Stepped hyperplane and extension of the three distance Theorem.

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Nicolas Chevallier [1]

Abstract

The three distance Theorem states that given a real number θ and an integer n, the fractional parts of $\theta, 2\theta, ..., n\theta$ divide the unit interval $[0,1]$ into $n+1$ intervals having three different lengths at most. Several two-dimensional extensions using triangulations instead of intervals already exist. We give a result improving these extensions. This result should be close to optimality.

1 Introduction

Let θ be in $\mathbf{R}$ and let q be in $\mathbf{N}$. The points $0, \{\theta\}, \{2\theta\}, ..., \{q\theta\}$ divide the interval $[0,1]$ into $q+1$ intervals having at most three lengths ($\{x\}$ denotes the fractional part of the real number x). This property is known as the three distance Theorem and was first proven by V. T. Sòs in 1957. Since then many extensions have been proven. The first extension is due to K. R. Chung and R. L. Graham [4]; they have shown that if $\theta_1, ..., \theta_d$ are d real numbers, then the points $\{k_i\theta_i\}$, for $1 \leq i \leq d$ and $0 \leq k_i \leq q_i$, cut the interval $[0,1]$ into intervals having at most $3d$ lengths. There is another kind of extension of the three distance Theorem: given θ in the two-dimensional torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ and q in $\mathbf{N}$, is it possible to describe the relative positions of the elements of $\{0, \theta, ..., q\theta\}$? Recently, S. Vijay [8] has found a nice property about the mutual distances between the elements of $\{0, \theta, ..., q\theta\}$ (see also [3]). The distances are used to define natural neighboors of a point in $\{0, \theta, ..., q\theta\}$ but a triangulation can be used as well and in the present work, we will use triangulations. We consider triangulations whose set of vertices is $\{0, \theta, ..., q\theta\}$ and we wonder whether there exists such a triangulation with at most C different triangles up to translation? It is understood that the constant C must be independent of θ and q. This problem has already been solved in [2]. Nevertheless, the constant C which can be computed with the method of proof of this article is very large and very far from being optimal. Here we use a different approach which leads to the far better value $C = 10$. Moreover extensions in dimension greater than 3 seem possible with this approach while the method of [2] can hardly be used in dimension 3.

We state our result in $\mathbf{R}^2$ instead of $\mathbf{T}^2$ with a $\mathbf{Z}^2$-invariant triangulation:

Theorem 1 *Let $\theta = (\theta_1, \theta_2) \in \mathbf{R}^2$. Suppose that $1, \theta_1, \theta_2$ are linearly independent over the rational numbers. Then, for all integer q, there exists a triangulation $\mathcal{T}_q$ of $\mathbf{R}^2$ whose set of vertices is $\{0, \theta, ..., q\theta\} + \mathbf{Z}^2$ such that:*
i. $\mathcal{T}_q$ is invariant under $\mathbf{Z}^2$-translations,
ii. $\mathcal{T}_q$ has at most 10 different triangles up to translation,
iii. the maximum of diameters of triangles of $\mathcal{T}_q$ goes to zero when q tends to infinity.

From a dynamical systems point of view, ii is better formulated with Rokhlin towers. Going down to $\mathbf{T}^2$:
it is possible to tile the torus with ten triangles and their translates by θ.

[1] Nicolas Chevallier, Faculté des Sciences et Techniques, Université de Haute Alsace, 68100 Colmar, France

What about the shape of the triangles? The only general information we can give is the last point of the Theorem. For particular θ and/or particular q it is possible to give more informations (see [3]).

Our main tool is due to J. Kwapisz [6]. In fact, Kwapisz studied diffeomorphisms of the two dimensional torus with a single rotation number, a particular case of which are translations. In this latter case, Kwapisz's result implies that for infinitely many q, the constant $C = 6$ works (actually, Kwapisz used 3 parallelograms instead of 6 triangles). It has also been proved in [2] that this value $C = 6$ works for infinitely many q (not for the same set of q). The advantage of Kwapisz's approach is that it is far easier to fill the gap between the integers q for which $C = 6$ works.

Except for the condition about diameters, it is straightforward to extend our result in higher dimension because it is almost entirely contained in Kwapisz's result. The complete extension would require a new inequality about a well chosen multidimensional continued fraction algorithm (see lemma 7 below).

In section 2 we state the part of Kwapisz'result we need and prove it in the appendix. The third section is devoted to results about multidimensional continued fraction algorithms and finally Theorem 1 is proved in section 4.

2 Kwapisz's result for translation

We identify $\mathbf{R}^{d+1}$ with $\mathbf{R}^d \times \mathbf{R}$.

