A SURVEY OF BEST SIMULTANEOUS DIOPHANTINE APPROXIMATIONS

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1 Introduction

Given a irrational number θ there exists a unique sequence $a_0 \in \mathbf{Z}$, $a_1 > 0$, $a_2 > 0, \ldots$ of integers such that the sequence of irreducible fractions

$$\frac{p_0}{q_0} = a_0,\ \frac{p_1}{q_1} = a_0 + \frac{1}{a_1},\ \frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \ldots$$

converges to θ. This sequence of fractions, called the continued fraction expansion of θ, enjoys many remarkable properties, this is the reason why, since Jacobi's first extension, many tries have been made to define multidimensional generalizations. Most of these generalizations start with one of the three following properties of the continued fraction expansion.

1. The sequence $(a_n)_{n\geq 0}$ can be easily computed from the iterates of the Gauss map $T :]0,1[\to [0,1]$, $x \to \{\frac{1}{x}\}$.

2. For all $n \in \mathbf{N}$, $\det \begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \pm 1$ (unimodularity property).

3. The set of denominators q_n, $n \geq 0$, is the set of integers $q \geq 1$ such that, $\forall 1 \leq k < q$, $d(kx, \mathbf{Z}) > d(qx, \mathbf{Z})$ (best approximation property).

Property 1 leads to classical multidimensional continued fraction expansions such as Jacobi-Perron's expansion, Brun's expansion, Selmer's expansions....

Poincaré ([Poi]) introduced a geometric viewpoint which enlights the unimodularity property. Many works use this geometric viewpoint and Brentjes defined a multidimensional continued fraction expansion of an element θ in $\mathbf{R}^d$ as a sequence of $\mathbf{Z}^{d+1}$-basis whose positive cone contains the half-line $\mathbf{R}_+(\theta, 1)$. One basis is deduced from the previous one adding to one of the basis vectors a integer multiple of another basis vector (see [Bren]).

Fewer works start with property 3 which leads to best simultaneous Diophantine approximations. Lagarias was the first to study best Diophantine approximations for their own sake. The goal of our paper is to give an overview of the works on best simultaneous Diophantine approximations with a special emphasis on Lagarias multidimensional continued fraction expansion.

The first two kinds of generalization are closely related. On the one hand, the classical continued fraction expansion admit geometric definitions. On the other hand, the generalized Gauss maps are piecewise unimodular Möbius transforms and hence, their iterates define sequences of basis of $\mathbf{Z}^{d+1}$. However, best simultaneous Diophantine approximations cannot be easily related to unimodularity.

In the first part of the paper, after the definition of best Diophantine approximations , we give results explaining the incompatibility with unimodularity. Next, we give some properties of best simultaneous Diophantine approximations that are partial generalization of well known properties of the one-dimensional continued fraction expansion. Then, we describe connections between the best simultaneous approximations of an element θ in $\mathbf{R}^d$ and the geometric properties of the sequence $n\theta \bmod 1$. The last part of the paper is devoted to Lagarias' multidimensional expansion. We adopt a more general presentation than Lagarias original one's.

While, in the first parts, we choose to include a very few proofs, the last part is nearly self contained, all important results leading to Lagarias' expansion are proved.

At last, we must say that there is almost nothing in this paper about best Diophantine approximations to a linear form or to a set of linear forms.

2 Best Diophantine approximations

2.1 Definitions

Let N be a norm on $\mathbf{R}^d$ and denote $d(.,.)$ the distance associated with N.

Définition 1 *Let $\theta \in \mathbf{R}^d$.*
1. A positive integer qis a best simultaneous Diophantine approximation denominator of θ (associated with the norm N) if

$$\forall k \in \{1, ..., q-1\},\ d(q\theta, \mathbf{Z}^d) < d(k\theta, \mathbf{Z}^d).$$

2. An element (P, q) in $\mathbf{Z}^d \times \mathbf{Z}$ is a best Diophantine approximation vector of θ if q is a best simultaneous Diophantine approximation denominator of θ and if

$$N(q\theta - P) = d(q\theta, \mathbf{Z}^d).$$

For short, we will always write best Diophantine approximation instead of best simultaneous Diophantine approximation denominator.

If $\theta \notin \mathbf{Q}^d$, the set of best Diophantine approximations of θ is infinite. Ordering this set, we get a sequence $q_0 = q_0(\theta) = 1 < q_1 = q_0(\theta) < ... < q_n = q_n(\theta) < ....$ When $d = 1$, by the best approximation property, the best Diophantine approximations $q_0, q_1, ..., q_n, ...$ are the denominators of the ordinary continued fraction expansion of θ. The only slight difference is that in the ordinary continued fraction expansion it can happen that $q_0 = q_1 = 1$. In this case, the indices are shifted by one.

The first drawback is that the sequence $(q_n)_{n\geq 0}$ depends on the norm as soon as the dimension is not 1 (see section 2.4 for an inequality between best Diophantine approximations associated with two different norms).

Notation Denote by $r_n = r_n(\theta)$ the distance from $q_n\theta$ to $\mathbf{Z}^d$, and by P_n the point in $\mathbf{Z}^d$ such $d(q_n\theta, P_n) = r_n(\theta)$. With these notations, (P_n, q_n) is a best Diophantine approximation vector. The *remainder vector* $q_n\theta - P_n$ is denoted by ε_n.

To our knowledge, C.A. Rogers in 1951 [Rog] was the first to define best Diophantine approximations associated with the sup norm; he noticed that two consecutive remainder vectors cannot lie in the same quadrant. This initial work on remainder vectors has been continued by V. T. Sós and G. Szekeres [SóSz], and by Moshchevitin [Mosh2].

In “Introduction to Diophantine Approximation” [Cas], Cassels defines the continued fraction expansion of a real number starting with the best approximation property. Then, he derived the unimodularity and constructed the Gauss map using only the best approximation property. As we will see in next section, this program cannot be realized in dimension ≥ 2.

The study of best Diophantine approximations actually began in 1979 with the works of J. C. Lagarias [Lag1,2,3,4,5]. He defined best Diophantine approximations for any norm and studied the unimodularity property, the growth rate of the denominators and their computational complexity. Beside these works he also defined best Diophantine approximations to a set of linear forms. Later in 1994, he proposed a geodesic multidimensional continued fraction expansion.

Negative results about unimodularity are due both to Lagarias and N. Moshchevitin [Mosh3,4] who disproved a conjecture of Lagarias (see also the survey [Mosh1]).

Many authors use implicitly best Diophantine approximations especially through the following lemma which shows that best Diophantine approximations are indeed good approximations. They are at least of the quality of approximations given by Dirichlet's pigeonhole principle. The inequality of the lemma may be seen as an alternative to Dirichlet's theorem.

Lemma 2 *There exists a constant C_N depending only on N such that for all $\theta \in \mathbf{R}^d$, and for all $n \in \mathbf{N}$,*

$$q_{n+1} r_n^d \leq C_N.$$

Proof. We begin with the sup norm $N = N_\infty$. In the d-dimensional torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$, the open balls $B_\infty(k\theta, \frac{r_n}{2})$, $k = 0, ..., q_{n+1} - 1$ are disjoint, therefore, the sum of their volumes is less than 1. Furthermore $r_n \leq \frac{1}{2}$, hence the volume of each of these balls is r_n^d and $q_{n+1} r_n^d \leq 1$.

If N is not the sup norm, there exists a constant C such that $N \leq C N_\infty$. Therefore, $B_\infty(k\theta, \frac{r_n}{2C}) \subset B_N(k\theta, \frac{r_n}{2})$. By definition of best Diophantine approximations, $d(0, \theta + \mathbf{Z}^d) \geq d(0, q_n\theta + \mathbf{Z}^d)$, hence the set $\theta + \mathbf{Z}^d$ does not meet neither the open ball $B_N(0, r_n)$ nor the open ball $B_\infty(0, \frac{r_n}{C})$. Therefore $\frac{r_n}{C} \leq \frac{1}{2}$. As before, $1 \geq q_{n+1} V(B_\infty(0, \frac{r_n}{2C})) = q_{n+1} \frac{r_n^d}{C^d}$. QED

Remark For all integer n sufficiently large, i.e. such that $r_n < d(0, \mathbf{Z}^d \backslash \{0\})$, using Minkowski's convex body theorem, the constant C_N can be chosen depending only on the volume of the unit ball associated with the norm N.

