Examples of recurrent or transient stationary walks in $\mathbb{R}^d$ over a rotation of $\mathbb{T}^2$

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ABSTRACT. We give examples of recurrent and non-recurrent cocycles in $\mathbb{R}^d$ over rotations of the torus $\mathbb{T}^2$.

Introduction

Let $(E, \mathcal{A}, \mu)$ be a probability space, τ a measurable map from E into itself leaving the measure μ invariant and φ a measurable function on E with values in $\mathbb{R}^d$.
The function φ generates over the dynamical system (E, μ, τ) a *cocycle* with values in $\mathbb{R}^d$, denoted by (φ, τ) or $(\varphi_n)_{n\in\mathbb{N}}$, defined by $\varphi_n := \sum_{k=0}^{n-1} \varphi \circ \tau^k$, for $n \geq 0$, and a map τ_φ acting on $E \times \mathbb{R}^d$ by

$$\tau_\varphi : (x, y) \to (\tau x, \varphi(x) + y). \tag{1}$$

The map τ_φ leaves invariant the measure $\lambda := \mu \times m$ where m is the Lebesgue measure on $\mathbb{R}^d$. The cocycle (φ_n) can be viewed as a *stationary walk* in $\mathbb{R}^d$ "driven" by the dynamical system (E, μ, τ). It is *recurrent* if, for almost every x, the sequence $(\varphi_n(x))_{n\geq 0}$ returns infinitely often to every neighborhood of 0. The map τ_φ is then conservative for the measure λ.
This construction gives a way to get dynamical systems with an infinite measure and it is desirable to have recurrent examples. When φ is integrable with values in $\mathbb{R}$ and the system (E, τ, μ) is ergodic, $(\varphi_n)_{n\in\mathbb{N}}$ is recurrent if and only if $\mu(\varphi) = 0$. When φ has its values in $\mathbb{R}^d$, this is no more true for $d > 1$.
The aim of this note is to give examples of recurrent and non-recurrent cocycles in $\mathbb{R}^d$ over rotations of the torus $\mathbb{T}^2$. At the same time, we obtain some information on the size of the sum of a series related to 2-dimensional rotations.

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1. A sufficient condition of recurrence for stationary walks

The following general lemma gives a simple condition which implies recurrence. In what follows, we denote by $|\ \ |$ a norm on $\mathbb{R}^d$.

LEMMA 1.1. *Let (φ, τ) be a cocycle over a dynamical system $(E, \mathcal{A}, \mu, \tau)$ with values in $\mathbb{R}^d$. If there exists a strictly increasing sequence of integers (k_n) and a sequence of real numbers $(\delta_n > 0)$ such that:*

$$\lim_n \mu(x : |\varphi_{k_n}(x)| \geq \delta_n) = 0 \text{ and } \delta_n = o(n^{1/d}), \tag{2}$$

then the cocycle (φ, τ) is recurrent.

Proof If τ is not invertible, we can consider its natural (invertible) extension $(\tilde{E}, \tilde{\mathcal{A}}, \tilde{\mu}, \tilde{\tau})$ and extend the cocycle (φ, τ) into a cocycle $(\tilde{\varphi}, \tilde{\tau})$ which still satisfies (2) with $\tilde{\mu}$ instead of μ. Therefore, without loss of generality, we can assume that τ is invertible.

We have to show that, if B is a wandering set for τ_φ (i.e. such that the images $(\tau_\varphi^\ell B, \ell \geq 0)$ are pairwise disjoint), then $\lambda(B) = 0$. Assume $\lambda(B) > 0$. We can also assume that B is included in $E \times K$ where K is a compact set in $\mathbb{R}^d$.

Let $E_n = \{x : |\varphi_{k_n}(x)| \leq \delta_n\}$ and $B_n = (E_n \times K) \bigcap B$. If we choose n_0 such that $\lambda(B_j) \geq \frac{1}{2}\lambda(B)$, for $j \geq n_0$, then $\lambda(\cup_{j=n_0}^n \tau_\varphi^{k_j} B_j) \geq \frac{1}{2}(n - n_0)\lambda(B)$.

Set $\rho_n := \sup_{1 \leq k \leq n} \delta_k$. The increasing sequence (ρ_n) satisfies, as the sequence (δ_n), $\rho_n = o(n^{1/d})$.

On the other hand, the union $\cup_{j=n_0}^n \tau_\varphi^{k_j} B_j$ is included in $E \times (D(\rho_n) + K)$, where $D(\rho_n)$ is the ball centered at the origin with radius ρ_n. Therefore there exists a finite constant C such that

$$\lambda(\cup_{j=n_0}^n \tau_\varphi^{k_j} B_j) \leq C m(D(\rho_n)).$$

This implies $\liminf_n \dfrac{m(D(\rho_n))}{n} \geq \dfrac{1}{2C}\lambda(B) > 0$ and contradicts $\rho_n = o(n^{1/d})$. $\square$

COROLLARY 1.2. *If $\|\varphi_n\|_2 = o(n^{\frac{1}{d}})$, the cocycle is recurrent.*

Proof Let $E_n := \{x : |\varphi_n(x)| \leq \delta_n\}$, where (δ_n) is a sequence of reals > 0.

We have $\mu(E_n^c) \leq (\delta_n^{-1}\|\varphi_n\|_2)^2$. The conditions of Lemma 1.1 hold if the following conditions are satisfied:

$$\delta_n^{-1}\|\varphi_n\|_2 \to 0, \ n^{-1/d}\delta_n \to 0. \tag{3}$$

To have (3), it is sufficient to take $\delta_n := (n^{1/d}\|\varphi_n\|_2)^{1/2}$. $\square$

We are going to apply Corollary 1.2 to cocycles defined over rotations. For a dynamical system $(E, \mathcal{A}, \mu, \tau)$ defined by an irrational 1-dimensional rotation, the Denjoy-Koksma inequality ensures the recurrence of cocycles (φ, τ) with values in

$\mathbb{R}^d$ whose components are centered and have bounded variation, for any dimension d.