Definition 1 *Let $\theta \in \mathbf{R}^d \backslash \{0\}$. We say that a $(d+1) \times (d+1)$ matrix $M = \begin{pmatrix} P_1 & \cdots & P_{d+1} \\ q_d & \cdots & q_{d+1} \end{pmatrix}$ is a Farey matrix for θ if*
- the half line generated by $(\theta, 1)$ is in the positive quadrant span by the columns of M, i.e. $(\theta, 1) \in M\mathbf{R}_+^{d+1}$,
- $M \in SL(d+1, \mathbf{Z})$,
- $q_1, ..., q_{d+1} \geq 0$.

Definition 2 *The last coordinate of a vector in $\mathbf{Z}^{d+1}$ is called the denominator of this vector and the coefficients of the last row of a matrix in $SL(d+1, \mathbf{Z})$ are called the denominators of this matrix.*

Notations 1. Let $M = \begin{pmatrix} P_1 & \cdots & P_{d+1} \\ q_d & \cdots & q_{d+1} \end{pmatrix} \in SL(d+1, \mathbf{Z})$. The integer $\sum_{i=1}^{d+1} q_i$ is denoted by q_M and the point $\sum_{i=1}^{d+1} P_i \in \mathbf{Z}^d$ by P_M.
2. Let $\theta \in \mathbf{R}_+^d$. We denote by $\mathbf{p}_\theta : \mathbf{R}^{d+1} \to \mathbf{R}^d$ the projection defined by $\mathbf{p}_\theta(P, q) = P - q\theta$.
3. Let $M = (f_1, ..., f_{d+1}) \in SL(d+1, \mathbf{Z})$. For $i \in \{1, ..., d+1\}$, we denote by $F_i(M)$ the set

$$F_i(M) = \{\sum_{j \neq i} x_j f_j : x_j \in [0, 1]\}.$$

These are the bottom faces of the parallelepiped span by the vectors $f_1, ..., f_{d+1}$.

Next Theorem is a particular case of the aforementioned result of Kwapisz. The first (and the simplest) part of the proof of Kwapisz corresponds to this Theorem. We shall give a complete and elementary proof of this Theorem in the appendix; following Kwapisz, we will use stepped planes associated with Farey matrices.

Theorem 2 *Let $\theta \in \mathbf{R}^d \backslash \{0\}$ and $M = \begin{pmatrix} P_1 & \cdots & P_{d+1} \\ q_d & \cdots & q_{d+1} \end{pmatrix}$ be a Farey matrix for θ. Then the parallelepipeds*

$$P + k\theta + \mathbf{p}_\theta(-F_i(M)),\ P \in \mathbf{Z}^d,\ k = 0, ..., q_i - 1,\ i = 1, ..., d+1,$$

form a tiling of $\mathbf{R}^d$. Moreover the intersection of each of these parallelepipeds with the set $\{P + k\theta : P \in \mathbf{Z}^2,\ 0 \leq k < q_M\}$ is exactly its set of vertices.

One can decompose each d-dimensional parallelepiped $\mathbf{p}_\theta(-F_i(M))$ into $d!$ d-dimensional tetrahedrons. This leads to

Corollary 3 *There exists a d-dimensional triangulation of $\mathbf{R}^d$ whose set of vertices is $\mathbf{Z}^d + \{0, \theta, ..., (q_M - 1)\theta\}$ with at most $(d+1)!$ tetrahedrons up to translation.*

We shall need the two following simple results.

Lemma 4 *Let $\theta \in \mathbf{R}^d \backslash \{0\}$ and $M = (f_1, ..., f_{d+1})$ be a Farey matrix for θ with $(\theta, 1) = \sum_{i=1}^{d+1} x_i f_i$. Denote by i_0 an index such that $x_{i_0} = \max\{x_i : i = 1, ..., d+1\}$. Then*

$$\mathbf{p}_\theta(f_1 + ... + f_{d+1}) = \sum_{i \neq i_0} (1 - \frac{x_i}{x_{i_0}}) \mathbf{p}_\theta(f_i).$$

Consequently,

$$P_M - q_M \theta \in \mathbf{p}_\theta(F_{i_0}(M)).$$

Proof. Since

$$\begin{aligned} f_1 + ... + f_{d+1} &= \sum_{i=1}^{d+1} \frac{x_i}{x_{i_0}} f_i + \sum_{i=1}^{d+1} (1 - \frac{x_i}{x_{i_0}}) f_i \\ &= \frac{1}{x_{i_0}} (\theta, 1) + \sum_{i \neq i_0} (1 - \frac{x_i}{x_{i_0}}) f_i, \end{aligned}$$

$$\mathbf{p}_\theta(f_1 + ... + f_{d+1}) = \sum_{i \neq i_0} (1 - \frac{x_i}{x_{i_0}}) \mathbf{p}_\theta(f_i). \ \square$$