2.2 Unimodularity

Fix a norm N on $\mathbf{R}^d$. For $\theta = (\theta_1, ..., \theta_d) \in \mathbf{R}^d$ and $n \in \mathbf{N}$, set

$$D_n = \begin{pmatrix} p_{n,1} & \cdots & p_{n+d,1} \\ \vdots & \vdots & \vdots \\ p_{n,d} & \cdots & p_{n+d,d} \\ q_n & \cdots & q_{n+d} \end{pmatrix}$$

where the columns of D_n are $d+1$ best consecutive approximation vectors of θ. We would like to know whether $\det D_n = \pm 1$.

In the 2-dimensional case, if $\dim_{\mathbf{Q}}[1, \theta_1, \theta_2] = 3$, there always exist infinitely many integers n such that $\operatorname{rank} D_n = 3$. Indeed, suppose $\det D_n = 0$ for all n large enough. Since two best approximation vectors are never colinear, the subspace spanned by two consecutive best approximation vectors is independent on n for n large. The vector $(\theta, 1)$ is in this subspace F for $(\theta, 1) = \lim_{n\to\infty} \frac{1}{q_n}(p_{n,1}, p_{n,2}, q_n)$. Since F contains two linearly independent integer vectors, $\dim_{\mathbf{Q}}[1, \theta_1, \theta_2] = 2$.

The key argument of the previous way of reasoning is that two best approximation vectors are not colinear. In the 3-dimensional case this argument is not strong enough to prove that if $\det D_n = 0$ for all n large enough, then the space spanned by three consecutive best approximation vectors is independent on n. There is no way to circumvent this problem as shown by the following two results.

Theorem 1 *([Lag 4]). For any norm, there exists $\theta = (\theta_1, ..., \theta_d) \in \mathbf{R}^d$ such that $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] = d+1$ and for all integer N there exists k such that*

$$\det D_k = \det D_{k+1} = ... = \det D_{k+N} = 0.$$

Theorem 2 *([Mosh3,4]). Assume N is the sup norm and $d \geq 3$. There exists an uncountable family of $\theta = (\theta_1, ..., \theta_d)$ in $\mathbf{R}^d$ such that $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] = d+1$ and*

$$\operatorname{rank}(D_n) \leq 3$$

for all n large enough.

Moshchevitin's theorem disprove the following conjecture due to Lagarias.
For all $\theta \in \mathbf{R}^d \backslash \mathbf{Q}^d$ the two properties are equivalent :
- $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] \leq r$,
- there exits an integer $k_0 = k_0(\theta, N)$ such that for all $k \geq k_0$, $\operatorname{rank}(D_k) \leq r$.
Lagarias proved that these two properties are equivalent for $r = 2$.

These two negative results show that best simultaneous Diophantine approximations do not define an unimodular multidimensional continued fraction expansion. It is necessary to add intermediate approximations or to delete some of them.

After these bad news, we continue with positive results.

2.3 Periodic expansions

Let $p \geq 1$ be an integer. The positive solution of the equation $x^2 + px - 1 = 0$ is in the interval $[0, 1[$ and it is readily seen that

$$x = \frac{1}{p+x} = \frac{1}{p + \frac{1}{p+x}} \cdots$$

Hence, $x = [0, p, ..., p, ...]$ and the sequence $(q_n)_{n\geq 0}$ of denominators of the continued fraction expansion of x is such that

$$q_0 = 1,\ q_1 = p,\ q_{n+1} = pq_n + q_{n-1}$$

for all $n \geq 1$. An analogous result holds for best Diophantine approximations in the two-dimensional case.

Theorem 3 *([HM]). Let $P(x) = x^3 + bx^2 + ax - 1$ be an integer polynomial. Suppose that P has a unique real root β and that ($a \geq 0$ and $0 \leq b \leq a + 1$) or ($b = -1$ and $a \geq 2$). Then there exists Euclidean norm on $\mathbf{R}^2$ (Rauzy's norm) such that the sequence of best Diophantine approximations de $\theta = (\beta, \beta^2)$ satisfies*

$$q_0 = 1,\ q_1 = a,\ q_2 = a^2 + 1,\ q_{n+3} = aq_{n+2} + bq_{n+1} + q_n.$$

Lagrange's theorem about ultimately periodic expansions can also be partially extended to best Diophantine approximations in $\mathbf{R}^2$.

Theorem 4 *([Lag 5]). Let $1, \theta_1, \theta_2$ be a $\mathbf{Q}$-basis of non totally real cubic field $K = \mathbf{Q}(\theta_1, \theta_2)$ and $\|.\|$ a given a norm on $\mathbf{R}^2$. Let $P(x) = x^3 - a_2x^2 - a_1x \pm 1$ be the minimal polynomial of the fundamental unit of K. Then the best approximation vectors to $\theta = (\theta_1, \theta_2)$ with respect to $\|.\|$ are a subset of $u_m^{(j)} \in \mathbf{Z}^3$, $m \in \mathbf{N}$, $1 \leq j \leq p$ where the $u_m^{(j)}$ satisfy the third-order vector linear relation*

$$u_{m+3}^{(j)} + a_2 u_{m+2}^{(j)} + a_1 u_{m+1}^{(j)} \pm u_m^{(j)} = 0$$

for a finite set of initial conditions $u_0^{(j)}$, $u_1^{(j)}$, $u_2^{(j)}$, $1 \leq j \leq p$.

For particular θ, it is possible to prove a more precise result.

Theorem 5 *([Chev2]). Let $P(x) = x^3 + bx^2 + ax - 1$ be a integer polynomial. Suppose that P has an unique real root β and that $a, b \geq 0$. Then there exists an Euclidean norm on $\mathbf{R}^2$ and a finite number of best approximation vectors of $\theta = (\beta, \beta^2)$, $X_i = (P_i, q_i)$, $i = 1, ..., m$, such that the set*

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ a & 1 & b \end{pmatrix}^n \begin{pmatrix} P_i \\ q_i \end{pmatrix} : n \in \mathbf{N}, i = 1, ..., m \right\}$$

is included in the set of best approximations of θ and is equal to this set up to a finite number of elements.

2.4 Growth rate of best Diophantine approximations

2.4.1 Lower bound.

Let $\theta \in \mathbf{R}^d \backslash \mathbf{Q}^d$. In the one-dimensional case, it is well known that $q_{n+1} = a_{n+1}q_n + q_{n-1}$, hence $q_{n+1} \geq q_n + q_{n-1} \geq 2q_{n-1}$. It follows that best Diophantine approximations grow at least at the rate of a geometric progression. It is easy to prove that geometric growth rate occurs in all dimensions.

For the sup norm, Lagarias has proved that $q_{n+2^d} \geq q_{n+1} + q_n$. The weaker inequality $q_{n+2^d} \geq 2q_n$ is easy. One of the 2^d "quadrants" of $\mathbf{R}^d$ contains at least two of the remainder vectors $q_{n+k}\theta - P_{n+k}$, $k = 0, ..., 2^d$. The distance between these two vectors $q_{n+k_1}\theta - P_{n+k_1}$ and $q_{n+k_2}\theta - P_{n+k_2}$, is at most r_n, therefore, by definition of the best Diophantine approximations, $q_{n+k_2} - q_{n+k_1} \geq q_n$. $\square$

The following inequality and its nice proof are due to Lagarias.

Theorem 6 *([Lag3]). For any norm on* $\mathbf{R}^d$*, for all* $\theta \in \mathbf{R}^d \backslash \mathbf{Q}^d$*, and for all* $n \in \mathbf{N}$*,*

$$q_{n+2^{d+1}} \geq 2q_{n+1} + q_n.$$

Proof. Assume on the contrary that $q_{n+2^{d+1}} < 2q_{n+1} + q_n$. Among the integers $0, 1, ..., 2^{d+1}$, there are at least two of them, $i < j$, such that

$$(P_{n+i}, q_{n+i}) = (P_{n+j}, q_{n+j}) \mod 2.$$

The vector $(P, q) = \frac{1}{2}(P_{n+j} - P_{n+i}, q_{n+j} - q_{n+i})$ has integer coordinates and

$$0 < q \leq \frac{1}{2}(q_{n+2^{d+1}} - q_n) < q_{n+1}.$$

But,

$$\begin{aligned} N(P - q\theta) &\leq \frac{1}{2}(N(P_{n+j} - q_{n+j}\theta) + N(P_{n+i} - q_{n+i}\theta)) \\ &\leq \frac{1}{2}(r_{n+1} + r_n) < r_n \end{aligned}$$

which contradicts the definition of q_{n+1}. QED

There are similar results about r_n whose proofs are easy:

Proposition 3 *([Chev5], [Lag1]). For the sup norm, for all* $\theta \in \mathbf{R}^d \backslash \mathbf{Q}^d$*, and for all* $n \in \mathbf{N}$*,*

$$r_{n+3^d} \leq \frac{1}{3} r_n.$$

2. For any norm on $\mathbf{R}^d$*, for all* $\theta \in \mathbf{R}^d \backslash \mathbf{Q}^d$*, and for all* $n \in \mathbf{N}$*,*

$$r_{n+3^d} \leq \frac{1}{2} r_n.$$

This proposition allows to compare the growth rate of the sequences of best Diophantine approximations associated with two norms.