When the basis is a rotation on $\mathbb{T}^r$, with $r > 1$, the Denjoy-Koksma inequality doesn't hold anymore and there are cocycles with valued in $\mathbb{C}$ centered and analytical which are *dissipative* (cf. [**Yo80**]). However, we will see that, under a diophantine condition on the rotation, recurrence holds for cocycles with values in $\mathbb{R}^d$ defined by functions satisfying some assumption on their Fourier coefficients, like characteristic functions of polygonal sets. For simplicity, we will restrict to rotations of the 2-dimensional torus.

In studying extensions above rotations of the torus $\mathbb{T}^r$, Fraczek ([**Fr00**]) used bounds provided by estimates on the discrepancy for functions with bounded variation in the sense of Hardy and Krause. Note that the characteristic functions of polygonal sets do not belong to that class when the sides are not parallel to the axes, and the bounds for the ergodic sums require the estimates of the following section 2.

2. Series with small denominators

NOTATIONS 2.1. The Lebesgue measure on the torus $\mathbb{T}^2$ is denoted by m. We use the following notations:

. $\mathbb{Z}^2_* = \mathbb{Z}^2 \backslash \{0\}$,
. for a real x, $|x|_0 := \max(1, |x|)$ and $||x|| := \inf(\{x\}, \{1-x\}) = \inf_{n \in \mathbb{Z}} |x-n|$,
. $R_0 : \mathbb{R}^2 \to \mathbb{R}$ is defined by $R_0(h_1, h_2) := |h_1|_0 \ |h_2|_0$.

When ℓ_1 and ℓ_2 are two linear forms on $\mathbb{R}^2$, we define $R_{\ell_1,\ell_2} : \mathbb{R}^2 \to \mathbb{R}$ by

$$R_{\ell_1,\ell_2}(h) := |\ell_1(h)|_0 \ |\ell_2(h)|_0 .$$

In the following, we consider a subset Δ of the torus $\mathbb{T}^2$ whose boundary is a finite union of segments. Let 1_Δ be the characteristic function of Δ. We set $\varphi_\Delta := 1_\Delta - m(\Delta)$.

LEMMA 2.2. *There exist a constant C and linear forms $\ell_1, \ell_2, ...\ell_p$, pairwise linearly independent, such that, for every $h \in \mathbb{Z}^2$,*

$$|\widehat{\varphi}_\Delta(h)| \leq C \sum_{1 \leq i < j \leq p} \frac{1}{R_{\ell_i,\ell_j}(h)}.$$

Proof Up to the boundary, Δ is a finite union of triangles $\Delta_1, ..., \Delta_n$ with disjoint interiors which are the projections of triangles $T_1, ..., T_n$ of $[0,1]^2$. For each of these triangles Δ_k, it is sufficient to compute the Fourier transform of the characteristic function of T_k. Through an affine change of variable we are reduced to the case of the triangle $T = \{(x,y) \in [0,1]^2 : x \leq y\}$. For this triangle one has:

$$\widehat{1_T}(\xi_1, \xi_2) = \frac{1}{2i\pi\xi_1}\left(\frac{1 - e^{-2i\pi\xi_2}}{2i\pi\xi_2} - \frac{1 - e^{-2i\pi(\xi_1+\xi_2)}}{2i\pi(\xi_1+\xi_2)}\right)$$

and we can verify that

$$\left|\widehat{1_T}(\xi_1, \xi_2)\right| \leq C\left(\frac{1}{|\xi_1|_0 \, |\xi_2|_0} + \frac{1}{|\xi_1|_0 \, |\xi_1+\xi_2|_0} + \frac{1}{|\xi_2|_0 \, |\xi_1+\xi_2|_0}\right). \square$$

Lemma 2.3. *Let ℓ_1 and ℓ_2 be two independent linear forms on $\mathbb{R}^2$. There are constants C_1, C_2 such that, for every $\delta > 0$,*

$$\sum_{h \in \mathbb{Z}^2_*} \frac{1}{R_{\ell_1,\ell_2}(h)^{1+\delta}} \in [C_1\delta^{-2}, C_2\delta^{-2}].$$

Proof By considering the map $L = (\ell_1, \ell_2)$ and the image Λ of the lattice $\mathbb{Z}^2$ by L, we are reduced to prove the convergence of the series $\sum_{h\in\Lambda} \frac{1}{R_0(h)^{1+\delta}}$. For that series, it is enough to remark that the number of elements of the lattice Λ in a square $x + [0,1]^2$ is bounded by a number M which is independent of $x \in \mathbb{R}^2$. Therefore we have for some constant C_2:

$$\sum_{h\in\Lambda} \frac{1}{R_0(h)^{1+\delta}} \leq M \sum_{h\in\mathbb{Z}^2} \frac{1}{R_0(h)^{1+\delta}} < C_2\delta^{-2}.$$

Likewise we get the lower bound $C_1\delta^{-2}$. □

A diophantine condition

In the following, we fix ℓ_1, ℓ_2 two independent linear forms on $\mathbb{R}^2$. For $\alpha \in \mathbb{T}^2$ and $\delta > 0$, we set:

$$K_\delta(\alpha) := \sup_{h\in\mathbb{Z}^2_*} \{R_{\ell_1,\ell_2}(h)^{-(1+\delta)} \|h.\alpha\|^{-1}\} \tag{4}$$

and for $h \in \mathbb{Z}^2$ and $\varepsilon > 0$, $A_{h,\delta,\varepsilon} := \{\alpha \in \mathbb{T}^2 : \|h.\alpha\| \leq \frac{\varepsilon}{R_{\ell_1,\ell_2}(h)^{1+\delta}}\}$.

Remark that it is well known (Khinchin) that, for every $\varepsilon > 0$, for almost every $\alpha \in \mathbb{T}^2$ there is $c > 0$ such that:

$$\|k.\alpha\| \geq c|k|^{-(2+\varepsilon)}, \ \forall k \in \mathbb{Z}^2_*.$$

The quantity K_δ is adapted to the sums with small denominators related to the diophantine properties of α that we consider below. It follows from the Borel-Cantelli lemma that $K_\delta(\alpha) < +\infty$, for a.e. α. Thanks to the following lemma, we can obtain a more precise information about the size of K_δ. We will write simply R for R_{l_1,l_2}.