Corollary 5 *For all integer k such that $q_M \leq k < q_M + q_{i_0}$, $k\theta - P_M$ is in $(k - q_M)\theta + \mathbf{p}_\theta(-F_{i_0}(M))$.*

Proof. $k\theta - P_M = (k - q_M)\theta + (q_M \theta - P_M) \in (k - q_M)\theta + \mathbf{p}_\theta(-F_{i_0}(M))$. $\square$

3 Continued fraction expansions

Let $M = (f_1, ..., f_{d+1}) \in SL(d+1, \mathbf{Z})$. Denote by $\mathcal{C}_M$ the positive quadrant defined by $f_1, ..., f_{d+1}$

$$\mathcal{C}_M = \{\sum_{i=1}^{d+1} x_i f_i : x_1, ..., x_{d+1} \geq 0\}.$$

Definition 6 *(Brentjes, [1]) Given a line $l = \mathbf{R}X$ of $\mathbf{R}^{d+1}$, a continued fraction expansions of l is a sequence*

$$M_n = (f_{n,1}, ..., f_{n,d+1})$$

of matrices in $SL(d+1, \mathbf{Z})$ such that for all n, $X \in \mathcal{C}_{M_n}$ and there exist two integers (depending on n) $i_0 \neq i_1$ in $\{1, ..., d+1\}$ and a positive integer a_n such that for $i \neq i_1$, $f_{n+1,i} = f_{n,i}$ and

$$f_{n+1,i_1} = f_{n,i_1} + a_n f_{n,i_0}.$$

Most of the time, continued fraction expansions deal with a line l span by a non zero vector $X = (\theta, 1)$ in $\mathbf{R}^d \times \mathbf{R}$.

3.1 Brun's algorithm

Brun's algorithm associates with each matrix $M \in SL(d+1, \mathbf{Z})$ and each line $\mathbf{R}X$ where $X = \sum_{i=1}^{d+1} x_i f_i$ is a non-zero element of $\mathcal{C}_M$ a new matrix $M' = B(M, X)$ in $SL(d+1, \mathbf{Z})$:
Let i_0 be the index such that $x_{i_0} = \max\{x_i : i = 1, ..., d+1\}$ and let i_1 be the index such that $x_{i_1} = \max\{x_i : i \neq i_0\}$. The matrix $M' = (f'_1, ..., f'_{d+1})$ is defined by
- $f'_i = f_i$ for $i \neq i_1$,

- $f'_{i_1} = f_{i_1} + f_{i_0}$.

If $x_1, ..., x_{d+1}$ are linearly independent over $\mathbf{Q}$ the indices i_0 and i_1 are unique.

Starting with an initial matrix M_0, Brun's algorithm gives rise to the sequence of matrices $(M_n)_n$ defined by $M_{n+1} = M'_n$. In the following we will use Brun's algorithm combined with another continued fraction expansion. The only important things about Brun's algorithm are the relations between X, M and $M' = B(M, X)$.

First, by definition,

$$\begin{aligned} X = \sum_{i=1}^{d+1} x_i f_i &= x_{i_0} f_{i_0} + x_{i_1} f_{i_1} + \sum_{i \neq i_0, i_1} x_i f_i \\ &= (x_{i_0} - x_{i_1}) f_{i_0} + x_{i_1}(f_{i_1} + f_{i_0}) + \sum_{i \neq i_0, i_1} x_i f_i \\ &= \sum_{i=1}^{d+1} x'_i f'_i \end{aligned}$$

where $x'_i = x_i$ for $i \neq i_0$ and $x'_{i_0} = x_{i_0} - x_{i_1}$. Next:

Lemma 7 *Suppose $d = 2$. Endow $\mathbf{R}^3$ with the Euclidean distance. Then*

$$\max_{i=1,2,3} d(f'_i, \mathbf{R}X) \leq \max_{i=1,2,3} d(f_i, \mathbf{R}X).$$

Proof. This lemma is stated without proof in the book of Brentjes [1]. For convenience of the reader, we give its proof. Consider the plane H containing the points f_i, $i = 1, 2, 3$. Denote by r the maximum of the distances $d(f_i, \mathbf{R}X)$, $i = 1, 2, 3$, and denote by C the cylinder