Corollary 4 *([Chev5]). Suppose that* $\mathbf{R}^d$ *is endowed with two norms* N *and* N'*. Denote by* $(q_n)_{n\geq 0}$ *and* $(q'_n)_{n\geq 0}$ *the best Diophantine approximations associated with the norms* N *and* N'*. There exists a constant* k *depending only on the norms* N *and* N' *such that for all* $\theta \in \mathbf{R}^d \backslash \mathbf{Q}^d$*, and for all* $n \in \mathbf{N}$*, there exists* $m \in \mathbf{N}$ *such that*

$$q_n \leq q'_m \leq q_{n+k}.$$

2.4.2 Upper bound.

In the following, "almost all" always refers to the Lebesgue measure on $\mathbf{R}^d$.

In the one dimensional case, the following theorem due to Levy shows that almost surely, best Diophantine approximations grow at most at the rate of a geometric progression (independently from Levy, Khinchin proved an inequality strong enough to ensure the same geometric growth rate).

Theorem 7 *For almost all* θ *in* $\mathbf{R}$*,* $\lim_{n\to\infty} \frac{1}{n} \ln q_n = \frac{\pi^2}{12 \ln 2}$.

Theorem 8 *([Chev3,4]). There exists a constant* C_N *depending only on the norm* N *such that, for almost all* $\theta \in \mathbf{R}^d$*,*

$$\limsup_{n\to\infty} \frac{1}{n} \ln q_n \leq C_N.$$

Actually, this result has been proved for the sup norm or the Euclidean norm, but by corollary 2.11, given two norms N_1 and N_2, there exists a constant $C = C(N_1, N_2)$ such that the number of best Diophantine approximations associated with N_1 between two consecutive best Diophantine approximations associated with N_2, is at most C. Hence the geometric growth rates for the norm N_1 and N_2 are equivalent.

In [Chev3] the above Theorem is derived from an asymptotic estimate by W. M. Schmidt [Schm] of the number of solutions of some Diophantine inequalities (actually, a less general result due to Susz is enough). In [Chev1] the result is proved for best Diophantine approximations to a set of linear forms. The proof follows a different way because it seems that there is no appropriate generalization of Schmidt's result to simultaneous approximations to a set of linear forms. As in many works, the proof rests on some ergodic theory and a diagonal action on the homogeneous space $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ ($n = d+$the number of linear forms). Up to a renormalization this diagonal action is the same as the diagonal action used by Lagarias to define his multidimensional expansion (see below).

2.5 Extension of Borel-Bernstein Theorem

Theorem 9 ***Borel-Bernstein***. *Let (α_n) be a sequence of positive integers.*
1. If $\sum_{n\geq 1} \frac{1}{\alpha_n} < +\infty$, then for almost all real number $\theta = [a_0, a_1, ..., a_n, ...]$, there are finitely many integers n such that $a_n \geq \alpha_n$.
2. If $\sum_{n\geq 1} \frac{1}{\alpha_n} = +\infty$, then for almost all real number $\theta = [a_0, a_1, ..., a_n, ...]$, there are infinitely many integers n such that $a_n \geq \alpha_n$.

In order to generalize this Theorem to all dimensions, we need to define the partial quotients associated with the best Diophantine approximations. In the one dimensional case, it is well known that the partial quotients $a_0, ..., a_n, ...$ of a real number can be recovered from the denominators or from the remaindors:

$$a_n = \left[\frac{q_n(x)}{q_{n-1}(x)}\right] = \left[\frac{r_{n-2}(x)}{r_{n-1}(x)}\right]$$

($[x]$ denote the integer part of the real number x). It suggests two definitions of the partial quotients of θ in $\mathbf{R}^d$,

$$a_n(\theta) = \left[\frac{q_n(\theta)}{q_{n-1}(\theta)}\right] \text{ or } b_n(\theta) = \left[\frac{r_{n-2}(\theta)}{r_{n-1}(\theta)}\right]^d$$

(the integer part is not really important). Therefore, we have two natural definitions of partial quotients. The only simple relation (known) between a_n and b_n is:

$$\frac{q_{n+1}}{q_n} \geq \left[\frac{r_{n-1}}{r_n}\right]$$

for the sup norm. However, the coefficients b_n seem to have a stronger geometrical meaning than the coefficients a_n: each b_n is the quotient of the volume of two balls in the torus $\mathbf{T}^d$. Furthermore, Borel-Bernstein theorem can be stated in all dimensions with b_n.

Theorem 10 *([Chev3]). Let (α_n) be a nondecreasing sequence of positive real numbers.*
1. If $\sum_{n\geq 1} \frac{1}{\alpha_n} < +\infty$, then for almost all θ in $\mathbf{R}^d$, there are finitely many integers n such that $\left(\frac{r_{n-1}(\theta)}{r_n(\theta)}\right)^d \geq \alpha_n$.
2. If $\sum_{n\geq 1} \frac{1}{\alpha_n} = +\infty$, then for almost all θ in $\mathbf{R}^d$, there are infinitely many integers n such that $\left(\frac{r_{n-1}(\theta)}{r_n(\theta)}\right)^d \geq \alpha_n$.

Note that in the previous Theorem we assume that the sequence (α_n) is nondecreasing while this assumption is not necessary in Borel-Bernstein's Theorem.

2.6 Badly approximable vectors and singular vectors

Recall that θ in $\mathbf{R}^d$ is badly approximable if

$$\lim_{q\to\infty}\inf q^{\frac{1}{d}} d(q\theta, \mathbf{Z}^d) > 0,$$

and that θ is singular (Khinchine) if

$$\lim_{N\to\infty} N^{1/d} \min\{d(k\theta, \mathbf{Z}^d) : k = 1, ..., N\} = 0.$$

These two concepts are easy to translate in terms of best Diophantine approximations :
- θ is badly approximable if and only if $\liminf_{n\to\infty} q_n r_n^d > 0$.
- θ is singular if and only if $\lim_{n\to\infty} q_{n+1} r_n^d = 0$.

Indeed, if θ is badly approximable, then for all integers n, $r_n \geq c q_n^{-1/d}$, hence $q_n r_n^d \geq c^d$. Conversely, if $\liminf_{n\to\infty} q_n r_n^d > 0$, there exists a positive real number a such that for all integers n, $q_n r_n^d \geq a$. Therefore for all $q_n \leq q < q_{n+1}$, we have $d(q\theta, \mathbf{Z}^d) \geq r_n \geq (\frac{a}{q_n})^{1/d} \geq (\frac{a}{q})^{1/d}$.

In the same way, if θ is singular then for $N = q_{n+1} - 1$,

$$N^{1/d} \min\{d(k\theta, \mathbf{Z}^d) : k = 1, ..., N\} = (q_{n+1} - 1)^{1/d} r_n,$$

hence, $q_{n+1} r_n^d$ goes to 0. Conversely, if $\lim_{n\to\infty} q_{n+1} r_n^d = 0$ then $q_n \leq N < q_{n+1}$,

$$N^{1/d} \min\{d(k\theta, \mathbf{Z}^d) : k = 1, ..., N\} \leq q_{n+1}^{1/d} r_n$$

which goes to 0.

Remark If θ is badly approximable, then the sequences of partial quotients (a_n) and (b_n) are bounded. The converse doesn't hold. To see it, it suffices to take θ with rationally dependent coordinates.