Lemma 2.4. *For every $\delta > 0$, we have:*

$$m(\bigcup_{h\in\mathbb{Z}^2_*} A_{h,\delta,\varepsilon}) \geq \frac{1}{3}\min(1, \frac{\delta}{2}\sum_{h\in\mathbb{Z}^2_*} m(A_{h,\delta,\varepsilon})). \tag{5}$$

Proof Let us fix an integer $N \geq 1$. Let Λ_d be the set of lattice vectors $h = (h_1, h_2)$ such that $\gcd(h_1, h_2) = d$. Since $\Lambda_d = d\Lambda_1$, for all d, we have:

$$\begin{aligned}
\sum_{0<|h|\leq N} m(A_{h,\delta,\varepsilon}) &= \sum_{d\geq 1} \sum_{h\in\Lambda_d,\, 0<|h|\leq N} m(A_{h,\delta,\varepsilon}) = \sum_{d\geq 1} \sum_{h\in d\Lambda_1,\, 0<|h|\leq N} \frac{2\varepsilon}{R(h)^{1+\delta}} \\
&\leq \sum_{d\geq 1} \sum_{h\in\Lambda_1,\, 0<|h|\leq N} \frac{2\varepsilon}{R(dh)^{1+\delta}} \leq \sum_{h\in\Lambda_1,\, 0<|h|\leq N} \sum_{d\geq 1} \frac{2\varepsilon}{d^{1+\delta}R(h)^{1+\delta}} \\
&\leq \frac{2}{\delta} \sum_{h\in\Lambda_1,\, 0<|h|\leq N} \frac{2\varepsilon}{R(h)^{1+\delta}} = \frac{2}{\delta} \sum_{h\in\Lambda_1,\, 0<|h|\leq N} m(A_{h,\delta,\varepsilon}).
\end{aligned}$$

Set

$$A^{(N)} = \bigcup_{h \in \Lambda_1,\, 0<|h|\le N} A_{h,\delta,\varepsilon}.$$

By the Cauchy-Schwarz inequality, we have:

$$\int_{A^{(N)}} \sum_{h \in \Lambda_1,\, 0<|h|\le N} 1_{A_{h,\delta,\varepsilon}} dm \le (m(A^{(N)}))^{1/2} (\int_{A^{(N)}} (\sum_{h \in \Lambda_1,\, 0<|h|\le N} 1_{A_{h,\delta,\varepsilon}})^2 dm)^{1/2},$$

hence

$$m(A^{(N)}) \ge \frac{(\sum_{h \in \Lambda_1,\, 0<|h|\le N} m(A_{h,\delta,\varepsilon}))^2}{D},$$

where

$$D = \sum_{h \ne k \in \Lambda_1,\, 0<|h|,|k|\le N} m(A_{h,\delta,\varepsilon} \cap A_{k,\delta,\varepsilon}) + \sum_{h \in \Lambda_1,\, 0<|h|\le N} m(A_{h,\delta,\varepsilon}).$$

If h and k are two linearly dependent elements of Λ_1, then $h = \pm k$. When h and k are two linearly independent vectors of $\mathbb{Z}^2$ and a and b are two nonnegative real numbers, we have (cf. V. G. Sprindzuk [**Sp79**], p. 35):

$$m(\{\alpha : \|h.\alpha\| \le a\} \cap \{\alpha : \|k.\alpha\| \le b\}) = m(\{\alpha : \|h.\alpha\| \le a\})\ m(\{\alpha : \|k.\alpha\| \le b\}).$$

Therefore:

$$\begin{aligned} &\sum_{h \ne k \in \Lambda_1,\, 0<|h|,|k|\le N} m(A_{h,\delta,\varepsilon} \cap A_{k,\delta,\varepsilon}) \\ &= \sum_{h \ne \pm k \in \Lambda_1,\, 0<|h|,|k|\le N} m(A_{h,\delta,\varepsilon})\ m(A_{k,\delta,\varepsilon}) + \sum_{h \in \Lambda_1,\, 0<|h|\le N} m(A_{h,\delta,\varepsilon}), \end{aligned}$$

and

$$D \le (\sum_{h \in \Lambda_1,\, 0<|h|\le N} m(A_{h,\delta,\varepsilon}))^2 + 2 \sum_{h \in \Lambda_1,\, 0<|h|\le N} m(A_{h,\delta,\varepsilon}).$$

Since $\dfrac{v^2}{v^2+2v} \ge \dfrac{1}{3}\min(1, v), \forall v > 0$, it follows that

$$\begin{aligned} m(A^{(N)}) &\ge \frac{1}{3}\min(1, \sum_{h \in \Lambda_1,\, 0<|h|\le N} m(A_{h,\delta,\varepsilon})) \\ &\ge \frac{1}{3}\min(1, \frac{\delta}{2} \sum_{0<|h|\le N} m(A_{h,\delta,\varepsilon})). \end{aligned}$$

Letting N tend to $+\infty$, we obtain (5). $\square$

We prove now that K_δ is in $L^s[0,1]$ for every $s < 1$, but is not integrable.

PROPOSITION 2.5. *There are constants C, C' such that for $\delta > 0$ and $s \in]0,1[$,*

$$\frac{C}{\delta^s}\frac{s}{1-s} \le \int_{\mathbb{T}^2} K_\delta(\alpha)^s\ d\alpha \le 1 + \frac{C'}{\delta^2}\frac{1}{1-s}. \tag{6}$$

Proof 1) Denote $\sigma_\delta := \sum_{h \in \mathbb{Z}^2_*} R(h)^{-(1+\delta)}$. We know that there are constants $0 < C_1 < C_2$ such that $\sigma_\delta \delta^2 \in [C_1, C_2]$. Let $s \in]0,1[$.
For every $t > 0$, the set $A^s_t := \{\alpha \in \mathbb{T}^2 \mid K_\delta(\alpha)^s > t\}$ is the union $\bigcup_{h \ne 0} A_{h,\delta,t^{-\frac{1}{s}}}$.