$$C = \{p \in \mathbf{R}^3 : d(p, \mathbf{R}X) = r\}$$

Observe that $\mathbf{R}X$ is the axis of C. We have to show that $f_{i_0} + f_{i_1}$ is inside the cylinder C where $x_{i_0} = \max\{x_i : i = 1, 2, 3\}$ and $x_{i_1} = \max\{x_i : i \neq i_0\}$. We look at the picture in plane H. The curve $\mathcal{E} = H \cap C$ is an ellipse whose interior contains the three points $A_0 = f_{i_0}$, $A_1 = f_{i_1}$ and A_2 the remaining point among f_1, f_2 and f_3. Denote by O the center of the ellipse $\mathcal{E}$ and by G the center of gravity of the triangle $\mathrm{conv}(A_0, A_1, A_2)$. Let A_3 be symmetric to A_2 with respect to O. Denote by O' and G' the points such that $\overrightarrow{OO'} = \overrightarrow{OA_0} + \overrightarrow{OA_1}$ and $\overrightarrow{GG'} = \overrightarrow{GA_0} + \overrightarrow{GA_1}$. The definitions of i_0 and i_1 imply that O is in the triangle $\mathrm{conv}(G, A_0, A_1)$. On the one hand, this implies that A_2 and O are not on the same side of each of the lines A_0G and A_1G. Therefore, the triangle $\mathrm{conv}(A_0, A_1, A_3)$ contains G'. On the other hand, this implies that O' is in the triangle $\mathrm{conv}(G', A_0, A_1)$. It follows that O' is in the triangle $\mathrm{conv}(A_0, A_1, A_3)$. Since the points A_0, A_1, and A_3 are inside the ellipse $\mathcal{E}$, O' is also in $\mathcal{E}$. At last,

$$\begin{aligned} f_{i_0} + f_{i_1} &= (O + \overrightarrow{OA_0}) + (O + \overrightarrow{OA_1}) \\ &= 2O + \overrightarrow{OO'} \end{aligned}$$

is in the cylinder C. □

3.2 Strong convergence

Definition 8 *Let $X = (\theta, 1) \in \mathbf{R}^d \times \mathbf{R}$. A sequence of matrices $M_n = (f_{n,1}, ..., f_{n,d+1}) \in SL(d+1, \mathbf{Z})$ strongly converges to the line $\mathbf{R}X$ if*

$$\lim_{n \to 0} \max_{i=1}^{d} d(f_{n,i}, \mathbf{R}X) = 0.$$

Since the projection $\mathbf{p}_\theta$ maps the subspace orthogonal to X isomorphically on $\mathbf{R}^d$, the strong convergence is equivalent to

$$\lim_{n \to 0} \max_{i=1}^{d} |q_{n,i}\theta - P_{n,i}| = 0$$

where $M_n = \begin{pmatrix} P_{n,1} & \cdots & P_{n,d+1} \\ q_{n,1} & \cdots & q_{n,d+1} \end{pmatrix}$. Indeed, there is a positive constant $c = c(\theta)$ such that

$$\frac{1}{c} d(f_{n,i}, \mathbf{R}X) \leq |q_{n,i}\theta - P_{n,i}| \leq c d(f_{n,i}, \mathbf{R}X).$$

Most of the time, it is difficult to know whether a given multidimensional continued fraction expansion is strongly convergent or not. For instance, there exists $\theta = (\theta_1, \theta_2)$ such $1, \theta_1, \theta_2$ are independent over $\mathbf{Q}$ and such that the Brun's expansion of θ does not strongly converges ([1] p. 44-49). In any dimension ≥ 2, the same kind of examples for Jacobi-Perron's algorithm go back to Perron (see [1] p. 33-34, [7] p. 120). Nevertheless,

Theorem 3 *(Ferguson and Forcade) Let $\theta = (\theta_1, ..., \theta_d) \in \mathbf{R}^d$ such that $1, \theta_1, ..., \theta_d$ are linearly independent over the rational numbers and let M_0 be a Farey matrix for θ. Then there exists a strongly convergent continued fraction expansion of the line $\mathbf{R}(\theta, 1)$ starting with the matrix M_0.*

In fact Ferguson and Forcade have given an algorithm which, either stops in finitely many steps and find a linear relation between $1, \theta_1, ..., \theta_d$ with rational coefficients, or is strongly convergent (see [1], p. 123-127).

4 Proof of Theorem 1

Let $\theta = (\theta_1, \theta_2) \in \mathbf{R}^2$ and suppose that $1, \theta_1, \theta_2$ are linearly independent over the rational numbers. We can assume that θ_1 and θ_2 are in $]0, 1[$. By Theorem 3, there exists a sequence of matrices $M_k = \begin{pmatrix} P_{k,1} & P_{k,2} & P_{k,3} \\ q_{k,1} & q_{k,2} & q_{k,3} \end{pmatrix}$, $k \in \mathbf{N}$, of matrices in $SL(3, \mathbf{Z})$ such that $\lim_{k\to 0} \max_{i=1,2,3} |q_{k,i}\theta - P_{k,i}| = 0$. We choose the first matrix $M_0 = Id$, hence all the denominators $q_{k,i}$ are non negative.

