< \|p_- h\|\,|X|_- + \|p_- h p_+\|\,\lambda \\ &\leq (\|p_- h\| + \|p_- h p_+\|\,\frac{\lambda}{\eta})\,|X|_-\,. \end{aligned}$$

We can choose W in order that $\|p_- h\| + \|p_- h p_+\|\,\frac{\lambda}{\eta} \leq 1+\eta$. We also have

$$|hX'|_- \geq \|p_- h p_- X'\| - \|p_- h p_+\|\,\|X'\|\,.$$

We can choose W in order that $\|p_- h p_+\|\,\frac{\lambda}{\eta} \leq \eta$ and $\|p_- - p_- h p_-\| \leq \eta$. With this choice, we have

$$|hX'|_- \geq |X'|_- - \eta\,|X'|_- - \eta\,|X'|_-\,.$$

It follows that

$$\frac{|g_t hX|_-}{|g_t hX'|_-} = \frac{|hX|_-}{|hX'|_-} \leq \frac{|X|_-}{|X'|_-} \times \frac{1+\eta}{1-2\eta} \leq \frac{(1+\eta)^2}{1-2\eta} \leq 1+5\eta$$

when η is small enough. Inverting the role of X and X' we get the inequality $\frac{|hX'|_-}{|hX|_-} \leq 1 + 5\eta$ and we are done. $\square$

8.5 Proofs of Theorems 1 and 2.1.

We begin by the proof of Theorem 1 which is more difficult. We want to prove that for almost all θ in $\mathrm{M}_{d,c}(\mathbb{R})$,

$$\lim_{n\to\infty} \frac{1}{n} \ln q_n(\theta) = L_{d,c} = \frac{1}{\mu_S(S)} \int_S \rho(\Lambda)\ d\mu_S(\Lambda)$$

and that

$$\lim_{n\to\infty} \frac{-1}{n} \ln r_n(\theta) = \frac{c}{d} L_{d,c}.$$

By Khintchin-Groshev Theorem, the convergence almost everywhere of $\frac{1}{n} \ln q_n(\theta)$ to $L_{d,c} = \frac{1}{\mu_S(S)} \int_S \rho(\Lambda)\ d\mu_S(\Lambda)$, implies the convergence almost everywhere of $\frac{-1}{n} \ln r_n(\theta)$ to $\frac{c}{d} L_{d,c}$. Therefore the proof of Theorem 1 reduces in the first almost everywhere limit.

As soon as Theorem 1 is proven, the formula in the introduction

$$L_{d,c} = \frac{d}{\mu_S(S)} \int_S \tau\ d\mu_S = \frac{d \times \mu(\mathcal{L}_{d+c})}{\mu_S(S)}$$

is a consequence of Proposition 21 and Lemma 17. Indeed, by Proposition 21 and by Lemma 17, $\frac{c}{d} L_{d,c} = L^*_{d,c} = \frac{1}{\mu_S(S)} \int_S \rho^*(\Lambda)\ d\mu_S(\Lambda)$ and $\tau(\Lambda) = \frac{1}{d+c}(\rho(R(\Lambda)) + \rho^*(\Lambda))$ for Λ in $S \setminus \mathcal{N}$, hence

$$\begin{aligned} L_{d,c} &= \frac{d}{d+c} \times \frac{1}{\mu_S(S)} \int_S (\rho(R(\Lambda)) + \rho^*(\Lambda)) d\mu_S(\Lambda) \\ &= \frac{d}{\mu_S(S)} \int_S \tau(\Lambda) d\mu_S(\Lambda) \\ &= \frac{d\ \mu(\mathcal{L}_{d+c})}{\mu_S(S)}. \end{aligned}$$

Let us now prove the first almost everywhere limit. We need two Lemmas. The first one is clear.

Lemma 32. *For all compact set K in $\mathbb{R}^{d+c}$ and all $\varepsilon > 0$ there exists $\alpha > 0$ such that for all $g \in B(I_{d+c}, \alpha)$ and all $x \in K$, $\mathrm{d}(gx, x) \leq \varepsilon$.*

Lemma 33. *Let Λ be in $S' \setminus \mathcal{N}$. Then return map $R = R_S$ is defined on neighborhood of Λ and is continuous at Λ.*

Proof. Consider the minimal vectors $X_0 = X_0(\Lambda)$ and $X_1 = X_1(\Lambda)$. By definition of S' the only nonzero vector $B(0, \lambda_1(\Lambda)) = \mathcal{C}(X_0)$ are $\pm X_0$. Therefore there exists $\varepsilon > 0$ such that all X in $\Lambda \setminus \{0, \pm X_0\}$ are at a distance $\geq \varepsilon$ from $\mathcal{C}(X_0)$. Since Λ is not in $\mathcal{N}$, $\pm X_0$ and $\pm X_1$ are the only nonzero vector of Λ in the cylinder $\mathcal{C}(X_0, X_1)$. Therefore reducing ε if necessary, all

X in $\Lambda \setminus \{0, \pm X_0, \pm X_1\}$ are at a distance $\geq \varepsilon$ from $\mathcal{C}(X_0, X_1)$. By the above Lemma we can choose $\delta > 0$ such that $\forall g \in B(I_{d+c}, \delta),\ \forall X \in \mathcal{C}(X_0, X_1) + B(0,1)$,

$$\max(\mathrm{d}(g^{-1}X, X), \mathrm{d}(gX, X)) \leq \varepsilon/3.$$

It follows that for all $g \in B(I_{d+c}, \delta)$, $\pm gX_0$ are the only nonzero vector of $g\Lambda$ in $\mathcal{C}(gX_0)$ and that $\pm gX_0$ and $\pm gX_1$ are the only nonzero vectors of Λ in $\mathcal{C}(gX_0, gX_1)$. It follows that if the lattice $\Gamma = g\Lambda$ is in the set of lattices $B(I_{d+c}, \delta)\Lambda \cap S'$ then $X_0(\Gamma) = gX_0$ and $X_1(\Gamma) = gX_1$. By definition of S the return times are from Λ and Γ are well defined we have

$$\tau(\Lambda) = \frac{1}{d+1} \ln \frac{|X_1|_-}{|X_0|_+},$$
$$\tau(\Gamma) = \frac{1}{d+1} \ln \frac{|gX_1|_-}{|gX_0|_+},$$

hence $R(\Lambda)$ is defined and

$$\begin{aligned}|\tau(\Lambda) - \tau(\Gamma)| &= \frac{1}{d+1} \left| \ln \frac{|X_1|_-}{|X_0|_+} \frac{|gX_0|_+}{|gX_1|_-} \right| \\ &\leq \frac{1}{d+1} (\left| \ln \frac{|X_1|_-}{|gX_1|_-} \right| + \left| \ln \frac{|gX_0|_+}{|X_0|_+} \right|).\end{aligned}$$