Since $m(A_{h,\delta,t^{-\frac{1}{s}}}) = \min(1, 2R(h)^{-(1+\delta)}t^{-\frac{1}{s}})$, we have

$$\begin{aligned}\int_{\mathbb{T}^2} K_\delta(\alpha)^s \, d\alpha &= \int_0^\infty m(A_t^s)\, dt \leq 1 + 2\int_1^{+\infty} \sum_{h\in\mathbb{Z}^2_*} R(h)^{-(1+\delta)} \ t^{-\frac{1}{s}} dt \\ &< 1 + \frac{2}{1-s}\sum_{h\in\mathbb{Z}^2_*} R(h)^{-(1+\delta)} \leq 1 + \frac{1}{1-s}\frac{C_2}{\delta^2}.\end{aligned}$$

2) We consider now the lower bound.

From (5) we have:

$$\begin{aligned}\int_{\mathbb{T}^2} K_\delta(\alpha)^s \, d\alpha &= \int_0^\infty m(A_t^s)\, dt \\ &\geq \int_0^\infty \frac{1}{3}\min(1, \frac{\delta}{2}\sum_{h\in\mathbb{Z}^2_*} \min(1, 2R(h)^{-(1+\delta)} \ t^{-1/s}) \ dt \\ &\geq \frac{\delta}{3}\int_0^\infty \sum_{h\in\mathbb{Z}^2_*} \min(\frac{1}{\delta\sigma_\delta}R(h)^{-(1+\delta)}, \frac{1}{2}, R(h)^{-(1+\delta)} \ t^{-1/s})dt \\ &\geq \frac{\delta}{3}\sum_{h\in\mathbb{Z}^2_*}\int_0^\infty \min(\frac{1}{\delta\sigma_\delta}R(h)^{-(1+\delta)}, R(h)^{-(1+\delta)} \ t^{-1/s})dt \\ &\geq \frac{\delta}{3}\sum_{h\in\mathbb{Z}^2_*} R(h)^{-(1+\delta)}\int_{(\delta\sigma_\delta)^s}^\infty \ t^{-1/s} \ dt \ \geq \ \frac{1}{3}\frac{C_1^s}{\delta^s}\frac{s}{1-s}.\end{aligned}$$

□

Convergence of a series

The following proposition relies on a method due to Niederreiter (cf. [**Ni72**]):

PROPOSITION 2.6. *Let ℓ_1 and ℓ_2 be two independent linear forms on $\mathbb{R}^2$. Let $\gamma > 0$. For almost every $\alpha \in \mathbb{R}^2$, we have:*

$$\sum_{h\in\mathbb{Z}^2_*}\frac{1}{R_{l_1,l_2}(h)^2\,\|h.\alpha\|^{2-\gamma}} < +\infty.$$

<u>Proof</u> Let $\delta \in]0, \gamma/3[$. Again we write simply R for R_{l_1,l_2}. We have

$$R(h)^{1+\delta}\,\|h.\alpha\| \geq c, \ \forall h \in \mathbb{Z}^2_*, \tag{7}$$

where c is the constant $K_\delta^{-1}(\alpha)$, which is strictly positive for a.e. α by Proposition 2.5. Set $\lambda := \gamma - \delta$. We have by (7):

$$\sum_{h\in\mathbb{Z}^2_*}\frac{1}{R(h)^2\,\|h.\alpha\|^{2-\gamma}} \leq \sum_{h\in\mathbb{Z}^2_*}\frac{1}{R(h)^2(cR(h)^{-(1+\delta)})^{1-\gamma}} \times \frac{1}{\|h.\alpha\|} \tag{8}$$

$$\leq B_1\sum_{h\in\mathbb{Z}^2_*} R(h)^{-(1+\lambda)} \times \frac{1}{\|h.\alpha\|}. \tag{9}$$

Now we estimate, for $n = (n_1, n_2)$ such that $n_i \geq 1$, the sum

$$\sum_{h\in\mathbb{Z}^2_*,\ |\ell_i(h)|\leq n_i}\frac{1}{\|h.\alpha\|}.$$

For every pair (h, h'), $h \neq h'$, of points of the lattice $\mathbb{Z}^2$ in the domain of summation, we have:

$$\begin{aligned} |\|h.\alpha\| - \|h'.\alpha\|| &\geq \min(\|(h+h').\alpha\|, \|(h-h').\alpha\|) \\ &\geq c \min(R(h+h')^{-(1+\delta)}, R(h-h')^{-(1+\delta)}) \\ &\geq c(R_0(2n))^{-(1+\delta)}. \end{aligned}$$

Let $d := c(R_0(2n))^{-(1+\delta)}$. The elements h of the domain of summation are different from 0. Therefore they satisfy also the inequality $\|h.\alpha\| \geq d$. On the other hand, as $\|x\| \leq 1/2$ for every real x, the numbers $\|h.\alpha\|$ belong to one of the intervals

$$[d, 2d[,\ [2d, 3d[,\ ...,\ [rd, (r+1)d[,$$

where $r = [\frac{1}{2d}]$ and each of these intervals contains at most one element of the form $\|h.\alpha\|$.

This implies

$$\begin{aligned} \sum_{h \in \mathbb{Z}^2_*,\ |\ell_i(h)| \leq n_i} \frac{1}{\|h.\alpha\|} &\leq \sum_{k=1}^{r} \frac{1}{kd} \leq \frac{1}{d}(1 + \ln r) \\ &\leq \frac{1}{c} R_0(2n)^{1+\delta}(1 + \ln \frac{R_0(2n)^{1+\delta}}{c}) \leq B_2 R_0(n)^{1+2\delta}, \end{aligned}$$

where the constant B_2 does not depend on n.