Consider the parallelograms

$$\mathcal{P}_{k,i} = -\mathbf{p}_\theta(F_i(M_k)).$$

By Kwapisz's theorem, the parallelograms

$$\mathcal{P}_{k,i} + q\theta + P, \ q = 0, ..., q_{k,i} - 1, \ i = 1, 2, 3, \ P \in \mathbf{Z}^2$$

form a tiling of $\mathbf{R}^2$. Each parallelogram $\mathcal{P}_{k,i}$ can be decomposed into two triangles, this leads to $\mathbf{Z}^2$-invariant triangulations of $\mathbf{R}^2$ with at most 6 different triangles up to translation. Moreover, the vertices of this triangulation are the points $q\theta + P$ with $0 \leq q < q_{k,1} + q_{k,2} + q_{k,3} = q_{M_k}$. Therefore the Theorem is proved for all the integers of the form $n = q_{M_k} - 1$.

Suppose now that n is in the interval $[q_{M_k}, q_{M_{k+1}}[$. Denote by f_1, f_2 and f_3 the columns of M_k. We have $(\theta, 1) = \sum_{i=1}^{d+1} x_i f_i$. Let i_0 the index such that $x_{i_0} = \max\{x_i : i = 1, 2, 3\}$. By lemma 4, the parallelogram $\mathcal{P}_{k,i_0}$ contains the point $A = q_{M_k}\theta - P_{M_k}$. Moreover, A cannot be on the boundary of $\mathcal{P}_{k,i_0}$ for it would implies that $1, \theta_1$ and θ_2 are dependent over the rational numbers. Therefore, each parallelogram $\mathcal{P}_{k,i_0} + q\theta$ contains the point $A + q\theta$ in its interior for all q.

First suppose that n is in the interval $\{q_{M_k}, ..., q_{M_k} + q_{k,i_0} - 1\}$ and set $N = n - q_{M_k}$. For $q = 0, ..., N$ and $P \in \mathbf{Z}^2$, we decompose the parallelograms $\mathcal{P}_{k,i_0} + P + q\theta$ into the 4 triangles defined by the 4 sides of the parallelogram and the point $A + q\theta + P$. This leads to a $\mathbf{Z}^2$-invariant triangulation of the plane whose set of vertices is $\mathbf{Z}^2 + \{0, \theta, ..., n\theta\}$ and whose triangles are:
- two triangles for each parallelogram $\mathcal{P}_{k,i} + q\theta + P$ where $i \neq i_0$, $q \in \{0, ..., q_{k,i} - 1\}$ and $P \in \mathbf{Z}^2$,
- four triangles for each parallelogram $\mathcal{P}_{k,i_0} + q\theta + P$ where $q \in \{0, ..., N\}$ and $P \in \mathbf{Z}^2$,
- two triangles for each parallelogram $\mathcal{P}_{k,i_0} + q\theta + P$ where $q \in \{N + 1, ..., q_{k,i_0} - 1\}$.

Up to translation, there are 10 different triangles at most.

When $n = q_{M_k} + q_{k,i_0}$, we use Brun's algorithm to find the new Farey matrix $M' = B(M_k, (\theta, 1))$. By definition of Brun's algorithm, $q_{M'} = q_{M_k} + q_{k,i_0}$, thus, as before, we are able to find the desired triangulation for all the integers $n \in \{q_{M'}, q_{M'} + q'_{i_0} - 1\}$ where q'_{i_0} is the denominator of the vector f'_{i_0} associated with M' by Brun's algorithm. We can continue this process until $n = q_{M_{k+1}}$ and then restart with M_{k+1} instead of M_k. This shows that for all integers $n \geq 0$, there is a $\mathbf{Z}^2$-invariant triangulation with at most 10 different triangles up to translation.

It remains to see that the diameters of the triangles go to zero when n goes to infinity. By lemma 7, we know that for all $n \in \{q_{M_k}, ..., q_{M_{k+1}} - 1\}$, the diameters of the triangles are less than

$$2c \max_{i=1,2,3} d(f_{k,i}, \mathbf{R}(\theta, 1)).$$

Now, by our choice the of the sequence $(M_k)_k$ of Farey matrices, $\lim_{k\to\infty} \max_i |q_{k,i}\theta - P_{k,i}| = 0$, hence

$$\lim_{k\to\infty} \max_{i=1,2,3} d(f_{k,i}, \mathbf{R}(\theta, 1)) = 0. \square$$

5 Appendix : proof of theorem 2 and stepped Stepped hyperplane

In this section $M = \begin{pmatrix} P_1 & ... & P_{d+1} \\ q_1 & ... & q_{d+1} \end{pmatrix} = (f_1, ..., f_{d+1})$ is a matrix in $SL(d+1, \mathbf{Z})$ with non negative denominators $q_1, ..., q_{d+1}$.