One of the main difference between the one-dimensional case and the higher dimensions is that singular systems do not exist in one dimension, whereas Khinchin proved the existence of singular $\theta = (\theta_1, ..., \theta_d)$ in $\mathbf{R}^d$ with $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] = d + 1$, as soon as the dimension is at least two. This phenomena implies that an inequality of the form $q_{n+1} r_n^d \geq c > 0$ can hold for all θ with $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] = d + 1$, only if $d = 1$. In fact, most θ in $\mathbf{R}^d$, $d \geq 2$, are both "regular" and "singular":

Theorem 11 *1. ([Chev3]) (N =norm sup) if $d \geq 2$, then for almost all θ in $\mathbf{R}^d$,*

$$\lim_{n\to\infty}\inf q_{n+1} r_n^d = 0.$$

2. There exists a constant $c = c(d) > 0$ such that for almost all θ in $\mathbf{R}^d$,

$$\lim_{n\to\infty}\sup q_{n+1} r_n^d \geq c.$$

One can wonder whether there are other ways to extend the one-dimensional inequality $q_{n+1} r_n \geq \frac{1}{2}$ to higher dimensions. Y. Cheung found such an extension (see next subsection) however the geometrical meaning of this extension is not as clear as a lower bound on $q_{n+1} r_n^d$. In the two dimensional case, a positive lower bound on

$$q_{n+1} r_n r_{n-1} \geq c$$

would have a quite clear geometrical meaning. In [Chev4] it is proved that for the sup norm and for all $\theta = (\theta_1, \theta_2)$ in $\mathbf{R}^2$ such that $\dim_{\mathbf{Q}}[1, \theta_1, \theta_2] = 3$,

$$q_{n+1} r_n r_{n-1} \geq \frac{1}{100}$$

for infinitely many integers n. But the set of θ in $\mathbf{R}^2$ such that

$$\lim_{n\to\infty}\inf q_{n+1} r_n r_{n-1} = 0$$

contains a countable intersection of dense open subsets in $\mathbf{R}^2$. In the three dimensional case the situation is even worse, for there exists $\theta = (\theta_1, \theta_2, \theta_3)$ in $\mathbf{R}^d$ with $\dim_{\mathbf{Q}}[1, \theta_1, \theta_2, \theta_3] = 4$ such that

$$\lim_{n\to\infty} q_{n+1} r_n r_{n-1} r_{n-2} = 0$$

(see [Chev3]).

3 Distribution of the sequence $n\theta$ mod 1

3.1 Lattice and subgroup associated to a best Diophantine approximation

Let N be a norm on $\mathbf{R}^d$ and let θ be in $\mathbf{R}^d$. In this subsection, we give a few easy properties that give some geometric informations about the set

$$E_n = \{0, \theta, ..., (q_n - 1)\theta\} + \mathbf{Z}^d.$$

Let (P_n, q_n) be the best approximation vector associated with q_n and let $\varepsilon_n = q_n\theta - P_n$ be the remainder vector. The rational approximation associated with q_n is

$$\theta_n = \frac{1}{q_n} P_n = \theta - \frac{\varepsilon_n}{q_n}.$$

In the torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$, we have $q_n\theta_n = 0$, hence the subgroup $\langle\theta_n\rangle$ generated by θ_n is finite. The lifting of this subgroup in $\mathbf{R}^d$ is the lattice

$$\Lambda_n = \mathbf{Z}\theta_n + \mathbf{Z}^d = \{0, \theta_n, ..., (q_n - 1)\theta_n\} + \mathbf{Z}^d.$$

Since q_n is a best Diophantine approximation of θ, θ_n is a good approximation of θ! Hence the lattice Λ_n is close to the set E_n, and the study of the geometrical properties of Λ_n should enlight those of E_n. It is worth noting that in the one-dimensional case, the situation is crystal clear : there is only one subgroup in $\mathbf{T}^1$ with a given cardinality or equivalently, only one lattice in $\mathbf{R}$ with a given determinant. In higher dimensions, the geometry of a lattice is no longer determined by its determinant, the successive minima are needed to know quantitative informations about the geometry of a lattice. The existence of singular θ in dimension ≥ 2 is strongly related to this observation.

The following properties give the connections between Λ_n and E_n, they are easy to prove (see below) and are often used inside proofs.

(P1). *In the torus $\mathbf{T}^d$, the set $\{0, \theta, ..., (q_n - 1)\theta\}$ and the subgroup $\langle\theta_n\rangle$ generated by θ_n are close :*

$$\forall k \in \{0, ..., q_n - 1\},\ d_{\mathbf{T}^d}(k\theta, k\theta_n) \leq r_n(\theta)$$

(hence, the Hausdorff distance $\{0, \theta, ..., (q_n - 1)\theta\}$ between $\langle\theta_n\rangle$ is smaller than r_n).
Notation Let E be a subset of a metric space. Denote $r(E) = \inf\{d(x, y) : x \neq y \in E\}$.
In $\mathbf{T}^d$, we have $r_{n-1}(\theta) = r(\{0, \theta, ..., (q_n - 1)\theta\})$.

(P2). *The minimal distance between two points of the subgroup $\langle\theta_n\rangle$ is of the same order of size than the distances between two points of the set E_n :*

$$\frac{1}{2} r_{n-1}(\theta) \leq r(\Lambda_n) = r(\langle\theta_n\rangle) = \lambda_1(\Lambda_n) \leq 2r_{n-1}(\theta)$$

($\lambda_1(\Lambda_n)$ is the first minimum of Λ_n).

(P3). $\forall k \in \{1, ..., q_n - 1\}$, $k\theta_n \neq 0$, *hence the cardinality of the subgroup $\langle\theta_n\rangle$ is q_n.*

(P4). $\det \Lambda_n = \frac{1}{q_n}$.

(P5). Yitwah Cheung [Cheu]. *Let $\Delta_n = (\delta_{n,1}, ..., \delta_{n,d})$ be the vector of $\mathbf{R}^d$ whose coordinates are the determinants $\delta_{n,i} = \det \begin{pmatrix} p_{n-1,i} & p_{n,i} \\ q_{n-1} & q_n \end{pmatrix}$ $i = 1, ..., d$. Then*

$$\frac{1}{2} N(\Delta_n) \leq q_n r_{n-1} \leq 2N(\Delta_n).$$

Proof. 1. In the torus $\mathbf{T}^d$, for $k \leq q_n$,

$$d_{\mathbf{T}^d}(k\theta_n, k\theta) = d_{\mathbf{R}^d}(k(\theta + \frac{\varepsilon_n}{q_n}), k\theta + \mathbf{Z}^d) \leq \frac{k}{q_n} N(\varepsilon_n) \leq N(\varepsilon_n) = r_n.$$

2. Let $k \in \{1, ..., q_n - 1\}$ such that $d_{\mathbf{T}^d}(k\theta_n, 0) = r(\langle\theta_n\rangle)$. Since in the torus $q_n\theta_n = 0$, $d_{\mathbf{T}^d}((q_n - k)\theta_n, 0) = d_{\mathbf{T}^d}(k\theta_n, 0) = r(\langle\theta_n\rangle)$. Hence we can assume that $k \leq \frac{q_n}{2}$, therefore

$$\begin{aligned}
d_{\mathbf{T}^d}(k\theta_n, 0) = r(\langle\theta_n\rangle) &= d_{\mathbf{T}^d}(k(\theta - \frac{\varepsilon_n}{q_n}), 0) = d_{\mathbf{R}^d}(k(\theta - \frac{\varepsilon_n}{q_n}), \mathbf{Z}^d) \\
&\geq d_{\mathbf{R}^d}(k\theta, \mathbf{Z}^d) - d_{\mathbf{R}^d}(k\theta, k\theta - k\frac{\varepsilon_n}{q_n}) \\
&\geq r_{n-1} - \frac{k}{q_n}N(\varepsilon_n) \geq r_{n-1} - \frac{r_n}{2} \geq \frac{r_{n-1}}{2}.
\end{aligned}$$

We also have $r(\langle\theta_n\rangle) \leq d_{\mathbf{T}^d}(q_{n-1}\theta_n, 0) \leq N(q_{n-1}\theta_n - P_{n-1}) \leq N(q_{n-1}(\theta_n - \theta)) + N(q_{n-1}\theta - P_{n-1}) \leq r_n + r_{n-1} \leq 2r_{n-1}$.
3. If $k \in \{1, ..., q_n - 1\}$, by the previous computation we have $d_{\mathbf{T}^d}(k\theta_n, 0) \geq \frac{r_{n-1}}{2} > 0$, therefore, $k\theta_n \neq 0$.
4. $\det \Lambda_n = \frac{1}{q_n}$ for $\operatorname{card} \Lambda_n/\mathbf{Z}^d = \operatorname{card}\langle\theta_n\rangle = q_n$.
5. We have

$$r(\langle\theta_n\rangle) \leq N(q_{n-1}\theta_n - P_{n-1}) = N(q_{n-1}\frac{P_n}{q_n} - P_{n-1}) = \frac{1}{q_n}N(\Delta_n)$$

hence by **P2**, $q_n r_{n-1} \leq 2N(\Delta_n)$.
Moreover

$$\begin{aligned}
N(\Delta_n) &= N(q_n P_{n-1} - q_{n-1}P_n) \\
&= q_n q_{n-1} N(\theta_{n-1} - \theta_n) \\
&\leq q_n q_{n-1}(N(\theta_{n-1} - \theta) + N(\theta_n - \theta)) \\
&= q_n q_{n-1}(\frac{r_{n-1}}{q_{n-1}} + \frac{r_n}{q_n}) \leq 2q_n r_{n-1}.
\end{aligned}$$

QED

Remark Property **P5** extends the one dimensional inequality $q_n r_{n-1} \geq \frac{1}{2}$, since in that case, $\Delta_n = \pm 1$. Actually, Cheung gives the slightly better lower bound $(q_n + q_{n-1})r_{n-1} \geq \Delta_n$ which is easily deduced from the above proof.