We have

$$\sum_{n=(n_1,n_2),\ n_i \geq |\ell_i(h)|_0} (n_1 n_2)^{-(2+\lambda)} \geq \frac{(1+\lambda)^{-2}}{(|\ell_1(h)|_0\, |\ell_2(h)|_0)^{1+\lambda}} = (1+\lambda)^{-2} R(h)^{-(1+\lambda)},$$

so that, by permuting the order in the summation, we get:

$$\begin{aligned} \sum_{h \in \mathbb{Z}^2_*} R(h)^{-(1+\lambda)} \times \frac{1}{\|h.\alpha\|} &\leq (1+\lambda)^2 \sum_{n_1=1, n_2=1}^{\infty} (n_1 n_2)^{-(2+\lambda)} \\ &\quad \times \sum_{h=(h_1,h_2) \in \mathbb{Z}^2_*,\ |\ell_i(h)| \leq n_i} \frac{1}{\|h.\alpha\|}. \end{aligned}$$

Finally, using (9) and the previous inequality, we have for constants B_3, B_4:

$$\begin{aligned} \sum_{h \in \mathbb{Z}^2_*} \frac{1}{R(h)^2 \|h.\alpha\|^{2-\gamma}} &\leq B_3 \sum_{n_1=1, n_2=1}^{\infty} (n_1 n_2)^{-(2+\lambda)} \times \sum_{h=(h_1,h_2) \in \mathbb{Z}^2_*,\ |\ell_i(h)| \leq n_i} \frac{1}{\|h.\alpha\|} \\ &\leq B_4 \sum_{n_1=1, n_2=1}^{\infty} (n_1 n_2)^{-(2+\lambda)} R_0(n_1, n_2)^{1+2\delta} \\ &= B_4 \sum_{n_1=1, n_2=1}^{\infty} (n_1 n_2)^{-1-\gamma+3\delta}. \end{aligned}$$

As $\delta < \frac{\gamma}{3}$, the series converges. $\square$

Remark 2.7. Making use of Proposition 2.5, one can get some information about the size of the sum (as a function of α)

$$C_\gamma(\alpha) := \sum_{h\in\mathbb{Z}^2_*} \frac{1}{R_{l_1,l_2}(h)^2\, \|h.\alpha\|^{2-\gamma}}.$$

Indeed, looking at the constants in the proof of the above lemma, it can be shown that, for a constant C_0 depending only on (ℓ_1, ℓ_2), for every $a > 0$ and $\varepsilon > 0$ such that $a(2-\gamma) = 1-\varepsilon$, we have:

$$\int_{\mathbb{T}^2} C_\gamma(\alpha)^a \, d\alpha \leq \frac{C_0}{\varepsilon \gamma^{2(1+a)}}(\frac{1}{\gamma^a} + \frac{1}{\varepsilon^a}). \tag{10}$$

On the other hand, we have $C_\gamma(\alpha) \geq K_\delta(\alpha)^{2-\gamma}$, with $\delta = \frac{\gamma}{2}\frac{1}{1-\frac{\gamma}{2}}$, so that for a constant C_0',

$$\int_{\mathbb{T}^2} C_\gamma(\alpha)^a \, d\alpha \geq \int_{\mathbb{T}^2} K_\delta(\alpha)^{1-\varepsilon} \, d\alpha \geq \frac{C_0'}{\delta^{1-\varepsilon}} \frac{1-\varepsilon}{\varepsilon}.$$

3. Growth in norm $\| \|_2$ of the ergodic sums and recurrence

Let Δ be a subset of the 2-torus whose boundary is a finite union of segments as in 2.1 and let $\varphi_\Delta := 1_\Delta - m(\Delta)$. Given $\alpha = (\alpha_1, \alpha_2) \in \mathbb{T}^2$, our aim is to bound the rate of growth in $\| \|_2$ norm of the cocycle generated by φ_Δ over the rotation by α. We denote by

$$S_N^\alpha \varphi_\Delta(x,y) := \sum_{k=0}^{N-1} \varphi_\Delta(x + k\alpha_1, y + k\alpha_2)$$

the ergodic sums of the function φ_Δ for the rotation by α. The following result shows that for a.e. α the rate of growth is slow enough to insure recurrence for a class of $\mathbb{R}^d$-valued cocycles.

Theorem 3.1. *For every $\gamma > 0$, for almost every $\alpha \in \mathbb{T}^2$, there is a finite constant $A_\gamma(\alpha)$ such that*

$$\|S_N^\alpha \varphi_\Delta\|_2 \leq A_\gamma(\alpha) N^\gamma. \tag{11}$$

<u>Proof</u>

$$\begin{aligned}
\|S_N^\alpha \varphi_\Delta\|_2^2 &\leq \sum_{h\in\mathbb{Z}^2_*} |\hat{\varphi}_\Delta(h)|^2 |\sum_{k=0}^{N-1} e^{2\pi i k(h.\alpha)}|^2 \\
&\leq \sum_{h\in\mathbb{Z}^2_*} |\hat{\varphi}_\Delta(h)|^2 \inf(N^2, \frac{1}{||h.\alpha||^2}) \\
&\leq N^{2\gamma} \sum_{h\in\mathbb{Z}^2_*} |\hat{\varphi}_\Delta(h)|^2 \frac{1}{||h.\alpha||^{2(1-\gamma)}}.
\end{aligned}$$

By Lemma 2.2 and Proposition 2.6, for every $\gamma > 0$ the series converges for almost every α. $\square$

The set $\{\alpha : A_\gamma(\alpha) < +\infty\}$ depends on the function φ_Δ. Note also that $A_\gamma = C_\gamma^{\frac{1}{2}}$ and the remark 2.7 gives an information about the size of A_γ.

Application to recurrence

We consider a family $(\varphi_{\Delta_i}, i = 1, \cdots, d)$ with φ_{Δ_i} as in 2.1. From Theorem 3.1 and Corollary 1.2 we deduce:

THEOREM 3.2. *For a.e.* $\alpha = (\alpha_1, \alpha_2)$*, the* $\mathbb{R}^d$*-cocycle defined over the rotation on* $\mathbb{T}^2$ $(x, y) \to (x + \alpha_1, y + \alpha_2)$ *by* $\Phi = (\varphi_{\Delta_i})_{i=1,\cdots,d}$ *is recurrent.*

REMARK 3.3. For a 1-dimensional rotation and some functions like $1_{[0,\frac{1}{2}[} - \frac{1}{2}$, it is possible to find a lower estimate of the growth of the corresponding ergodic sums (cf. [**Is06**]) along some subsequences. A natural question is to obtain such an information for 2-dimensional rotations and functions like φ_Δ.

REMARK 3.4. For the cocycle defined by $\Phi(x, y) = (\varphi_1(x), \varphi_2(y))$, with φ_1 and φ_2 BV functions, a bound like (11) for a.e. rotation (α_1, α_2) follows from bounds for the ergodic sums associated to rotations in one variable. Indeed if α_1 is of type 1 and if f is a BV function on $\mathbb{T}^1$, then, for every $\varepsilon > 0$, we have uniformly in x: $\sum_{k=0}^{n-1} f(x + k\alpha_1) = O(n^\varepsilon)$.