which goes to zero δ goes to zero. □

End of proof of Theorem 1. We use Theorem 24 with S' and the function $\varphi : S' \to \mathbb{R}_{\geq 0}$ defined by $\varphi(\Lambda) = \rho \circ R(\Lambda) = \ln \frac{q_1(\Lambda)}{q_0(\Lambda)}$ when $R(\Lambda)$ is defined and by $\varphi(\Lambda) = 0$ otherwise. Since ρ is continuous on S and R is continuous on $S' \setminus \mathcal{N}$, φ is almost everywhere continuous on S'. We need to find a uniformly continuous function $f : \mathcal{L} \to \mathbb{R}$ such that $|\varphi| \leq f$. Observe that φ is nonnegative. By Minkowski convex body Theorem, for all lattice $\Lambda \in \mathcal{L}$.

$$q_1(\Lambda)^c r_0(\Lambda)^d \leq C = C_{d,c}$$

where $C_{d,c}$ depends only on c and d. It follows that for all Λ in S' we have

$$\begin{aligned}\varphi(\Lambda) &= \ln \frac{q_1(\Lambda)}{q_0(\Lambda)} \\ &= \ln \frac{q_1(\Lambda) r_0(\Lambda)^{d/c}}{q_0(\Lambda) r_0(\Lambda)^{d/c}} \\ &\leq \ln C^{d/c} - \ln q_0(\Lambda) r_0(\Lambda)^{d/c}.\end{aligned}$$

For Λ is in S' we have $q_0(\Lambda) = r_0(\Lambda) = \lambda_1(\Lambda)$. Therefore

$$\varphi(\Lambda) \leq \ln C^{d/c} - \frac{d+c}{c} \ln \lambda_1(\Lambda).$$

It is well known that the function $\ln \lambda_1$ is uniformly continuous and integrable on $\mathcal{L}_{c+d}$, consequently we can use Theorem 24 with S' and φ. It follows that for almost all θ in $\mathrm{M}_{d,c}(\mathbb{R})$ we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R_{S'}^k(\Lambda_\theta) = \frac{1}{\mu_{S'}(S')} \int_{S'} \varphi \, d\mu_{S'}$$

where $R_{S'}$ is the first return map on S'. Now by Lemmas 5 and 15, for almost all θ, there is an integer k_0 such that

$$\varphi \circ R_{S'}^k(\Lambda_\theta) = \ln \frac{q_{k+k_0+1}(\theta)}{q_{k+k_0}(\theta)}$$

for all large enough k, where k_0 depend only on θ. It follows that for almost all θ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R_{S'}^k(\Lambda_\theta) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \frac{q_{k+k_0+1}(\theta)}{q_{k+k_0}(\theta)} = \lim_{n\to\infty} \frac{1}{n} \ln q_n(\theta)$$

So that the only thing left is the equality

$$\int_S \rho \; d\mu_S = \int_{S'} \varphi \; d\mu_{S'}.$$

Now, the image of $\mu_{S'}$ by R is μ_S, hence

$$\begin{aligned} \int_{S'} \varphi \; d\mu_{S'} &= \int_{S'} \rho \circ R \; d\mu_{S'} \\ &= \int_S \rho \; d\mu_S. \end{aligned}$$

□

Proof of Theorem 2.1. Consider the map $F : S \to \mathbb{R}$ defined by

$$F(\Lambda) = q_1^c(\Lambda) r_0^d(\Lambda) = |v_1^S(\Lambda)|_-^c |v_0^S(\Lambda)|_+^d$$

and call $\nu = \nu_{d,c}$ the image of the measure $\frac{1}{\mu_S(S)}\mu_S$ by F. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. We want to prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\beta_k(\theta)) = \int_{\mathbb{R}} \varphi(x) \; d\nu(x)$$

for almost all $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$.

Now by Lemmas 5 and 15, for almost all θ,

$$F(R^k(\Lambda_\theta)) = q_{k+k_0+1}^c(\theta) r_{k+k_0}^d(\theta) = \beta_{k+k_0}(\theta)$$

for all k large enough. Now the function $\varphi \circ F$ is bounded and continuous, thus by Theorem 24 (or Proposition 26) we have for almost all θ,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ F(R^k(\Lambda_\theta)) &= \frac{1}{\mu_S(S)} \int_S \varphi \circ F(\Lambda) \, d\mu_S(\Lambda) \\ &= \int_{\mathbb{R}} \varphi(x) \, d\nu(x) \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\beta_k(\theta)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ F(R^k(\Lambda_\theta)) = \int_{\mathbb{R}} \varphi(x) \; d\nu(x)$$

and finishes the proof of Theorem 2.1. The proof of Theorem 2.2. is postponed at the end of section 9. □

9 On $\liminf q_{n+k}^c r_n^d$

For each θ in $\mathrm{M}_{d,c}(\mathbb{R})$, we consider the sequence of best approximation denominators $(Q_n(\theta))_{n\in\mathbb{N}}$, their norms $q_n = \|Q_n(\theta)\|_{\mathbb{R}^c}$, and the sequence $(r_n)_{n\geq 0}$ defined by

$$r_n = d_{\mathbb{R}^d}(\theta Q_n, \mathbb{Z}^d).$$

For a nonnegative integer k, call Bad_k the subset of $\mathrm{M}_{d,c}(\mathbb{R})$ defined by

$$Bad_k(d,c) = Bad_k = \{\theta \in \mathrm{M}_{d,c}(\mathbb{R}) \setminus \mathrm{M}_{d,c}(\mathbb{Q}) : \inf_{n\in\mathbb{N}} q_{n+k}^c r_n^d > 0\}$$

(if $r_n = 0$ for some integer n, θ is not in Bad_k). The sequence of sets $(Bad_k)_{k\geq 0}$ is clearly nondecreasing and the set Bad_0 is the usual set of badly approximable matrices. When $d = c = 1$, the classical inequality $q_{n+1} r_n \geq \frac{1}{2}$ shows that $Bad_1 = \mathbb{R} \setminus \mathbb{Q}$ while in [9] it has been shown that for $c = 1$ and $d \geq 2$, Bad_1 is negligible. Our first goal is to show that $Bad_1 \setminus Bad_0$ is nonempty for $c = 1$ and $d = 2$. Next we will prove that the set

$$\mathcal{B}(d,c) = \mathcal{B} = \cup_{k\geq 0} Bad_k$$

is negligible and does not depend on the choice of the norm.