3.2 Dual Lattice

All the result of this subsection can be found in [Chev5]. We assume that $\mathbf{R}^d$ is endowed with Euclidean norm $\|.\|$.

In the previous subsection we have seen that the lattices Λ_n associated with the sequence of best approximations (q_n) of an element θ in $\mathbf{R}^d$ are well suited to study the sets

$$E_n = \{0, \theta, ..., (q_n - 1)\theta\} + \mathbf{Z}^d.$$

If want to study the transition between the sets E_n and E_{n+1}, i.e the sets

$$\{0, \theta, ..., q\theta\} + \mathbf{Z}^d$$

with $q_n \leq q < q_{n+1}$, the dual lattice

$$\Lambda_n^* = \{x \in \mathbf{R}^d : \forall y \in \Lambda_n,\ x.y \in \mathbf{Z}\}$$

(the dot denote the scalar product) provides some important geometric informations. Let x_n^* be the shortest vector of Λ_n. The net of hyperplans $\mathcal{H}_n = \{x \in \mathbf{R}^d : x.x_n \in \mathbf{Z}\}$ is the best possible net of hyperplans that contains Λ_n. By the property **P1** above, the set E_n is close to the net of hyperplans $\mathcal{H}_n$. Note that the distance d_n between two consecutive hyperplan of $\mathcal{H}_n$ is $\frac{1}{\|x_n^*\|}$ and by Minkowski's theorem on minima of a lattice,

$$\|x_n^*\|^d \leq \lambda_1(\Lambda_n^*)...\lambda_1(\Lambda_n^*)^d \ll \det \Lambda_n^* = \frac{1}{q_n},$$

hence

$$\|x_n^*\| \ll q_n^{1/d},\ d_n \gg \frac{1}{q_n^{1/d}}.$$

The property **P1** shows that the closeness of E_n to $\mathcal{H}_n$ compared to d_n is bounded above by $q_n^{1/d} r_n(\theta)$ (up to a multiplicative constant).

The vector x_n^* allows to regroup best approximations : either $x_n^* \in \Lambda_{n+1}^*$, or $x_n^* \notin \Lambda_{n+1}^*$. In the two cases described below, we assume that $q_n r_n^d(\theta)$ is small.

Case 1: $x_n^* \in \Lambda_{n+1}^*$. E_{n+1} is still close to $\mathcal{H}_n$, hence all the points $q\theta$, $q < q_{n+1}$ are close to $\mathcal{H}_n$. More precisely, for all $q < q_{n+1}$,

$$d(q\theta.x_n^*, \mathbf{Z}) \ll q_n^{1/d} r_{n+1}.$$

Case 2: $x_n^* \notin \Lambda_{n+1}^*$. At the time $q = q_{n-1}$ the trajectory $\{0, \theta, ..., q\theta\} + \mathbf{Z}^d$ is close to $\mathcal{H}_n$. It can be proved that when q increases from q_n to q_{n+1}, the points $q\theta + \mathbf{Z}^d$ fill the gap between the hyperplans of $\mathcal{H}_n$. This gap between the hyperplans is filled in a very simple way : the trajectory move away from E_n by small jump. Indeed, for all $a < q_n$, and $k = 0, ..., [\frac{q_{n+1}}{q_n}]$,

$$\begin{aligned}(kq_n + a)\theta &= a\theta + kq_n(\theta - \theta_n) \\ &\equiv a\theta + k\varepsilon_n \bmod \mathbf{Z}^d\end{aligned}$$

where $\varepsilon_n = q_n\theta - P_n$ and $\|\varepsilon_n\| = r_n(\theta)$.

We have seen in the subsection about singular systems that the one-dimensional inequality $q_{n+1} r_n \geq \frac{1}{2}$ is difficult to extend to higher dimension. But using the shortest vector x_n^* enable to prove a partial extension of this inequality.

Theorem 12 *There exists a positive constant $c(d)$ such that for all $\theta \in \mathbf{R}^d$ such that* $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] = d + 1$*, then either $q_n r_n^d(\theta) \geq c(d)$ or $q_{n+1} d(x_n^*.\theta, \mathbf{Z}) \geq c(d)$ for infinitely many n.*

4 Lattices in $\mathbf{R}^{d+1}$ associated with θ in $\mathbf{R}^d$

Endow $\mathbf{R}^d$ with a norm N_d and $\mathbf{R}^{d+1}$ with a norm N_{d+1}. For each θ in $\mathbf{R}^d$, we consider the family of lattices $(\Lambda_s(\theta))_{s>0}$ defined by

$$\Lambda_s(\theta) = M_s(\theta)\mathbf{Z}^{d+1}$$

where

$$M_s(\theta) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\theta_1 \\ 0 & 1 & \cdots & 0 & -\theta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & 1 & -\theta_d \\ 0 & \cdots & \cdots & 0 & s \end{pmatrix} \in GL(d+1, \mathbf{R})$$

Since for $P \in \mathbf{Z}^d$ and $q \in \mathbf{Z}$, $M_s(\theta)\begin{pmatrix} P \\ q \end{pmatrix} = \begin{pmatrix} P - q\theta \\ sq \end{pmatrix}$, a short vector of $\Lambda_s(\theta)$ with $q \neq 0$ provides a good rational approximation of θ.

Remark. Generally a renormalized version of $M_s(\theta)$ is considered : $g_t M_1(\theta)$ where g_t is the diagonal matrix

$$g_t = \begin{pmatrix} e^t & 0 & \cdots & 0 & 0 \\ 0 & e^t & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & e^t & 0 \\ 0 & \cdots & \cdots & 0 & e^{-dt} \end{pmatrix}$$

and $s = e^{-(d+1)t}$. The advantage of this renormalized version is that the lattices $g_t M_1(\theta)\mathbf{Z}^d$ are unimodular, and the space of unimodular lattices $SL(d+1, \mathbf{R})/SL(d+1, \mathbf{Z})$ has finite Haar measure which allows to use tools from ergodic theory. In what follows, we shall keep the non normalized version because computations are slightly simpler. Furthermore, the map $s > 0 \to M_s(\theta)^t\, M_s(\theta)$, can be interpreted as a geodesic in the space of positive definite quadratic forms (see [Lag1]).

4.1 Euclidean norm

Next lemma connects best approximation to the lattices Λ_s, and was first stated in [Lag 1].

Lemma 5 *(Lagarias) Assume that $N_{d+1} = \|.\|_{\mathbf{R}^{d+1}}$ and $N_d = \|.\|_{\mathbf{R}^d}$ are the usual Euclidean norms. If $v_s = (P_s - q_s\theta, sq_s)$ is a shortest vector of Λ_s and if $q_s > 0$, then q_s is a best Diophantine approximation of θ.*

Proof. If q is an integer between 1 and $q_s - 1$ then for all $K \in \mathbf{Z}^d$,

$$\|(K\theta - q, sq)\|^2_{\mathbf{R}^{d+1}} \geq \|v_s\|^2_{\mathbf{R}^{d+1}}$$

hence

$$\|K - q\theta\|^2_{\mathbf{R}^d} + q^2 s^2 \geq \|P_s - q_s\theta\|^2_{\mathbf{R}^d} + q_s^2 s^2$$

and therefore

$$\|K - q\theta\|^2_{\mathbf{R}^d} > \|P_s - q_s\theta\|^2_{\mathbf{R}^d}.$$

QED

With the hypothesis of the previous lemma, it is easy to prove that if $s > t > 0$ then $q_s \leq q_t$. This lemma is the main observation leading to a weak form of Lagarias multidimensional expansion :

For each $s > 0$, compute the shortest vector (P_s, q_s) of $\Lambda_s(\theta)$. The set of these q_s is a subsequence of the sequence of all best Diophantine approximations of θ.