Notations.

1. Let X be in $\mathbf{R}^{d+1}$ and not in $\mathbf{R}^d \times \{0\}$. We denote by $\mathbf{p}_X : \mathbf{R}^{d+1} \to \mathbf{R}^{d+1}$ the projection on $\mathbf{R}^d \times \{0\}$ along the line $\mathbf{R}X$.
2. We denote by Π^- and Π^+ the half spaces defined by

$$\Pi^- = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y \le 0\},$$
$$\Pi^+ = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} : y \ge 0\},$$

and by Q the hypercube

$$Q = [0, 1]^{d+1}.$$

3. Three subsets of $\mathbf{R}^{d+1}$ are associated with a matrix M. The first two are the "half-spaces"

$$H^- = H^-(M) = \cup(X + MQ)$$

where the union is taken over all the lattice point $X \in \mathbf{Z}^{d+1}$ such that the parallelepiped $X + MQ$ is included in Π^- and

$$H^+ = H^+(M) = \cup(X + MQ)$$

where the union is taken over all the lattice point $X \in \mathbf{Z}^{d+1}$ such that the parallelepiped $X + MQ$ is not included in Π^- (this means that $X + \sum_{i=1}^{d+1} f_i$ is in the interior of Π^+). The third subsets is the boundary ∂H^- of H^-, it is call the *stepped hyperplane* associated with M. Let us quote a few facts.

- $H^- \cup H^+ = \mathbf{R}^{d+1}$.
- Since $\det M = 1$, the interiors of two parallelepipeds $X + MQ$ and $Y + MQ$ with $X \neq Y$ both in $\mathbf{Z}^{d+1}$, are disjoint. It follows that the interiors of the parallelepipeds defining H^- do not intersect the interiors of the parallelepiped defining H^+, hence $(H^-)^o \cap (H^+)^o = \emptyset$. Therefore, $H^- \cap H^+ = \partial H^- = \partial H^+$.

We outline the proof of Theorem 2:
Step 1. A point $Y = (y_1, ..., y_{d+1}) \in \mathbf{Z}^{d+1}$ is in ∂H^- if and only if $-q_M < y_{d+1} \le 0$.
Step 2. For each X in the interior of $\mathcal{C}_M = M\mathbf{R}_+^{d+1}$, $\mathbf{p}_X$ is an homeomorphism of ∂H^- onto $\mathbf{R}^d \times \{0\}$.
Step 3. Denote by $\mathcal{F}_i$ the set of faces $Y - F_i(M)$, $Y \in \mathbf{Z}^{d+1}$, such that $Y - F_i(M) \subset \partial H^-$. A face $F = Y - F_i(M) \in \mathcal{F}_i$ if and only if $Y \in \mathbf{Z}^{d+1}$ and the vertices Y and $Y - \sum_{j\neq i} f_j$ of F are both in ∂H^-.
Step 4. A face $Y - F_i(M)$ with $Y = (y_1, ..., y_{d+1}) \in \mathbf{Z}^{d+1}$ is in $\mathcal{F}_i$ if and only if $q_i < y_{d+1} \le 0$.
Step 5. $\partial H^- = \cup_{i=1}^{d+1} \cup_{F\in\mathcal{F}_i} F$.
Step 6. Finally,

$$\partial H^- = \cup_{i=1}^{d+1} \cup_{q=0}^{q_i-1} \cup_{P\in\mathbf{Z}^{d+1}} [(P, -q) - F_i(M)],$$
$$\mathbf{R}^d \times \{0\} = \cup_{i=1}^{d+1} \cup_{q=0}^{q_i-1} \cup_{P\in\mathbf{Z}^{d+1}} [P + q\theta - \mathbf{p}_\theta(F_i(M))].$$

Although, these steps are geometrically clear, we prove them.

Step 1.

Let $Y = (y_1, ..., y_{d+1})$ be in $\mathbf{Z}^{d+1}$. On the one hand, H^- is included in $\mathbf{R}^d \times \mathbf{R}_-$, hence $Y \in \partial H^-$ implies $y_{d+1} \leq 0$. On the other hand H^+ is included in $\mathbf{R}^d \times [1 - q_M, +\infty[$, hence $Y \in \partial H^-$ implies $y_{d+1} > -q_M$.