Proposition 34. *If $c = 1$ and $d = 2$ then $Bad_1 \setminus Bad_0$ contains uncountably many elements.*

Remark 7. The set $\mathbb{Z}\theta + \mathbb{Z}^2$ is everywhere dense in $\mathbb{R}^2$ for all in $\theta \in Bad_1$. Indeed it is known that the first minimum of the lattice

$$\Lambda_n = \mathbb{Z}^2 + \mathbb{Z}\frac{p_n}{q_n}$$

is $\asymp r_{n-1}$ where p_n is an integer vector such that $r_n = \mathrm{d}(q_n\theta, \mathbb{Z}^2)$. This implies that the second minimum of this lattice is $\lambda_2(\Lambda_n) \asymp \frac{1}{q_n r_{n-1}}$. Now a lower bound $q_n r_{n-1}^2 \geq \alpha > 0$ implies that $\frac{1}{q_n r_{n-1}} \leq \frac{r_{n-1}}{\alpha}$ which goes to zero when $n \to \infty$. The convergence to zero of $\lambda_2(\Lambda_n)$ implies that $\mathbb{Z}\theta + \mathbb{Z}^2$ is everywhere dense in $\mathbb{R}^2$ (see [11] or [8]).

Proof. We assume that $\mathbb{R}^2$ is equipped with the standard Euclidean norm. Set $\theta_0 = (0,0)$ and $\theta_1 = (\frac{1}{5}, \frac{1}{5})$. We construct inductively a sequence $(\theta_n)_{n\geq 0}$ of rational vectors in $\mathbb{R}^2$. For each n in $\mathbb{N}$, let $\Lambda_n = \mathbb{Z}^2 + \theta_n\mathbb{Z}$ be the lattice associated with θ_n. Observe that the least common denominator Q_n of the coordinates of the rational vector θ_n is the inverse of the volume of the lattice Λ_n, $\det \Lambda_n = \frac{1}{Q_n}$ (even for $n = 0$). For $1 \leq i \leq n$, set

$$\begin{aligned} M_{i,n} &= \min\{\mathrm{d}(q\theta_n, \mathbb{Z}^2) - \mathrm{d}(Q_{i-1}\theta_n, \mathbb{Z}^2) : Q_{i-1} < q < Q_i\}, \\ m_{i,n} &= \mathrm{d}(Q_{i-1}\theta_n, \mathbb{Z}^2) - \mathrm{d}(Q_i\theta_n, \mathbb{Z}^2) \end{aligned}$$

The sequence $(\theta_n)_{n\geq 0}$ is constructed such that the following properties hold for all $n \geq 1$:

1. $Q_0 = 1 < Q_1 = 5 < Q_2 < \cdots < Q_n$ are the best approximations (denominators) of θ_n,
2. $Q_n > 2nQ_{n-1}$ and given $\theta_0, \theta_1, \ldots, \theta_{n-1}$, there are at least two possible choices of θ_n leading to two different values of Q_n (to ensure that we construct an uncountable set),

3. for all $1 \leq i \leq n-1$, $M_{i,n} > 0$ (we need to avoid the situation where $\mathrm{d}(q\theta_n, \mathbb{Z}^2) = \mathrm{d}(Q_{i-1}\theta_n, \mathbb{Z}^2)$ for some q between Q_{i-1} and Q_i),
4. $\|\theta_n - \theta_{n-1}\| \leq \frac{1}{8Q_{n-1}} \min\{M_{i,j} : 1 \leq i < j \leq n-1\}$,
5. $\|\theta_n - \theta_{n-1}\| \leq \frac{1}{8Q_{n-1}} \min\{m_{i,j} : 1 \leq i \leq j \leq n-1\}$,
6. $\varepsilon_{n-1} = Q_{n-1}(\theta_n - \theta_{n-1})$ is a shortest vector of Λ_n, i.e. $\lambda_1(\Lambda_n) = \|\varepsilon_{n-1}\|$, and $(-1)^{n-1}\varepsilon_{n-1}$ has positive coordinates,
7. $2\lambda_1(\Lambda_n) \leq \lambda_2(\Lambda_n) \leq 30\lambda_1(\Lambda_n)$.

Observe that, with our choices of θ_0 and θ_1 all these conditions holds for $n = 1$ (the conditions 3 and 4 are empty for $n = 1$).

First, let us show that the above conditions imply that the sequence $(\theta_n)_{n\in\mathbb{N}}$ converges to θ in $Bad_1 \setminus Bad_0$. By 2 and 4, the sequence $\|\theta_{n-1} - \theta_n\|$ converges to 0 at least at a geometric rate, hence the sequence $(\theta_n)_{n\geq 1}$ converge to $\theta \in \mathbb{R}^2$. Furthermore, by 4, for all $n \geq 2$,

$$\begin{aligned}
\|\theta - \theta_n\| &\leq \sum_{p\geq n+1} \|\theta_p - \theta_{p-1}\| \\
&\leq \sum_{p\geq n+1} \frac{1}{8Q_{p-1}} \min\{M_{i,j} : 1 \leq i < j \leq p-1\} \\
&\leq \frac{1}{4Q_n} \min\{M_{i,n} : 1 \leq i < n\}.
\end{aligned}$$

Using 5 instead of 4, we obtain

$$\|\theta - \theta_n\| \leq \frac{1}{4Q_n} \min\{m_{i,n} : 1 \leq i \leq n\}$$

as well. It follows that for all $1 \leq i \leq n-1$ and all $Q_{i-1} < q < Q_i$, we have

$$\begin{aligned}
\mathrm{d}(q\theta, \mathbb{Z}^2) &\geq \mathrm{d}(q\theta_n, \mathbb{Z}^2) - q\,\|\theta - \theta_n\| \\
&\geq \mathrm{d}(Q_{i-1}\theta_n, \mathbb{Z}^2) + M_{i,n} - q\,\|\theta - \theta_n\| \\
&\geq \mathrm{d}(Q_{i-1}\theta, \mathbb{Z}^2) - Q_{i-1}\,\|\theta - \theta_n\| + M_{i,n} - q\,\|\theta - \theta_n\| \\
&\geq \mathrm{d}(Q_{i-1}\theta, \mathbb{Z}^2) + M_{i,n} - 2Q_i\,\|\theta - \theta_n\|.
\end{aligned}$$