These best Diophantine approximations are called *Hermite* best Diophantine approximations ([Lag1]). We denote by $(h_m)_{m\geq 0}$ the increasing sequence of Hermite best Diophantine approximations and by H_m the corresponding best Diophantine approximation vector. The subsequence $(h_m)_{m\geq 0}$ is generally a strict subsequence of the sequence of all best Diophantine approximations $(q_n)_{n\geq 0}$ even for $d = 1$ (Humbert). However, the following properties proved in [Chev1] show that the sequence of Hermite best Diophantine approximations is not a too sparse subsequence of $(q_n)_{n\geq 0}$.

1. *There exists a constant m_0 depending only on the dimension d such that for all θ in $\mathbf{R}^d$ and all integers m, the cardinality of the set of n with*

$$h_m < q_n < h_{m+1}$$

is at most m_0.

2. *There exists a constant C depending only on the dimension d such that for θ in $\mathbf{R}^d$ and all integers m,*

$$h_{m+1} d(h_m\theta, \mathbf{Z}^d)^d \leq C.$$

3. *There exists a positive constant c depending only on the dimension d such that for θ in $\mathbf{R}^d$ and all best Diophantine approximation q_n of θ, there exists an Hermite best Diophantine approximation h_m such that*

$$c\, d(h_m\theta, \mathbf{Z}^d) \leq d(q_n\theta, \mathbf{Z}^d) \leq d(h_m\theta, \mathbf{Z}^d).$$

These three properties can be deduced from a lemma due to Cheung (see next subsection) together with inequalities that connect best Diophantine approximations associated with two different norms.

If $d = 1$, Hermite proved that for all integers n

$$\mathrm{rank}\,(H_n, H_{n+1}) = 2$$

(two best approximation vectors are never colinear!).

If $d = 2$, it can happen that

$$\mathrm{rank}(H_n, H_{n+1}, H_{n+2}) < 3$$

([Lag1]). Hence, the unimodulary property does not hold for the sequence $(H_n)_{n\geq 0}$.

4.2 Cheung's norms.

Let N be a norm on $\mathbf{R}^d$. Y. Cheung ([Cheu]) considers the norm N' on $\mathbf{R}^{d+1}$ defined by

$$N'(X,x) = \max(N(X), |x|).$$

These norms appear to be better suited than Euclidean norms to link shortest vectors of $\Lambda_s(\theta)$ to best Diophantine approximations. The aim of Y. Cheung was to prove that the Hausdorff dimension of the set of singular couples (θ_1, θ_2) in $\mathbf{R}^2$ is $\frac{4}{3}$. Best Diophantine approximations are one of important the ingredients of his proof. The following result is essentially contained in [Cheu] and its proof is easy.

Lemma 6 *Assume that $\mathbf{R}^d$ is endowed with a norm N and $\mathbf{R}^{d+1}$ with the norm N'. Set $\delta_N = \min\{N(X) : X \in \mathbf{Z}^d\backslash\{0\}\}$. Let $V = (P, q)$ be in $\mathbf{Z}^{d+1}$ with $q > 0$.*
1. If V is a best Diophantine approximation vector of θ such that $N(P - q\theta) \leq \delta_N$ then $M_{s_0}(\theta)V$ is a shortest vector of $\Lambda_{s_0}(\theta)$ where $s_0 = \frac{N(P-q\theta)}{q}$.
2. Conversely, if there exists $s > 0$ such that $M_s(\theta)V$ is a shortest vector of $\Lambda_s(\theta)$ then there exists a best Diophantine approximation vector $V' = (P', q')$ of θ such that $N(P - q\theta) = N(P' - q'\theta)$. Moreover if N is the sup norm and if $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] = d + 1$, then V is a best Diophantine approximation vector of θ.

The lemma shows that for θ such that $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] = d + 1$, the set of best Diophantine approximation vectors of θ with respect to the sup norm and the set the shortest vectors of the lattices $\Lambda_s(\theta)$, $s > 0$, are equal.

4.3 Computation of best Diophantine approximations with the lattice $\Lambda_s(\theta)$.

Lagarias studied ([Lag2]) the complexity of the computation of best Diophantine approximations. Here we only explain one method to compute them.

By Lagarias lemma, a shortest vector of $\Lambda_s(\theta)$ gives rise to a best Diophantine approximation of θ. The LLL algorithm is likely to be the most efficient way to compute such a vector (see e.g. [G,L,S] for LLL algorithm). Assume that $\mathbf{R}^d$ is endowed with the Euclidean norm $\|.\|$. Use the LLL with the lattice $\Lambda_s(\theta)$ as input, the output is a "reduced" basis $(e_1, ..., e_d)$ of $\Lambda_s(\theta)$ whose first vector is almost a shortest vector of $\Lambda_s(\theta)$:

$$\|e_1\| \leq 2^{(d-1)/2}\lambda_1$$

where λ_1 is the first minimum of $\Lambda_s(\theta)$. It seems that, in practice, the length of the vector e_1 is often very close to λ_1 which means we have a very good Diophantine approximation of θ. In order to get a shortest vector of $\Lambda_s(\theta)$ it is possible to use the following result due to Babai ([Ba]):
for all $k \in \{1, ..., d\}$, the sinus of the angle between e_k and the sub-space generated by the other vectors of the basis, is $\geq (\sqrt{3}/2)^d$.
It follows that, if the absolute value of one of the coordinates of a vector X in $\Lambda_s(\theta)$ is $> (\frac{2}{\sqrt{3}})^d$, then its norm is $> \lambda_1$. Therefore, the shortest vector of $\Lambda_s(\theta)$ is among the vectors $X = \sum_{i=1}^d x_i e_i$ with $|x_i| \leq (\frac{2}{\sqrt{3}})^d$, $i = 1, ..., d$. Hence, $(2 \times \frac{2}{\sqrt{3}})^{d^2}$ computations are enough to find the shortest vector.

5 Multidimensional expansions and lattice reduction

5.1 General definition

Interpreting the lattices in $\mathbf{R}^n$ as points in $GL(n, \mathbf{R})/GL(n, \mathbf{Z})$, a reduction theory for lattices is given by a subset $\mathcal{B}_n$ in $GL(n, \mathbf{R})$ which contains a fundamental domain for the right action of $GL(n, \mathbf{Z})$; that is, for all matrix M in $GL(n, \mathbf{R})$ there exists P in $GL(n, \mathbf{Z})$ such that $MP \in \mathcal{B}_n$. A matrix in $\mathcal{B}_n$ is to be seen as a good basis of the underlying lattice, hence matrices in $\mathcal{B}_n$ must enjoy some geometrical properties depending on the aim of the reduction theory, e.g. the vectors of the basis must be as short as possible. To such a set of reduced matrices one can associated a multidimentional expansion :

Définition 7 *Suppose that $n = d+1$. Let $\mathcal{B}_n$ be a subset of $GL(n, \mathbf{R})$ which contains a fondamental domain for the right action of $SL(n, \mathbf{Z})$. Let $\theta \in \mathbf{R}^d$. A $\mathcal{B}_n$-expansion of θ is a map $Q = Q_\theta : s \in]0, +\infty[\to Q(s) \in \mathcal{B}_n$ such that $M_s(\theta)Q(s) \in \mathcal{B}_n$ for all $s \in]0, +\infty[$.*

An expansion is associated with any classical sets of reduced basis, e.g. Minkowski reduced basis, Korkine-Zolotarev reduced basis, Lovasz reduced basis, Siegel domains,....
The definition of the matrices $M_s(\theta)$ shows that the elements of the last row of the matrices $Q(s)$ are the denominators of the expansion.