Suppose now that $-\sum_{i=1}^{d+1} q_i < y_{d+1} \leq 0$. The parallelepiped $Y - \sum_{i=1}^{d+1} f_i + MQ$ is included in Π^-, hence Y is in H^-; the parallelepiped $Y + MQ$ is not included in Π^-, hence Y is in H^+. It follows that $Y \in H^+ \cap H^- = \partial H^-$.

Step 2.

Lemma 9 *1. If Y is in H^+ then $Y + \mathcal{C}_M$ is included in H^+.*
2. If Y is in H^- then $Y - \mathcal{C}_M$ is included in H^-.

Proof. The proofs of 1 and 2 are very similar and we will only prove 1. It is enough to prove that $Y + \mathcal{C}_M^o \subset H^+$. Let X be a non zero vector of $\mathbf{R}^{d+1}$ lying in the interior of $\mathcal{C}_M$. We have to prove that for all positive t, $Y + tX$ is in H^+. Since the set of non negative real numbers t such that $Y + tX$ is in H^+ is closed, it is enough to prove that for all such t, there exists $\varepsilon > 0$ such that for all $s \in [0, \varepsilon]$, $Y + (t+s)X$ is in H^+ .

Let t be a non negative real number such that $Y + tX \in H^+$. By definition of H^+, there exists a lattice point A such that $Y + tX \in A + MQ$ and $A + \sum_{i=1}^{d+1} f_i \notin \Pi^-$. By definition of MQ,

$$Y + tX = A + \sum_{i=1}^{d+1} t_i f_i$$

where $t_i \in [0, 1]$, $i = 1, ..., d+1$. Let J the set of indices i with $t_i = 1$. Clearly, the point $Y + tX$ is in the parallelepiped $\mathcal{P} = A + \sum_{i \in J} f_i + MQ$ and there exists $\varepsilon > 0$ such that $Y + (t+s)X \in \mathcal{P}$ for all $s \in [0, \varepsilon]$. Now, $\mathcal{P}$ contains the point $A + \sum_{i=1}^{d+1} f_i$ which is not in Π^-, hence $A + (t+s)X$ is in H^+ for all $s \in [0, \varepsilon]$. □

As a immediate consequence, we have:

Corollary 10 *If X is an interior point of $\mathcal{C}_M$ then $\mathbf{p}_X$ is a homeomorphism from ∂H to $\mathbf{R}^d \times \{0\}$.*

Proof. By the previous lemma, a line parallel to X cut ∂H^- in exactly one point, therefore $\mathbf{p}_X$ is a bijection between ∂H^- and $\mathbf{R}^d$. Since $\mathbf{p}_X$ is continuous, and since ∂H^- is closed and is inside the strip $\mathbf{R}^d \times [-\sum_{i=1}^{d+1} q_i, 0]$, it must be an homeomorphism. □

Step 3.

The only thing to prove is that if the vertices $Y \in \mathbf{Z}^{d+1}$ and $Z = Y - \sum_{j \neq i} f_j$ of the face $F = Y - F_i(M)$ are both in ∂H^- then $F \subset \partial H^-$. Since $H^- \subset \Pi^-$, $Y \in \Pi^-$. It follows that $Y - MQ \subset \Pi^-$. By assumption, Z is in $\partial H^- = \partial H^+$, therefore at least one of the parallelepipeds adjacent to Z must intersect the interior of the half space Π^+. Now the parallelepipeds adjacent to Z are of the shape $Z + MQ - \sum_{i \in I} f_i$ with $I \subset \{1, ..., d+1\}$, hence $Z + MQ$ intersects the interior of $\Pi^+$and therefore $Z + MQ \subset H^+$. Moreover, if $X = Y - \sum_{j \neq i} x_j f_j \in Y - F_i(M)$ then

$$\begin{aligned} X &= Y - \sum_{j \neq i} f_j + \sum_{j \neq i} (1 - x_j) f_j \\ &= Z + \sum_{j \neq i} (1 - x_j) f_j \in Z + MQ. \end{aligned}$$

Thus $Y - F_i(M)$ is the common face of the cubes $Y - MQ$ and $Z + MQ$ which shows that $Y - F_i(M) \subset H^- \cap H^+ = \partial H^-$.

Step 4.

Let $Y = (y_1, ..., y_{d+1}) \in \mathbf{Z}^{d+1}$ and $F = Y - F_i(M)$. By step 3, F is in $\mathcal{F}_i$ if and only if Y and $Y - \sum_{j \neq i} f_j$ are in ∂H^-. By the first step, this is equivalent to $-q_M < y_{d+1} \leq 0$ and $-q_M < y_{d+1} - \sum_{j \neq i} q_j \leq 0$. Hence, $F \in \mathcal{F}_i$ if and only if

$$0 \geq y_{d+1} > -q_M + \sum_{j \neq i} q_j = -q_i.$$

Step 5.