Since $\|\theta - \theta_n\| \leq \frac{1}{4Q_n} M_{i,n}$, $\mathrm{d}(q\theta, \mathbb{Z}^2) > \mathrm{d}(Q_{i-1}\theta, \mathbb{Z}^2)$. For all $1 \leq i \leq n-1$, we also have

$$\begin{aligned}
\mathrm{d}(Q_i\theta, \mathbb{Z}^2) &\leq \mathrm{d}(Q_i\theta_n, \mathbb{Z}^2) + Q_i\,\|\theta - \theta_n\| \\
&\leq \mathrm{d}(Q_{i-1}\theta_n, \mathbb{Z}^2) - m_{i,n} + \frac{Q_i}{4Q_n} m_{i,n} \\
&\leq \mathrm{d}(Q_{i-1}\theta, \mathbb{Z}^2) + Q_{i-1}\,\|\theta - \theta_n\| - m_{i,n} + \frac{Q_i}{4Q_n} m_{i,n} \\
&\leq \mathrm{d}(Q_{i-1}\theta, \mathbb{Z}^2) + \frac{Q_{i-1}}{4Q_n} m_{i,n} - m_{i,n} + \frac{Q_i}{4Q_n} m_{i,n} \\
&< \mathrm{d}(Q_{i-1}\theta, \mathbb{Z}^2).
\end{aligned}$$

It follows that $Q_0, Q_1, ..., Q_{n-1}$ are the first n best approximations of θ. Therefore $(Q_n)_{n\geq 0}$ is the sequence of best approximations of θ. The standard inequality (see for instance [11])

$$\lambda_1(\Lambda_n) \asymp \mathrm{d}(Q_{n-1}\theta, \mathbb{Z}^2)$$

together with 7 imply that $\theta \in Bad_0 \setminus Bad_1$.

Let n be integer ≥ 1. Let us explain the construction θ_{n+1} given that $\theta_0, ..., \theta_n$ are already constructed. First choose a primitive point $\alpha_n = k_n\theta_n + (a_n, b_n)$ of Λ_n with $0 \leq k_n < Q_n$ and $(a_n, b_n) \in \mathbb{Z}^2$, in either $\mathbb{R}^2_{>0}$ when n is even or in $\mathbb{R}^2_{<0}$ when n is odd. Just take α_n a point of Λ_n in a square $[x, x+1[\times]0, 1]$ with minimal ordinate when n is even and a point of Λ_n in a square $[x, x+1[\times[-1, 0[$ with maximal ordinate when n is odd. Observe that $\|\alpha_n\|$ can be made arbitrarily large by choosing$|x|$ large enough.

Call $L_n = \|\alpha_n\|$ the length of the segment $[0, \alpha_n]$. The (Euclidean) distance between two consecutive lines of the set $\mathcal{H}_n = \Lambda_n + \mathbb{R}\alpha_n$ is

$$d_n = \frac{\det \Lambda_n}{L_n} = \frac{1}{Q_n L_n}.$$

We can choose α_n such that

$$L_n^2 \geq n \det \Lambda_n,$$

hence $\frac{L_n}{d_n} = \frac{L_n^2}{\det \Lambda_n} \geq n$. There are at least two integers $p_n \geq 2$ such that

$$10\frac{L_n}{d_n} \leq p_n \leq 20\frac{L_n}{d_n}.$$

Suppose p_n is one of these and set

$$\varepsilon_n = \frac{1}{p_n - \frac{k_n}{Q_n}}\alpha_n,$$

$$\theta_{n+1} = \theta_n + \frac{\varepsilon_n}{Q_n},$$

and

$$Q_{n+1} = Q_n p_n - k_n.$$

Since by 1, $Q_0 = 1 < Q_1 = 5 < ... < Q_j$ are the best approximations of θ_j, $j = 1, ..., n$, the real number $\min\{m_{i,j} : 1 \leq i \leq j \leq n\}$ is strictly positive. Moreover,

$$\|\varepsilon_n\| \leq \frac{L_n}{p_n - 1} \leq \frac{L_n}{\frac{p_n}{2}} \leq 2\frac{L_n}{10\frac{L_n}{d_n}} \leq \frac{d_n}{5}$$

and α_n can be chosen in order that d_n is arbitrarily small, hence we can choose α_n such that $\|\varepsilon_n\| < \|\varepsilon_{n-1}\|$ and

$$\|\theta_{n+1} - \theta_n\| = \frac{1}{Q_n}\|\varepsilon_n\| \leq \frac{1}{8Q_n}\min\{m_{i,j} : 1 \leq i \leq j \leq n\}$$

which is condition 5. Next by 3, $\min\{M_{i,j} : 1 \le i < j \le n\}$ is strictly positive. As above, it follows that α_n can be chosen such that

$$\|\theta_{n+1} - \theta_n\| = \frac{1}{Q_n} \|\varepsilon_n\| \le \frac{1}{8Q_n} \min\{M_{i,j} : 1 \le i < j \le n\}$$

which is condition 4. Clearly $Q_{n+1} \ge Q_n(p_n - 1) \ge 2nQ_n$. Notice that the lattice $\Lambda_{n+1} = \mathbb{Z}\theta_{n+1} + \mathbb{Z}^2$ is included in $\mathcal{H}_n$. Next observe that

$$\begin{aligned}
Q_{n+1}\theta_{n+1} &= (Q_n p_n - k_n)(\theta_n + \frac{\varepsilon_n}{Q_n}) \\
&= (Q_n p_n - k_n)(\theta_n + \frac{\alpha_n}{Q_n(p_n - \frac{k_n}{Q_n})}) \\
&= Q_n p_n \theta_n - k_n \theta_n + \alpha_n \\
&= p_n Q_n \theta_n + (a_n, b_n) \in \mathbb{Z}^2.
\end{aligned}$$

It follows that $Q_{n+1} \det \Lambda_{n+1} = l \in \mathbb{N}$. On the other hand, consider the one dimensional lattice $\Lambda_{n+1} \cap \mathbb{R}\alpha_n$. Because $Q_n\theta_n \in \mathbb{Z}^2$ and $Q_n\theta_{n+1} = Q_n\theta_n + \varepsilon_n$, it contains ε_n and is spanned by a vector $v_n = \frac{\varepsilon_n}{m}$ where m is an integer. Next observe that $\theta_n \in \Lambda_{n+1} + \mathbb{R}\alpha_n$, hence $\Lambda_{n+1} + \mathbb{R}\alpha_n = \mathcal{H}_n$. It follows that

$$\begin{aligned}
\frac{l}{Q_{n+1}} &= \det \Lambda_{n+1} = \|v_n\| \, d_n \\
&= \frac{\|\varepsilon_n\|}{m} d_n = \frac{Q_n L_n}{m Q_{n+1}} d_n \\
&= \frac{1}{m Q_{n+1}},
\end{aligned}$$

which implies $m = l = 1$. Therefore $\det \Lambda_{n+1} = \frac{1}{Q_{n+1}}$ and

$$\Lambda_{n+1} = \{0, ..., Q_n - 1\}\theta_{n+1} + \mathbb{Z}\varepsilon_n + \mathbb{Z}^2.$$