The desired properties of such an expansion are:
E1. Unicity or finitness : for each $s > 0$, there is only one possible choice for $Q(s)$ or finitely many possible choices.
E2. Convexity : for a given matrix P in $\mathcal{B}_n$, the set of s such that $P = Q(s)$ is as simple as possible, e.g. one interval.
E3. Convergents may be associated with the expansion.
E4. $s \to Q(s)$ provides a strongly convergent expansion of θ : denoting $\begin{pmatrix} P_i(s) \\ q_i(s) \end{pmatrix}$ the columns of $Q(s)$,

$$\lim_{s \to 0} \max_{i=1}^{d+1} \|q_i(s)\theta - P_i(s)\| = 0$$

for $\theta = (\theta_1, ..., \theta_d)$ such that $\dim_{\mathbf{Q}}[1, \theta_1, ..., \theta_d] = d + 1$.
E4bis. The first column of $Q(s)$ is a best approximation vector of θ.
E5. Positivity: if $s > s'$ the positive cone defined by the columns of $Q(s)$ constains the positive cone defined by $Q(s')$.

We will see that Lagarias expansion conciliates all these properties except the last one, while it is known that classical multidimensional continued fraction expansions such as Jacobi-Perron algorithm, Brun algorithm,... are not strongly convergent but are positive (**E5**).

5.2 Lexicographically reduced basis

Lagarias expansion is defined with a set reduced matrices slightly smaller that the set $\mathcal{M}_n$ of Minskowski reduced basis. The reduced basis are called lexicographically reduced basis. In fact lexicographically reduced basis correspond to Hermite reduced quadratic forms (see [Tam1]).

$\mathbf{R}^n$ is endowed with the Euclidean norm $\|.\|$.

Définition 8 *A basis $(e_1, ..., e_n)$ of $\mathbf{R}^n$ is lexicographically reduced if the vector of norms*

$$(\|e_1\|, ..., \|e_n\|)$$

is minimal for the lexicographical order among all vectors of norms associated with the basis of $\Lambda = \mathbf{Z}e_1 \oplus ... \oplus \mathbf{Z}e_n$.

It is not difficult to show by induction that each lattice Λ admits a lexicographically reduced basis. Hence the set $\mathcal{L}_n$ of lexicographically reduced basis contains a fundamental domain.

The aim of this definition is to get basis with vectors as short as possible. Since a shortest vector of a lattice may be extended into a basis, the first vector of a lexicographically reduced basis is a shortest vector of Λ.

Recall that a basis $e_1, ..., e_n$ of a lattice Λ in $\mathbf{R}^n$ is Minkowski reduced if for $i = 1, ..., n$, e_i is a vector of minimal length among the vector x in Λ such that

$$(e_1, ..., e_{i-1}, x)$$

may be extended into a basis of Λ. Clearly, lexicographically reduced basis are Minkowski reduced. The above definition is not the one given by Lagarias who considers the minimum for the lexicographical order only among Minkowski's reduced basis. The next lemma is easy and connects the above definition with Lagarias definition.

Lemma 9 *For each basis $(f_1, ..., f_n)$ of the lattice Λ, there exists a Minkowski reduced basis $(e_1, ..., e_n)$ of Λ such that*

$$(\|e_1\|, ..., \|e_n\|) \preceq (\|f_1\|, ..., \|f_n\|)$$

for the lexicographical order.

The inclusions $\mathcal{M}_n^\circ \subset \mathcal{L}_n \subset \mathcal{M}_n$ holds in all dimensions and it is known that $\mathcal{L}_n = \mathcal{M}_n$ for $n \leq 6$ and that $\mathcal{L}_n \neq \mathcal{M}_n$ for $n > 6$ (see [Tam]). A quite simple example due to H.W. Lenstra shows that there exists Minkowski reduced basis that are not lexicographically reduced in dimension $d = 13$ (see [Lag1]).

5.3 Definition of Lagarias expansion

Définition 10 *Suppose that $n = d+1$ and that $\mathbf{R}^d$ and $\mathbf{R}^{d+1}$ are endowed with the Euclidean norms $\|.\|_{\mathbf{R}^d}$ and $\|.\|_{\mathbf{R}^{d+1}}$. Let θ be in $\mathbf{R}^d$. The Lagarias expansion of θ is the expansion $s \to Q_\theta(s)$ associated with the set $\mathcal{L}_n$ of lexicographically reduced basis.*

By definition, for all $s > 0$, the colums of $M_s(\theta)Q_\theta(s)$ form a lexicographically reduced basis of $\mathbf{R}^{d+1}$, hence the first column of $M_s(\theta)Q_\theta(s)$ is a shortest vector of the lattice $\Lambda_s(\theta)$ and by Lagarias lemma this first column of θ is a best approximation vector of θ.
Lagarias give a more precise definition based on main Theorem below.

5.4 Convexity properties of Lagarias expansion

To see that **E2** holds for Lagarias expansion, it is necessary to move $\mathcal{L}_n$ in the space $\mathcal{S}_n^+$ of symmetric positive definite matrices with the map

$$\begin{aligned} \varphi : GL(n, \mathbf{R}) &\to \mathcal{S}_n^+ \\ : M &\to M^t\, M \end{aligned}$$

Next theorem is proved in [Lag 1], and is a folklore result.

Theorem 13 *$\mathcal{Q}_n = \varphi(\mathcal{L}_n)$ is a convex set.*

Remark. It is clear that if q is in $\mathcal{Q}_n$ then for all $\lambda > 0$, λq is in $\mathcal{Q}_n$, hence $\mathcal{Q}_n$ is a convex cone. Thanks to the fact that the set of Minskowski reduced quadratic forms can be defined by finitely many linear inequalities (see [Waer]), we see that the convex cone $\mathcal{Q}_n$ has finitely many faces. When $n \leq 6$, Tammela [Tam1] gives all the inequalities defining the faces of $\mathcal{Q}_n$.

Notation Let θ be in $\mathbf{R}^d$. For each Q in $GL(d+1, \mathbf{Z})$, denote by $I(Q)$ the set of real numbers $s > 0$ such that

$$M_s(\theta)Q \in \mathcal{L}_{d+1}.$$

Next Theorem is proved in [Lag1]. We give its proof which is simple.

Theorem 14 *1. For all matrix Q in $GL(d+1, \mathbf{Z})$, $I(Q)$ is an interval.*
2. Let Q and Q' be in $GL(d+1, \mathbf{Z})$.
a. Then either $I(Q) = I(Q')$ or $I(Q) \cap I(Q')$ contains at most one element.
b. If $I(Q) \cap I(Q')$ contains at least 2 elements, then Q and Q' have the same last row up to signs.

Proof. 1. For $s > 0$, denote by B_s the diagonal matrix $(1, ..., 1, s)$. For all θ in $\mathbf{R}^d$ and all $s > 0$, we have $M_s(\theta) = B_s M_1(\theta) \in GL(d+1, \mathbf{R})$. Now, $s \in I(Q)$ means that the quadratic form

$$\varphi(M_s(\theta)Q) = Q^t M_1^t(\theta) B_{s^2} M_1(\theta) Q$$

is in $\mathcal{Q}_{d+1}$. Since $\varphi(M_s(\theta)Q)$ is an affine function of s^2, and since $\mathcal{Q}_{d+1}$ is convex, the set of positive real numbers s^2 such that $\varphi(M_s(\theta)Q) \in \mathcal{Q}_{d+1}$ is an interval. Hence $I(Q)$ is an interval.

2. Let $Q, Q' \in GL(d+1, \mathbf{Z})$ such that $I(Q) \cap I(Q')$ contains at least two elements $s_1 \neq s_2$. Denote by (P_i, q_i) the i-th column of Q and by (P_i', q_i') the i-th column of Q'. By definition of lexicographically reduced basis, the length of these two columns are equal, hence for $s \in \{s_1, s_2\}$,

$$\|P_i - q_i\theta\|^2_{\mathbf{R}^d} + s^2 q_i^2 = \|P_i' - q_i'\theta\|^2_{\mathbf{R}^d} + s^2 q_i'^2.$$

Therefore $|q_i| = |q_i'|$ and $\|P_i - q_i\theta\|^2_{\mathbf{R}^d} = \|P_i' - q_i'\theta\|^2_{\mathbf{R}^d}$. It follows that $I(Q) = I(Q')$. QED

5.5 Finitness property of expansions

Following Lagarias we will show that the finitness and the convergence of an expansion depend only on the following property of the reduction set :

Définition 11 *Let $\mathcal{R}$ be a subset of $GL(n, \mathbf{R})$. $\mathcal{R}$ is Minkowski-regular if there exists a constant C such that for any basis $(e_1, ..., e_n)$ in $\mathcal{R}$, and for $i = 1, ..., n$*

$$\|e_i\| \leq C\lambda_i(\Lambda)$$

where Λ is the lattice spanned by $e_1, ..., e_n$.