For instance consider $\Phi(x, y) = (2.1_{[0,\frac{1}{2}[}(x) - 1), 2.1_{[0,\frac{1}{2}[}(y) - 1)$. It defines a $\mathbb{Z}^2$-cocycle, or in other words a stationary walk in $\mathbb{Z}^2$ driven by a 2-dimensional rotation, which is recurrent for a.e. (α_1, α_2). A question is then the ergodicity with respect to the product of the uniform measure on $\mathbb{T}^2$ by the counting measure on $\mathbb{Z}^2$ of the skew-product

$$\tau_\Phi : (x, y, n) \to (x + \alpha_1, y + \alpha_2, n + \Phi(x, y)).$$

Remark also that one can construct examples of nonregular cocycles (Φ, τ) over 2-dimensional rotations, with functions Φ of this type. The corresponding skew-product is non ergodic in a strong sense: its ergodic decomposition involves σ-finite infinite ergodic measures on the basis.

In the next section, we construct $\alpha = (\alpha_1, \alpha_2)$ and a function Φ such that the associated cocycle is not recurrent: although recurrence occurs for each component, the return times of the components in a neighborhood of the origin are not "synchronized". In this way, we have a simple example of a transient stationary walk in $\mathbb{Z}^2$ over a two dimensional rotation.
As a consequence of the transience and Corollary 1.2, for this function Φ and this choice of α, we have for all strictly increasing sequences (k_n):

$$\limsup_n \frac{\|S_{k_n}^\alpha \Phi\|_2}{\sqrt{n}} > 0.$$

4. An example of transient cocycle

In [**Yo80**] J.-C. Yoccoz constructed an example of a centered dissipative cocycle given by an analytical function over a particular 2-dimensional rotation. Similarly we now construct a function, given by centered characteristic functions of regular domains, together with a minimal rotation, such that the corresponding cocycle defines a transient walk in the plane.

THEOREM 4.1. *Let* $\Phi : \mathbb{T}^2 \to \mathbb{R}^2$ *be the function*

$$\Phi(x) = \Phi(x_1, x_2) = (\varphi(x_1), \varphi(x_2)), \text{ where } \varphi := 1_{[0,\frac{1}{5}]} - \frac{1}{5}.$$

There exists a minimal rotation in $\mathbb{T}^2$, $x \to x + \alpha$, $\alpha = (\alpha_1, \alpha_2)$, *such that, for almost every* (x_1, x_2) *in* $\mathbb{T}^2$,

$$|\sum_{k=0}^{N-1} \varphi(x_1 + k\alpha_1)| + |\sum_{k=0}^{N-1} \varphi(x_2 + k\alpha_2)| \overset{N\to+\infty}{\longrightarrow} +\infty.$$

<u>Proof</u> The function Φ can be written

$$\Phi = (\varphi_1, \varphi_2) = (1_{\mathcal{R}_1} - \frac{1}{5}, 1_{\mathcal{R}_2} - \frac{1}{5}),$$

where $\mathcal{R}_1 = [0, \frac{1}{5}] \times [0, 1]$ and $\mathcal{R}_2 = [0, 1] \times [0, \frac{1}{5}]$.

We have to find an element $\alpha = (\alpha_1, \alpha_2)$ of $\mathbb{R}^2$ with $\mathbb{Z}\alpha + \mathbb{Z}^2$ everywhere dense in $\mathbb{R}^2$ such that, for a.e. $(x_1, x_2) \in \mathbb{T}^2$, either $|S_N^{\alpha_1}\varphi_1(x_1)|$ or $|S_N^{\alpha_2}\varphi_2(x_2)|$ is large when N is large.

Let $(k(n))_{n\geq 1}$ be a sequence of integers such that $k(1) = 1$ and

$$k(n+1) \geq 10 \times \frac{\ln 3}{\ln 2} \times k(n), \ \forall n \geq 1. \tag{12}$$

We are going to show that the following number α satisfies the conditions:

$$\alpha = (\sum_{n=1}^{\infty} 3^{-k(2n+1)}, \sum_{n=1}^{\infty} 2^{-k(2n)}).$$

Set $p_n = 2$ if n is even and $p_n = 3$ if n is odd. Consider the sequence $(\beta_n)_{n\geq 0}$ of rational approximations of α defined by

$$\beta_n := (\sum_{m<\frac{n}{2}} 3^{-k(2m+1)}, \sum_{m\leq\frac{n}{2}} 2^{-k(2m)}).$$

For every $n \geq 1$, β_n is a point of the torus of order $Q_n = p_{n-1}^{k(n-1)} p_n^{k(n)}$.
By definition $\alpha = \beta_n + \varepsilon_n$, where $\varepsilon_n = (\sum_{m\geq\frac{n}{2}} 3^{-k(2m+1)}, \sum_{m>\frac{n}{2}} 2^{-k(2m)})$ involves only exponents $k(\ell)$ with $\ell > n$. Thus

$$|\varepsilon_n| \leq \sum_{m\geq\frac{n}{2}} 3^{-k(2m+1)} + \sum_{m>\frac{n}{2}} 2^{-k(2m)} \leq 4 \times 2^{-k(n+1)}. \tag{13}$$

β_n has the form $(\frac{p_n}{q_n}, \frac{p'_n}{q'_n})$, with $q_n = 3^{k(n-1)}$, $q'_n = 2^{k(n)}$ for n even, or $q_n = 3^{k(n)}$, $q'_n = 2^{k(n-1)}$ for n odd. We have $Q_n = q_n q'_n$. Let $r(j)$ (resp. $r'(j)$) be the remainder of j mod q_n (resp. mod q'_n). The map $j \mapsto (r(j), r'(j))$ from $[0, q_n q'_n - 1]$ to $[0, q_n - 1] \times [0, q'_n - 1]$ is a bijection and the orbit $\{j\beta_n \bmod \mathbb{Z}^2, j \in \mathbb{Z}\}$ coincides with the set $\{\frac{j}{q_n} \bmod 1, 0 \leq j \leq q - 1\} \times \{\frac{j'}{q'_n} \bmod 1, 0 \leq j' \leq q'_n - 1\}$.