Choose X in the interior of $\mathcal{C}_M$. Clearly

$$\cup_{i=1}^{d+1} \cup_{F\in\mathcal{F}_i} F \subset \partial H^- \subset \cup_{i=1}^{d} \cup_{Y\in\mathbf{Z}^{d+1}} (Y - F_i(M)).$$

Let $E = E_{Y,i}$ be the relative boundary of a face $Y - F_i(M)$ (that is the boundary relatively to the hyperplane containing the face $Y - F_i(M)$). The projection $\mathbf{p}_X(E)$ is of dimension $\leq d-1$, hence $\mathbf{p}_X(E)$ is a subset of $\mathbf{R}^d$ with empty interior in $\mathbf{R}^d$. By step 2, $\mathbf{p}_X : \partial H^- \to \mathbf{R}^d$ is an homeomorphism, therefore the union of all $E_{Y,i} \cap \partial H^-$, $Y \in \mathbf{Z}^{d+1}$, $i = 1, ..., d+1$, is a subset of ∂H^- with empty interior. Moreover, $\cup_{i=1}^{d+1} \cup_{F\in\mathcal{F}_i} F$ is a closed subset of ∂H^-. Thus, the only thing to prove is that, if the relative interior of a face $Y - F_i(M)$ meets ∂H^- then this face is entirely included in ∂H^-. Denote by Q_Y^- the parallelepiped $Y - MQ$ and by Q_Y^+ the parallelepiped $Q_Y^- + f_i$. Let Z be in $\partial H^- \cap \operatorname{relint}(Y - F_i(M))$. Then, for $t > 0$ small enough, $Z - tX$ is the interior of Q_Y^- and $Z + tX$ is in the interior of Q_Y^+. Moreover by lemma 9, $Z - tX$ is in H^- and $Z + tX$ is in H^+. It follows that $Q_Y^+ \subset H^+$ and $Q_Y^- \subset H^-$. Therefore $F_i(M) \subset \partial H^-$.

Step 6 and end of proof of Theorem 2.

By steps 4 and 5,

$$\partial H^- = \cup_{i=1}^{d+1} \cup_{q=0}^{q_i-1} \cup_{P\in\mathbf{Z}^d} [(P, -q) - F_i(M)].$$

Hence, applying the projection $\mathbf{p}_\theta$,

$$\mathbf{R}^d = \cup_{i=1}^{d+1} \cup_{q=0}^{q_i-1} \cup_{P\in\mathbf{Z}^d} [P + q\theta - \mathbf{p}_\theta(F_i(M))].$$

Since the relative interiors of the faces

$$(P, -q) - F_i(M),\ P \in \mathbf{Z}^d,\ q = 0, ..., q_i - 1,\ i = 1, ..., d+1,$$

are disjoint and since by step 2, $\mathbf{p}_\theta : \partial H^- \to \mathbf{R}^d$ is an homeomorphism, the the parallelepipeds

$$P + q\theta + \mathbf{p}_\theta(-F_i(M)),\ P \in \mathbf{Z}^d,\ q = 0, ..., q_i - 1,\ i = 1, ..., d+1,$$

form a tiling of $\mathbf{R}^d$.

It remains to determine the vertices of this tiling. Let $G = A + k\theta + \mathbf{p}_\theta(-F_i(M))$ be such a parallelepiped and let V_G be its set of vertices. Since $0 \leq k < q_i$, by step 4, we know that $F = (A, -k) - F_i(M) \in \mathcal{F}_i$. Hence the set V_F of vertices of F is included in $\partial H^- \cap \mathbf{Z}^{d+1}$. Therefore

$$V_G = \mathbf{p}_\theta(V_F) \subset \mathbf{p}_\theta(\partial H^- \cap \mathbf{Z}^{d+1}),$$

and by step 1,

$$V_G \subset \{P + q\theta : P \in \mathbf{Z}^d,\ 0 \leq q < q_M\}.$$

Moreover, since $\det M = 1$, $F \cap \mathbf{Z}^{d+1} = V_F$. Hence,

$$\begin{aligned} G \cap \{P + q\theta : P \in \mathbf{Z}^2,\ 0 \leq k < q_M\} &= \mathbf{p}_\theta(F \cap \partial H^- \cap \mathbf{Z}^{d+1}) \\ &\subset \mathbf{p}_\theta(V_F) = V_G. \end{aligned}$$

Finally a point $P + q\theta$ with $P \in \mathbf{Z}^2$ and $0 \leq q < q_M$, belongs to at least one parallelepiped $A + k\theta + \mathbf{p}_\theta(-F_i(M))$ and therefore must be one of its vertices. □

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