Since $\|\varepsilon_n\| \le \frac{d_n}{5}$, and $\Lambda_{n+1} \subset \mathcal{H}_n$, ε_n is the shortest vector of Λ_{n+1}. The choice of the signs for α_n now implies that condition 6 holds. Next

$$\begin{aligned}
&\lambda_1(\Lambda_{n+1}) = \|\varepsilon_n\| \,, \\
&\quad 5 \|\varepsilon_n\| \le d_n \le \lambda_2(\Lambda_{n+1}) \le d_n + \|\varepsilon_n\| \,.
\end{aligned}$$

Since

$$\|\varepsilon_n\| \ge \frac{L_n}{p_n} \ge \frac{L_n}{20\frac{L_n}{d_n}} = \frac{d_n}{20},$$

we obtain

$$5\lambda_1(\Lambda_{n+1}) \le \lambda_2(\Lambda_{n+1}) \le 21 \|\varepsilon_n\| \le 30\lambda_1(\Lambda_{n+1})$$

which contains condition 7. Let us show that $Q_0, ..., Q_{n-1}$ are the first best approximations of θ_{n+1}.

For all $1 \leq i \leq n-1$ and all $Q_{i-1} < q < Q_i$, we have

$$\begin{aligned}
\mathrm{d}(q\theta_{n+1}, \mathbb{Z}^2) &\geq \mathrm{d}(q\theta_n, \mathbb{Z}^2) - q\left\|\theta_{n+1} - \theta_n\right\| \\
&\geq \mathrm{d}(Q_{i-1}\theta_n, \mathbb{Z}^2) + M_{i,n} - q\left\|\theta_{n+1} - \theta_n\right\| \\
&\geq \mathrm{d}(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2) - Q_{i-1}\left\|\theta_{n+1} - \theta_n\right\| + M_{i,n} - q\left\|\theta_{n+1} - \theta_n\right\| \\
&\geq \mathrm{d}(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2) + M_{i,n} - 2Q_i\left\|\theta_{n+1} - \theta_n\right\|.
\end{aligned}$$

Since $\|\theta_{n+1} - \theta_n\| \leq \frac{1}{8Q_n} M_{i,n}$, $\mathrm{d}(q\theta_{n+1}, \mathbb{Z}^2) > \mathrm{d}(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2)$ and hence $M_{i,n+1} > 0$. We also have

$$\begin{aligned}
\mathrm{d}(Q_i\theta_{n+1}, \mathbb{Z}^2) &\leq \mathrm{d}(Q_i\theta_n, \mathbb{Z}^2) + Q_i\left\|\theta_{n+1} - \theta_n\right\| \\
&\leq \mathrm{d}(Q_{i-1}\theta_n, \mathbb{Z}^2) - m_{i,n} + \frac{Q_i}{8Q_n} m_{i,n} \\
&\leq \mathrm{d}(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2) + Q_{i-1}\left\|\theta_{n+1} - \theta_n\right\| - m_{i,n} + \frac{Q_i}{8Q_n} m_{i,n} \\
&\leq \mathrm{d}(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2) + \frac{Q_{i-1}}{8Q_n} m_{i,n} - m_{i,n} + \frac{Q_i}{8Q_n} m_{i,n} \\
&< \mathrm{d}(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2).
\end{aligned}$$

It follows that $Q_0, Q_1, ..., Q_{n-1}$ are the first n best approximations of θ_{n+1}. The proof will be done once we will have explained that Q_n and Q_{n+1} are the only best approximations that follow Q_{n-1} and that $M_{n,n+1} > 0$. These are the places where the sign condition 6 plays a role. First observe that ε_n and $-\varepsilon_n$ are the only two shortest vectors of Λ_{n+1} and that

$$\varepsilon_n = Q_n\theta_{n+1} - Q_n\theta_n$$

and

$$\begin{aligned}
-\varepsilon_n &= (Q_{n+1} - Q_n)\theta_{n+1} - Q_{n+1}\theta_{n+1} + Q_n\theta_n \\
&= (Q_{n+1} - Q_n)\theta_{n+1} + \text{ a vector in } \mathbb{Z}^2.
\end{aligned}$$

Together with the inequality $Q_{n+1} - Q_n > Q_n$ this implies that Q_n is a best approximation of θ_{n+1} and that there is no best approximation of θ_{n+1} between Q_n and Q_{n+1}. Next, denoting by $\equiv$ the equivalence $\bmod\, \mathbb{Z}^2$, we have

$$\begin{aligned}
Q_{n-1}\theta_{n+1} &= Q_{n-1}(\theta_n + \frac{\varepsilon_n}{Q_n}) = Q_{n-1}(\theta_{n-1} + \frac{\varepsilon_{n-1}}{Q_{n-1}} + \frac{\varepsilon_n}{Q_n}) \\
&\equiv \varepsilon_{n-1} + \frac{Q_{n-1}}{Q_n}\varepsilon_n
\end{aligned}$$

and

$$\begin{aligned}
(Q_n - Q_{n-1})\theta_{n+1} &= (Q_n - Q_{n-1})(\theta_n + \frac{\varepsilon_n}{Q_n}) \\
&\equiv -Q_{n-1}\theta_n + (1 - \frac{Q_{n-1}}{Q_n})\varepsilon_n \\
&= -Q_{n-1}(\theta_{n-1} + \frac{\varepsilon_{n-1}}{Q_{n-1}}) + (1 - \frac{Q_{n-1}}{Q_n})\varepsilon_n \\
&\equiv -\varepsilon_{n-1} + (1 - \frac{Q_{n-1}}{Q_n})\varepsilon_n,
\end{aligned}$$

by the choice of the signs we obtain that

$$\mathrm{d}(Q_{n-1}\theta_{n+1}, \mathbb{Z}^2) < \mathrm{d}((Q_n - Q_{n-1})\theta_{n+1}, \mathbb{Z}^2).$$

If q is an integer $\neq Q_n - Q_{n-1}$ lying in $]Q_{n-1}, Q_n[$, then $q\theta_n$ cannot be $\equiv \pm\varepsilon_{n-1}$ which are the shortest vectors of Λ_n. Hence

$$\mathrm{d}(q\theta_n, \mathbb{Z}^2) \geq \min(2\,\|\varepsilon_{n-1}\|\,, \lambda_2(\Lambda_n)) = 2\,\|\varepsilon_{n-1}\|\,.$$