$\mathcal{M}_n$, $\mathcal{L}_n$, Lovazs reduced matrices, Siegle domain... are Minkowski-regular. A proof that $\mathcal{M}_n$ is Minkowski regular can be found in [Waer].

Theorem 15 *Let $\mathcal{R}$ be a Minkowski-regular subset of $GL(n, \mathbf{R})$. Let θ be in $\mathbf{R}^d$. For all $b > a > 0$, there exist finitely many matrices Q in $GL(n, \mathbf{Z})$ such that the basis $M_s(\theta)Q$ is in $\mathcal{R}$ for at least one $s \in [a, b]$.*

Proof. Consider a matrix Q in $GL(n, \mathbf{Z})$ such that $M_s(\theta)Q$ is in $\mathcal{R}$ for some $s \in [a, b]$. Let $e_1, ..., e_{d+1}$ be the columns of $M_s(\theta)Q$. Since $\mathcal{R}$ is regular, for $i = 1, ..., d+1$,

$$\|e_i\| \leq 2C\lambda_i(\Lambda_s(\theta)) \leq C\lambda_{d+1}(\Lambda_s(\theta))$$

where $\lambda_i(\Lambda_s(\theta))$ is the i-th minimum of $\Lambda_s(\theta)$. Now for all $t \leq b$, $\lambda_{d+1}(\Lambda_s(\theta)) \leq \max(1, b)$. Hence, with $e_i = (V_i, v_i)$, $\|M_s(\theta)e_i\|^2_{\mathbf{R}^{d+1}} = \|V - v\theta\|^2_{\mathbf{R}^d} + s^2v^2 \leq C^2 \max^2(1, b)$. Since $s \geq a$, $|v| \leq C\max(1, b)/a$ and V is in the union of balls $\cup_{|v| \leq \max(1,b)C/a} B(v\theta, \max(1, b)C)$. Therefore the number of matrices Q such that $M_s(\theta)Q$ is in $\mathcal{R}$ for at least one $s \in [a, b]$, is finite. QED

Theorem 16 *Let θ be in $\mathbf{R}^d$. For all $s_0 > 0$, there exist finitely many matrices Q in $GL(n, \mathbf{Z})$ such that the basis $M_s(\theta)Q$ is lexicographically reduced for at least one $s \geq s_0$.*

Proof. By the previous Theorem it is enough to consider the case $s > s_0 = 1$. For every non zero V in $\mathbf{Z}^{d+1}$, we have $\|M_s(\theta)V\|_{\mathbf{R}^{d+1}} \geq 1$ and if the last coordinate of V is not zero then $\|M_s(\theta)V\|_{\mathbf{R}^{d+1}} > 1$. Therefore all the lexicographically reduced basis are of the form:
- the first d vectors are $\pm M_s(\theta)e_i = \pm e_i$ where $i \leq d$ and e_i is the i-th vector of the canonical basis of $\mathbf{R}^{d+1}$
- the last vector is $M_s(\theta)V$ where $V = \pm(p_1, ..., p_n, 1)$ and p_i is such that $|p_i - \theta_i|$ is minimal.

Hence, the number of such basis is finite. QED

5.6 Main theorem for Lagarias expansion

From the previous Theorems, it follows immediately :

Theorem 17 ***(Main theorem)*** *If $\theta \notin \mathbf{Q}^d$, there exists an infinite sequence $\infty = s_{-1} > s_0 > s_1 > ... > s_n > ...$ going to 0 and a sequence of matrices $Q_0, ..., Q_n, ... \in GL(d+1, \mathbf{Z})$ such that $]s_n, s_{n-1}[= I(Q_n)^o$.*

The sequence of matrices $(Q_n)_{n\geq 0}$ is not uniquely defined but by Theorem 14, the last row is unique up to signs. The sequence of matrices $(Q_n)_{n\geq 0}$ is the Lagarias expansion of θ. The partial quotient associated with the sequence $(Q_n)_{n\geq 0}$ are defined by $Q_{n-1}^{-1}Q_n$. Since $\mathcal{M}_{d+1}Q_{n-1} \cap \mathcal{M}_{d+1}Q_n \neq \emptyset$, there are only finitely many possible partial quotients. Hence it is an additive expansion (see [Waer]). To obtain a multiplicative expansion it suffices to keep only the matrices Q_n at the times n where the first column of Q_n changes.

5.7 Convergence

Lagarias expansion is strongly convergent. Lagarias proof allows to show the more general result :

Theorem 18 *Let $\mathcal{R}$ be a Minkowski-regular subset of $SL(d+1,\mathbf{R})$ containing a fundamental domain. Let θ be in $\mathbf{R}^d$ be such that $\dim_{\mathbf{Q}}[1,\theta_1,...,\theta_d]=d+1$ and let $s\to Q(s)$ a $\mathcal{R}$-expansion associated with θ in $\mathbf{R}^d$. If $\dim_{\mathbf{Q}}[1,\theta_1,...,\theta_d]\geq r$, then the first r columns of the matrix $Q_\theta(s)$ strongly convergent to θ:*

$$\lim_{s\to 0}\max_{i=1}^{r}\|q_i(s)\theta-P_i(s)\|=0.$$

Proof. Denote by $\begin{pmatrix} P_{s,i} \\ q_{s,i}\end{pmatrix}$ the i-th column of the matrix $Q_\theta(s)$. The i-th column of $M_s(\theta)Q_\theta(s)$ is $\begin{pmatrix} P_{s,i}-q_{s,i}\theta \\ sq_{s,i}\end{pmatrix}$. By hypothesis,

$$\left\|\begin{pmatrix} P_{n,i}-q_{n,i}\theta \\ sq_{n,i}\end{pmatrix}\right\|_{\mathbf{R}^{d+1}}\leq C\lambda_{s,i}(\theta)$$

where $\lambda_{s,i}(\theta)$ is the i-th minimum of the lattice $\Lambda_s(\theta)=M_s(\theta)\mathbf{Z}^{d+1}$. Thus, it is enough to prove that

$$\lim_{s\to 0}\lambda_{s,r}(\theta)=0.$$

We will use the dual lattice $\Lambda_s^*(\theta)=\{Y\in\mathbf{R}^{d+1}:\forall X\in\Lambda_s(\theta),\ X.Y\in\mathbf{Z}\}$. The following lemma is standard (see, e.g. [Schm]).

Lemma 12 *Let Λ be a lattice in $\mathbf{R}^n$ and Λ^* its dual lattice. Then*

$$1\leq\lambda_i\lambda^*_{n+1-i}\leq(n+1)!.$$

Hence, it is enough to prove that the minimum $\lambda^*_{s,d+2-r}$ of $\Lambda_s^*(\theta)$ goes to infinity when s goes to zero. Suppose on the contrary that there exists a sequence $(s_n)_n$ going to 0 such that for all n, $\lambda^*_{s_n,d+2-r}\leq K$. The lattice $\Lambda_s^*(\theta)$ is spanned by the rows of the matrix

$$M_s^{-1}=\begin{pmatrix} 1 & 0 & \cdots & 0 & \theta_1 s^{-1}\\ 0 & 1 & \cdots & 0 & \theta_2 s^{-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \vdots & \vdots & & 1 & \theta_d s^{-1}\\ 0 & \cdots & \cdots & 0 & s^{-1}\end{pmatrix},$$

hence for all n, there exists $(A_{n,i},b_{n,i})\in\mathbf{Z}^{d+1}$, $i=1,...,d+2-r$, linearly independent such that the vector

$$v_{n,i}=(A_{n,i},b_{n,i})M_{s_n}^{-1}=(A_{n,i},\frac{1}{s_n}(b_{n,i}+A_{n,i}.\theta))$$

has a norm $\leq K$. If $s_n\leq 1$, then

$$\|A_{n,i}\|_{\mathbf{R}^d}\leq K$$

and

$$|b_{n,i}|\leq C+|A_{n,i}.\theta|\leq K'$$

where K' does not depend on n. Extracting a subsequence, we can assume $A_{n,i}=A_i$ and $b_{n,i}=b_i$ for all n with (A_i,b_i), $i=1,...,d+2-r$, linearly independent. Therefore,

$$|b_i+A_i.\theta|\leq Ks_n.$$

Now, $s_n\to 0$ hence $b_i+A_i.\theta=0$, $i=1,...,d+2-r$ which contradicts the assumption $\dim_{\mathbf{Q}}[1,\theta_1,...,\theta_d]\geq r$. QED

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