Since for all $q \leq Q_n$ we have by (12):

$$d(q\alpha, q\beta_n) = q\,|\varepsilon_n| \leq Q_n\,|\varepsilon_n| = p_{n-1}^{k(n-1)} p_n^{k(n)} \times 4 \times 2^{-k(n+1)} \to 0,$$

the subgroup $\mathbb{Z}\alpha + \mathbb{Z}^2$ is everywhere dense in $\mathbb{R}^2$.

For each $p \geq 1$, we define a family of segments in the torus according to the parity of p: when $p = 2n+1$, we denote by $s_{j,2n+1}$, for $j = 0, ..., 2^{k(2n)} - 1$, the projections of the horizontal segments $j\beta_{2n+1} + [0,1] \times \{0\}$; when $p = 2n$, we denote by $s_{j,2n}$, for $j = 0, ..., 3^{k(2n-1)} - 1$, the projections of the vertical segments $j\beta_{2n} + \{0\} \times [0,1]$.

Each integer q can be written as $q = ap_{n-1}^{k(n-1)} + j$ where a and j are integers and $0 \leq j < p_{n-1}^{k(n-1)}$. The point $q\beta_n$ is in the segment $s_{j,n}$. Therefore, for $q \leq Q_n^2$, the distance in the torus from $q\alpha$ to $s_{j,n}$ is bounded from above by

$$d_{\mathbb{T}^2}(q\alpha, s_{j,n}) = d_{\mathbb{T}^2}(q(\beta_n + \varepsilon_n), s_{j,n}) \leq 0 + q\,|\varepsilon_n| \leq Q_n^2\,|\varepsilon_n|\,.$$

We set

$$\delta_n := Q_n^2\,|\varepsilon_n|$$

and we denote by $B_{j,n}$ the strip made of the points of the torus at a distance $\leq 2\delta_n$ from the segment $s_{j,n}$.

Since the distance r_n between two consecutive segments (of the same orientation) $s_{j,n}$ and $s_{i,n}$ is $p_{n-1}^{-k(n-1)}$ and since by (13)

$$\begin{aligned} 4 \times \delta_n &\leq p_{n-1}^{2k(n-1)} p_n^{2k(n)} \times 16 \times 2^{-k(n+1)} \\ &\leq 3^{4k(n)+3} 2^{-k(n+1)} \leq p_{n-1}^{-6k(n)+3} < \frac{1}{2} p_{n-1}^{-k(n-1)}, \end{aligned}$$

the strips $B_{j,n}$, $j = 0, ..., p_n^{k(n-1)} - 1$, are disjoint.

Consider the rectangle $\mathcal{R}_i$ parallel to the segment $s_{j,n}$. The number of segments $s_{j,n}$ included in a translate $\mathcal{R}_i - x$ is $J_x = J_n$ or $J_n + 1$, where $J_n = \left[\frac{1/5}{r_n}\right]$.

Set $t_n = \frac{1}{5} - J_n r_n$ and $\rho_n = \min(t_n, r_n - t_n)$. We have

$$\rho_n = \min_{j \in \mathbb{Z}} \left| \frac{1}{5} - \frac{j}{p_{n-1}^{k(n-1)}} \right| \geq \frac{1}{5 \times p_{n-1}^{k(n-1)}}.$$

Set

$$I_n = \bigcup_{j=0}^{p_{n-1}^{k(n-1)}-1} B_{j,n}, \quad K_n = I_n \pm \rho_n e_n, \quad E_n = I_n \cup K_n,$$

where $e_n = (1,0)$ when n is even and $(0,1)$ when n is odd. The measure of the set E_n is less than $3p_{n-1}^{k(n-1)} \times$ (the width of the strips $B_{j,n}$), therefore less than $m_n = 12 \times p_{n-1}^{k(n-1)} \delta_n$.

Let N be an integer in the interval $[Q_{n-1}^2, Q_n^2[$.

Suppose n is even (we proceed in the same way for n odd). Take an element $x \in \mathbb{T}^2 \setminus E_n$. We have to estimate the number $M_{x,N}$ of integers $q < N$ such that $x + q\alpha \in \mathcal{R}_2$. Consider a segment $s_{j,n}$.

Since the distance in the torus is invariant under translations, we have

$$\begin{aligned} d_{\mathbb{T}^2}(x + s_{j,n}, \{0\} \times [0,1]) &= d_{\mathbb{T}^2}(x + j\beta_n, \{0\} \times [0,1]) \\ &= d_{\mathbb{T}^2}(x, -j\beta_n + \{0\} \times [0,1]) = d_{\mathbb{T}^2}(x, s_{i,n}) \end{aligned}$$

for an integer i in $\{0, ..., p_{n-1}^{k(n-1)} - 1\}$. By assumption x is not in I_n, thus $d_{\mathbb{T}^2}(x, s_{i,n}) > 2\delta_n$.

Therefore the distance between the two segments $x + s_{j,n}$ and $\{0\} \times [0,1]$ is greater than $2\delta_n$.

As above, we have

$$\begin{aligned} d_{\mathbb{T}^2}(x + s_{j,n}, \{\frac{1}{5}\} \times [0,1]) &= d_{\mathbb{T}^2}(x + j\beta_n, \{\frac{1}{5}\} \times [0,1]) \\ &= d_{\mathbb{T}^2}(x, -j\beta_n + \{\frac{1}{5}\} \times [0,1]) \\ &= d_{\mathbb{T}^2}(x, -j\beta_n + \{J_n r_n + t_n\} \times [0,1]) \\ &= d_{\mathbb{T}^2}(x, (-j\beta_n + (J_n p_{n-1}^{k(n-1)}, 0)) + \{t_n\} \times [0,1]) \\ &= d_{\mathbb{T}^2}(x, s_{i,n} + t_n e_n) \end{aligned}$$

and since x is not in K_n, $d_{\mathbb{T}^2}(x, s_{i,n} + t_n e_n) > 2\delta_n$.