It follows that

$$\begin{aligned} \mathrm{d}(q\theta_{n+1}, \mathbb{Z}^2) &\geq \mathrm{d}(q\theta_n, \mathbb{Z}^2) - q\,\|\theta_{n+1} - \theta_n\| \\ &\geq 2\,\|\varepsilon_{n-1}\| - \frac{q}{Q_n}\,\|\varepsilon_n\| \\ &\geq 2\,\|\varepsilon_{n-1}\| - \|\varepsilon_n\| > \|\varepsilon_{n-1}\| \\ &>= \|\varepsilon_{n-1} + \frac{Q_{n-1}}{Q_n}\varepsilon_n\| = \mathrm{d}(Q_{n-1}\theta_{n+1}, \mathbb{Z}^2) \end{aligned}$$

which implies both that Q_{n-1} and Q_n are consecutive best approximations of θ_{n+1} and that $M_{n,n+1} > 0$. $\square$

It is not clear whether the set Bad_1 depends on the norm. However, using an easy result about the relations between best approximation vectors associated with two norms, we can prove:

Proposition 35. *The set $\mathcal{B}(d, c)$ does not depend on the norms.*

Proof. We give the proof only in the case $c = 1$. When $c > 1$ one has to extend first, the following result about best approximations:

Consider two norms N and N' on $\mathbb{R}^d$. For $\theta \in \mathbb{R}^d$, call $(q_n)_{n\in\mathbb{N}}$ the sequence of best approximation denominators associated with the norm N and $(q'_n)_{n\in\mathbb{N}}$ the sequence associated with the norm N'. Then (see [11]) there exists an integer k depending only on the norms N and N' such that each interval $]q_n, q_{n+k}]$, contains a best approximation denominator q'_m associated with the norm N'.

It is enough to prove that $\mathbb{R}^d \setminus Bad_{kp} \subset \mathbb{R}^d \setminus Bad'_p$ for all p. Let θ be in $\mathbb{R}^d$ and $n \geq k$ be an integer. By the above result, their exists at least one best approximation denominator in each interval $]q_{n+(j-1)k}, q_{n+jk}]$, $j = 0, ..., p$. Let q'_{n_j} be the largest best approximation denominator in each of these intervals $]q_{n+(j-1)k}, q_{n+jk}]$. For each j we have

$$r'_{n_j} \leq C r_{n+jk}$$

where C is the constant involved in the norm equivalence. Making use of the above inequality with $j = 0$, we obtain $q'_{n_0+p} r'^{d}_{n_0} \leq C^d q'_{n_0+p} r^d_n$. Next $q'_{n_0+p} \leq q_{n+kp}$, hence,

$$q'_{n_0+p} r'^{d}_{n_0} \leq C^d q_{n+kp} r^d_n.$$

It follows that $\liminf_{n\to\infty} q_{n+kp} r^d_n = 0$ implies $\liminf_{n\to\infty} q'_{n+p} r'^{d}_n = 0$. $\square$

Theorem 36. *The set $\mathcal{B}(d, c) = \cup_{k\geq 0} Bad_k$ has zero measure.*

By the above Proposition $\mathcal{B}(d,c)$ doesn't depend on the norms and we can suppose that $\mathbb{R}^d$ and $\mathbb{R}^c$ are equipped with the standard Euclidean norms. Let us show that for each k, Bad_k has zero measure. We need two lemmas.

Lemma 37. *1. Let $a<b$ be two integers, let $\Lambda = M\mathbb{Z}^{d+c}$ be a lattice in S and let $(Y_n)_{n=a,\dots,b}$ be a sequence of vectors in $\mathbb{Z}^{d+c}$. Suppose that for $n=a,...,b$,*

- *$X_n(\Lambda) = MY_n$,*
- *the only nonzero points of Λ in the cylinder $C(X_n(\Lambda), X_{n+1}(\Lambda))$ are $\pm X_n(\Lambda)$ and $\pm X_{n+1}(\Lambda)$.*

Then there exists a open neighborhood W of M such that for all lattices $\Lambda' = M'\mathbb{Z}^{d+c} \in \mathcal{L}_{d+c}$ with M' in W, the vectors $Z_n = M'Y_n$ are consecutive minimal vectors of Λ' and

$$|Z_n|_+ \in [\frac{1}{2}r_n(\Lambda), 2r_n(\Lambda)],$$
$$|Z_n|_- \in [\frac{1}{2}q_n(\Lambda), 2q_n(\Lambda)]$$

for $n=a,...,b$.
2. Suppose furthermore that $a<0$, $b>1$ and $\Lambda \in S$. Then for all lattices $\Lambda' = M'\mathbb{Z}^{d+c} \in S$ with M' in W, we have $X_n(\Lambda') = M'Y_n$ for $n=a,...,b$..

Proof. 1. Consider a ball $B_{\mathbb{R}^{d+c}}(0,R)$ that contains all the points MY_n, $n=a,...,b$. There is a neighborhood ω of the identity matrix I_{d+c} such that for all A in ω and all X in $\mathbb{R}^{d+c}$,

$$\frac{1}{2}\|X\|_{\mathbb{R}^{d+c}} \le \|AX\|_{\mathbb{R}^{d+c}} \le 2\|X\|_{\mathbb{R}^{d+c}},$$

so that

$$X \notin B_{\mathbb{R}^{d+c}}(0,8R) \Rightarrow AX \notin B_{\mathbb{R}^{d+c}}(0,4R)$$
$$X \in B_{\mathbb{R}^{d+c}}(0,R) \Rightarrow AX \in B_{\mathbb{R}^{d+c}}(0,2R).$$

Two vectors $Z_n = AMY_n$ and $Z_{n+1} = AMY_{n+1}$ are consecutive minimal vectors of $A\Lambda$ as soon as

$$|Z_{n+1}|_- > |Z_n|_-\,,\ |Z_{n+1}|_+ < |Z_n|_+$$

and the cylinder $C(Z_n, Z_{n+1})$ contains no other nonzero vector of $A\Lambda$ than $\pm Z_n$ and $\pm Z_{n+1}$. Since, $|X_{n+1}|_- > |X_n|_-$, $|X_{n+1}|_+ < |X_n|_+$, by reducing ω, we can assume $|Z_{n+1}|_- > |Z_n|_-$ and $|Z_{n+1}|_+ < |Z_n|_+$, $n=a,\dots,b$. Since $C(Z_n,Z_{n+1}) = C(AX_n, AX_{n+1}) \subset B_{\mathbb{R}^{d+c}}(0,2R)$, the image by A of a vector of Λ that is not in the ball $B_{\mathbb{R}^{d+c}}(0,8R)$, cannot enter in the cylinder $C(AX_n, AX_{n+1})$. Therefore, there are only finitely many X in Λ such that AX is in $C(AX_n, AX_{n+1})$. Since by assumption all these vectors X, except $\pm X_n$ and $\pm X_{n+1}$, are at a positive distance from $C(X_n, X_{n+1})$, we obtain that Z_n and Z_{n+1} are consecutive minimal vectors by reducing once again ω. It follows that $Z_a,...,Z_b$ are consecutive minimal vectors of the lattice $A\Lambda$. A new reduction of ω ensures that the two inequalities of the Lemma hold.

2. We want to see that there is no shift on the indices. By the numbering convention (see section **??**),

$$|X_0(\Lambda)|_+ = |X_1(\Lambda)|_-\,,\ |X_{-1}(\Lambda)|_+ > |X_0(\Lambda)|_-\,,\ |X_1(\Lambda)|_+ < |X_2(\Lambda)|_-\,.$$

By a further reduction of ω, we can assume that the two inequalities hold for the vectors $Z_{-1} = AMY_{-1}$, $Z_0 = AMY_0$, $Z_1 = AMY_1$ and $Z_2 = AMY_2$. Therefore if $AM\mathbb{Z}^{d+c}$ is in S we must have

$$X_0(A\Lambda) = AMY_0 \text{ and } X_1(A\Lambda) = AMY_1$$

which implies that $X_n(A\Lambda) = Z_n$ for $n = a, ..., b$. $\square$

Lemma 38. *Assume that $d + c \geq 3$. Let Γ be a two dimensional lattice in $\mathcal{L}_2 \setminus \mathcal{N}_2$ which is in S_2 and let k be a non negative integer. Then for all positive real number δ, there exists $\varepsilon < 2\delta$ and a lattice Λ_ε in $S \setminus \mathcal{N}$ such that*

$$r_n(\Lambda_\varepsilon) \leq \varepsilon r_n(\Gamma)$$
$$q_n(\Lambda_\varepsilon) \leq \varepsilon q_n(\Gamma)$$

for $n = 0, ..., k$.

Proof. Let $\Gamma = A\mathbb{Z}^2$ be a lattice in $S_2 \setminus \mathcal{N}_2$ where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Consider the matrix $M_\delta \in \mathrm{SL}(d + c, \mathbb{R})$ defined by

$$M_\delta = \begin{pmatrix} \delta a_{11} & 0 & 0 & \dots & 0 & \delta a_{12} \\ 0 & \delta^{-\frac{2}{d+c-2}} & 0 & \dots & \dots & 0 \\ 0 & 0 & \delta^{-\frac{2}{d+c-2}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \delta^{-\frac{2}{d+c-2}} & 0 \\ \delta a_{21} & 0 & \dots & \dots & 0 & \delta a_{22} \end{pmatrix}.$$

Let $(U_n = (u_{n,1}, u_{n,2}))_{n\in\mathbb{Z}}$ be the sequence of vectors in $\mathbb{Z}^2$ such that $(X_n(\Gamma) = AU_n)_{n\in\mathbb{Z}}$ is the sequence of minimal vectors of Γ. For each $n \in \mathbb{Z}$, let Y_n be the element of $\mathbb{Z}^{d+c}$ defined by $y_1 = u_{n,1},\ y_2 = ... = y_{d+c-1} = 0$ and $y_{d+c} = u_{n,2}$. If $\delta > 0$ is small enough, then for all $Z \in \mathbb{Z}^{d+c}$ not in the $\mathbb{R}e_1 + \mathbb{R}e_{d+c}$-plane, we have

$$\|M_\delta Z\|_{\mathbb{R}^{d+c}} = \max(|M_\delta Z|_+, |M_\delta Z|_-) \geq \delta^{-\frac{2}{d+c-2}} \geq \max(2\delta r_{-1}(\Gamma), 2\delta q_{k+1}(\Gamma)).$$

It follows that none of these vectors $M_\delta Z$ are in one of the cylinders $C(M_\delta Y_n, M_\delta Y_{n+1})$, $n = -1, ..., k$. Therefore the vectors $X_n = M_\delta Y_n$, $n = -1, ..., k+1$ are all consecutive minimal vectors of $\Lambda_\delta = M_\delta \mathbb{Z}^{d+1}$ and Λ_δ is in S. With our numbering convention we have $X_n(\Lambda_\delta) = X_n$ for all $n = -1, ..., k+1$. Now we fix δ small enough. By the previous Lemma applied to Λ_δ, there is sequence of matrices $(M_p)_p$ in $S \setminus \mathcal{N}$ which converges to M_δ such for all p,

$$X_n(M_p\mathbb{Z}^{d+1}) = M_p Y_n$$

$n = 0, ..., k$. When p goes to infinity,

$$r_n(M_p\mathbb{Z}^{d+1}) = |M_p Z_n|_+ \to |M_\delta Z_n|_+ = \delta r_n(\Gamma),$$
$$q_n(M_p\mathbb{Z}^{d+1}) = |M_p Z_n|_- \to |M_\delta Z_n|_- = \delta q_n(\Gamma)$$

for $n = 0, ..., k$. So we can take $\Lambda_\varepsilon = M_p\mathbb{Z}^{d+1}$ for some p large enough. $\square$

End of proof of Theorem 36. Let k and $\eta > 0$ be fixed. We want to prove that the set of θ in $\mathrm{M}_{d,c}(\mathbb{R})$ such that
$$\liminf_{n\to\infty} q_{n+k}^c(\theta) r_n^d(\theta) \leq \eta$$
has full measure. By Lemma 5, it is enough to show that
$$\liminf_{n\to\infty} q_{n+k}^c(\Lambda_\theta) r_n^d(\Lambda_\theta) \leq \eta$$
for almost all θ. Fix a two-dimensional lattice Γ in $S_2 \setminus \mathcal{N}_2$ and let δ be a positive real number with $\delta \leq \frac{1}{4}(\frac{\eta}{q_k^c(\Gamma) r_0^d(\Gamma)})^{\frac{1}{d+c}}$. By Lemma 38, there exist $\varepsilon \leq 2\delta$ and a lattice Λ_ε in $S \setminus \mathcal{N}$ such that
$$r_n(\Lambda_\varepsilon) \leq \varepsilon r_n(\Gamma)$$
$$q_n(\Lambda_\varepsilon) \leq \varepsilon q_n(\Gamma)$$
for $n = 0, ..., k$. Hence,
$$q_k^c(\Lambda_\varepsilon) r_0^d(\Lambda_\varepsilon) \leq \frac{\eta}{2^{d+c}}.$$
By Lemma 37, there exists an open neighborhood W of Λ_ε such that for all Λ in W and some integer $m(\Lambda)$, we have both
$$r_{m(\Lambda)}(\Lambda) \leq 2\varepsilon r_0(\Gamma)$$
and
$$q_{m(\Lambda)+k}^c(\Lambda) r_{m(\Lambda)}^d(\Lambda) \leq \eta.$$
Let us show that if for a given lattice Λ, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ going to infinity such that $g_{t_n}\Lambda \in W$ for all $n \in \mathbb{N}$, then
$$\liminf_{n\to\infty} q_{n+k}^c(\Lambda) r_n^d(\Lambda) \leq \eta.$$
Indeed, if $g_{t_n}\Lambda \in W$, then for some integer $m(\Lambda, t_n)$ we have
$$(e^{-dt_n} q_{m(\Lambda,t_n)+k}(\Lambda))^c (e^{ct_n} r_{m(\Lambda,t_n)}(\Lambda))^d \leq \eta.$$
So, the only thing to see is that $m(\Lambda, t_n) \to \infty$ when $n \to \infty$. Now $e^{dt_n} r_{m(\Lambda,t_n)}(\Lambda) \leq 2\varepsilon r_0(\Gamma)$, hence $r_{m(\Lambda,t_n)}(\Lambda)$ goes to zero when m goes to infinity which implies that $m(\Lambda, t_n)$ goes to infinity.