Therefore the distance between the two segments $x + s_{j,n}$ and $\{\frac{1}{5}\} \times [0,1]$ is greater than $2\delta_n$. It follows that the strip $x + B_{j,n}$ is either entirely in $\mathcal{R}_2$ or entirely in the complement of $\mathcal{R}_2$. Since for each $j < p_{n-1}^{k(n-1)}$ the number of integers $q < N$ such that $q\alpha$ is in a strip $B_{j,n}$ is equal to A or to $A+1$, where $A := [(N-1) \times p_{n-1}^{-k(n-1)}]$, we have

$$J_x A \leq M_{x,N} \leq J_x(A+1).$$

It remains to find a lower bound for $|S_N\varphi_2(x)| = \left|M_{x,N} - \frac{1}{5}N\right|$. In case $J_x = J_n$, we have

$$\begin{aligned} \frac{1}{5}N - M_{x,N} &\geq \frac{1}{5}N - J_x(A+1) \geq \frac{1}{5}N - J_n(\frac{N}{p_{n-1}^{k(n-1)}} + 1) \\ &= (\frac{1}{5} - \frac{J_n}{p_{n-1}^{k(n-1)}})N - J_n \geq \rho_n N - J_n \\ &\geq \frac{1}{5 \times p_{n-1}^{k(n-1)}} Q_{n-1}^2 - \frac{1/5}{p_{n-1}^{-k(n-1)}} = \frac{p_{n-1}^{k(n-1)}}{5}(p_{n-2}^{2k(n-2)} - 1) \end{aligned}$$

and when $J_x = J_n + 1$, we have by the same computation:

$$M_{x,N} - \frac{1}{5}N \geq (J_n + 1)A - \frac{1}{5}N \geq \frac{p_{n-1}^{k(n-1)}}{5}(p_{n-2}^{2k(n-2)} - 1) - 1.$$

Therefore in both cases, we have:

$$(x \notin E_n) \Longrightarrow |S_N\varphi_2(x)| \geq \frac{p_{n-1}^{k(n-1)}}{5}(p_n^{2k(n)} - 1) - 1.$$

When n is odd, the same result holds for the sums $S_N\varphi_1(x)$.

Finally, since

$$\begin{aligned} \sum_{n\geq 1} m(E_n) &\leq 12 \sum_{n\geq 1} p_{n-1}^{k(n-1)} \delta_n = 12 \sum_{n\geq 1} p_{n-1}^{k(n-1)} Q_n^2 \, |\varepsilon_n| \\ &\leq 12 \sum_{n\geq 1} p_{n-1}^{k(n-1)} p_{n-1}^{2k(n-1)} p_n^{2k(n)} \times 4 \times 2^{-k(n+1)} < +\infty, \end{aligned}$$

almost every x in $\mathbb{T}^2$ belongs only to a finite number of sets E_n and this implies

$$\lim_{N\to+\infty} |S_N\varphi(x)| = +\infty.$$

$\square$

REMARK 4.2. The element α defined in the proof of Theorem 4.1 is singular in the following sense (see Cassels, singular systems [**Ca57**]) : for all $\varepsilon > 0$, there exists an integer N_ε such that for all $N \geq N_\varepsilon$, we have

$$\min\{\|q\alpha\| : 0 < q \leq N\} \leq \varepsilon N^{-\frac{1}{2}}.$$

Indeed, for $N \in [Q_n, Q_{n+1}[$, we have

$$\min\{\|q\alpha\| : 0 < q \leq N\} \leq |\varepsilon_{n-1}|\,,$$

$$|\varepsilon_n|\, N^{\frac{1}{2}} \leq (\sum_{m\geq\frac{n}{2}} 3^{-k(2m+1)} + \sum_{m>\frac{n}{2}} 2^{-k(2m)})Q_{n+1}^{\frac{1}{2}}.$$

For n even, we have

$$\begin{aligned}(\sum_{m\geq\frac{n}{2}} 3^{-k(2m+1)} + \sum_{m>\frac{n}{2}} 2^{-k(2m)})Q_{n+1}^{\frac{1}{2}} &\leq 2\sum_{m\geq\frac{n}{2}} 3^{-k(2m+1)}Q_{n+1}^{\frac{1}{2}}\\ &\leq 4\times 3^{-k(n+1)}\times 2^{\frac{1}{2}k(n)}3^{\frac{1}{2}k(n+1)}\end{aligned}$$

and for n odd

$$\begin{aligned}(\sum_{m\geq\frac{n}{2}} 3^{-k(2m+1)} + \sum_{m>\frac{n}{2}} 2^{-k(2m)})Q_{n+1}^{\frac{1}{2}} &\leq 2\sum_{m>\frac{n}{2}} 2^{-k(2m)}Q_{n+1}^{\frac{1}{2}}\\ &\leq 4\times 2^{-k(n+1)}3^{\frac{1}{2}k(n)}2^{\frac{1}{2}k(n+1)}.\end{aligned}$$

Therefore $|\varepsilon_n| \times Q_{n+1}^{\frac{1}{2}}$ goes to 0 when $n \to \infty$.
The set of singular elements of the torus $\mathbb{T}^2$ being of zero measure, to complete the previous result, the following question is natural: is the transformation τ_φ : $\mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{T}^2 \times \mathbb{R}^2$ associated with the function φ of Theorem 4.1, recurrent when α is not singular ?

References

[Ca57] Cassels (J. W. S): An introduction to Diophantine approximation. Facsimile reprint of the 1957 edition. Cambridge Tracts in Mathematics and Mathematical Physics, No. 45. Hafner Publishing Co., New York, 1972.

[Fr00] Fraczek (K.): Circle extensions of Z^d-rotations on the d-dimensional torus, J. London Math. Soc. (2) 61 (2000), no. 1, p.139-162.

[Ni72] Niederreiter (H.): On a number-theoretical integration method, Aequationes Math. 8 (1972), 304-311.

[Sp79] Sprindzuk (V. G.): *Metric Theory of Diophantine Approximations,* John Wiley & Sons 1979.

[Yo80] Yoccoz (J.-C.): Sur la disparition de propriétés de type Denjoy-Koksma en dimension 2, C. R. Acad. Sci. Paris Sér. A-B 291 (1980), no. 13.

[Is06] Isola (S.): Dispersion properties of ergodic translations. Int. J. Math. Math. Sci. 2006.

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