Making use of Birkhoff Theorem with the flow g_t, the proof would be already finished if our goal were $\liminf_{n\to\infty} q_{n+k}^c(\Lambda) r_n^d(\Lambda) \leq \eta$ for almost all lattices. However we want an "almost all" with respect of the Lebesgue measure of $\mathrm{M}_{d,c}(\mathbb{R})$.

Let U be a relatively compact nonempty open set in $\mathcal{L}_{d+c}$ such that $\overline{U} \subset W$. One can find a neighborhood V of I_{d+c} in $\mathcal{H}_{\leq}$ such that for all $\theta \in \mathrm{M}_{d,c}(\mathbb{R})$, all $t \geq 0$ and all $h \in V$, we have
$$g_t h \Lambda_\theta = (g_t h g_{-t}) g_t \Lambda_\theta \in U \Longrightarrow g_t \Lambda_\theta \in W.$$
Call $\mathcal{V}$ the set of θ such that $g_t\Lambda_\theta \notin W$ for all t large enough. By the choices of U and V, for all $h \in V$ and all $\theta \in \mathcal{V}$, $g_t h \Lambda_\theta \notin U$ for all t large enough. If the Lebesgue measure of $\mathcal{V}$ were nonzero then the set of lattices of the form $g_s h \Lambda_\theta$ with $s \in [0, 1]$, $h \in V$ and $\theta \in \mathcal{V}$, would have a nonzero measure. Now, by Birkhoff Theorem, for almost all lattices Λ, there exist a sequence $t_n \to \infty$ such that $g_{t_n}\Lambda \in U$ for all n, therefore $\mathcal{V}$ has zero measure. $\square$

Proof of Theorem 2. 2. By the proof of the first part of Theorem 2, we know that the measure $\nu_{d,c}$ is the image of the measure $\frac{1}{\mu_S(S)}\mu_S$ by the map $F : S \to \mathbb{R}$ defined by

$$F(\Lambda) = q_1^c(\Lambda) r_0^d(\Lambda) = |v_1^S(\Lambda)|_-^c |v_0^S(\Lambda)|_+^d.$$

We want to prove that the support of the measure $\nu_{d,c}$ contains zero, *i.e.*, that $\nu_{d,c}([0,\eta]) > 0$ for all $\eta > 0$. By Birkhoff Theorem and by definition of $\nu_{d,c}$, it is enough to prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{[0,\eta]}(F(R^i(\Lambda))) > 0$$

for almost all $\Lambda \in S$. By Lemmas 37 and 38, there exists a non empty open set W in S such that

$$\left|v_1^S(\Lambda)\right|_-^c \left|v_0^S(\Lambda)\right|_+^d \le \eta$$

for all $\Lambda \in W$. Hence $1_W \le 1_{[0,\eta]} \circ F$. By Birkhoff Theorem, for almost all Λ in S

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1_W \circ R^i(\Lambda) = \frac{1}{\mu_S(S)} \int_S 1_W \, d\mu_S = a > 0.$$

therefore,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{[0,\eta]}(F(R^i(\Lambda))) \ge a > 0.$$

□

10 Miscellaneous Questions

1. In Theorems 1 and 2, we assume that $\mathbb{R}^d$ and $\mathbb{R}^c$ are equipped with the standard Euclidean norms. Do these Theorems hold when $\mathbb{R}^d$ and $\mathbb{R}^c$ are equipped with any norms? If Theorem 1 holds for any norms, does the Levy's constant depend on the norms?

2. Is the measure $\nu_{d,c}$ in Theorem 2, absolutely continuous with respect to Lebesgue measure? Is the support of $\nu_{d,c}$ an interval?

3. Suppose $c = 1$. Consider a flow $(g_t)_{t\in\mathbb{R}}$ defined by the matrices

$$g_t = \mathrm{Diag}(e^{a_1 t}, \ldots, e^{a_d t}, e^{-dt}) \in \mathrm{SL}(d+1, \mathbb{R})$$

where the a_is are positive real numbers with sum d. Best approximation vectors of $\theta \in \mathbb{R}^d$ with respect to the flow $(g_t)_{t\in\mathbb{R}}$ can be defined as follow. A nonzero vector X in $\mathbb{Z}^{d+1}$ is a best approximation vector of θ if there exists $t \ge 0$ such that the interior of the ball $B(g_t M_\theta X) \subset \mathbb{R}^d \times \mathbb{R}$ contains no nonzero vector of the lattice $g_t M_\theta \mathbb{Z}^{d+1}$ (equivalently $\|g_t M_\theta X\|_{\mathbb{R}^{d+1}} = \lambda_1(g_t M_\theta \mathbb{Z}^{d+1})$). Arranging the set of best approximation vector according to their heights, we obtain a sequence $(X_n(\theta))_{n\in\mathbb{N}}$ of best approximation vectors associated with θ. Does Theorem 1 hold for these new sequences of best approximation vectors?

4. For a fixed $k \geq 1$, does the set

$$Bad_k(d,c) = Bad_k = \{\theta \in \mathrm{M}_{d,c}(\mathbb{R}) \setminus \mathrm{M}_{d,c}(\mathbb{Q}) : \inf_{n\in\mathbb{N}} q_{n+k}^c r_n^d > 0\}$$

depends on the norms used to define best approximations vectors?
Observe that by Proposition 35, the union $\cup_{k\geq 1} Bad_k$ does not depend on the choice of the norms.

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