NON COMPLETE AFFINE STRUCTURES ON LIE ALGEBRAS OF MAXIMAL CLASS

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January 23, 2001

Abstract

Every affine structure on Lie algebra $\mathfrak{g}$ defines a representation of $\mathfrak{g}$ in $aff(\mathbb{R}^n)$. If $\mathfrak{g}$ is irreducible of maximal class then the module associated is a sum of two irreducible submodules. We show that there exists on the model of Lie algebras of maximal class representations of second type which implies the existence of non complete affine structure on these Lie algebras of maximal class.

1 Affine structure on a nilpotent Lie algebra

1.1 Affine structure on nilpotent Lie algebras

Let $\mathfrak{g}$ be a n-dimensional Lie algebra over $\mathbb{R}$. A affine structure is given by a bilinear mapping

$$\nabla : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying

$$\left\{ \begin{array}{ll} 1) & \nabla(X,Y) - \nabla(Y,X) = [X,Y] \\ 2) & \nabla(X,\nabla(Y,Z)) - \nabla(Y,\nabla(X,Z)) = \nabla([X,Y],Z) \end{array} \right.$$

for all $X, Y, Z \in \mathfrak{g}$.

If $\mathfrak{g}$ is provided with an affine stucture, then the corresponding connected Lie group G is an affine manifold such that every left translation is an affine isomorphism of G. In this case, the operator ∇ is nothing that the connection operator of the affine connection on G.

Let $\mathfrak{g}$ be a Lie algebra with an affine structure ∇. Then the mapping

$$f: \quad \mathfrak{g} \rightarrow End(\mathfrak{g})$$

defined by

$$f(X)(Y) = \nabla(X,Y)$$

is a linear representation (non faithful) of $\mathfrak{g}$ satisfying

$$f(X)(Y) - f(Y)(X) = [X,Y] \qquad (*)$$

The adjoint representation $\widetilde{f}$ of $\mathfrak{g}$ satisfies

$$\widetilde{f}(X)(Y) - \widetilde{f}(Y)(X) = 2\,[X,Y]$$

and cannot correspond to an affine structure.

1.2 Classical examples of affine structure

i) Let $\mathfrak{g}$ be the n-dimensional abelian Lie algebra. Then the representation

$$\begin{array}{rccc} f: & \mathfrak{g} & \rightarrow & End(\mathfrak{g}) \\ & X & \mapsto & f(X) = 0 \end{array}$$

defines an affine structure.

ii) Let $\mathfrak{g}$ be an $2p$-dimensional Lie algebra endowed with a symplectic form :

$$\theta \in \Lambda^2 \mathfrak{g}^* such that d\theta = 0$$

with

$$d\theta(X,Y,Z) = \theta(X,[Y,Z]) + \theta(Y,[Z,X]) + \theta(Z,[X,Y]).$$

For every $X \in \mathfrak{g}$ we can define an unique endomorphism ∇_X by

$$\theta(adX(Y), Z) = -\theta(Y, \nabla_X(Z)).$$

Then $\nabla\,(X,Y) = \nabla_X(Y)$ is an affine structure on $\mathfrak{g}$.

iii) Following the work of Benoist $[Be]$ and Burde $[Bu]$, we know that exists nilpotent Lie algebra without affine structure.

1.3 Faithful representations associated to an affine structure

Let ∇ be an affine structure on an n-dimensional Lie algebra $\mathfrak{g}$. Let us consider the $(n+1)$-dimensional linear representation given by

$$\rho : \mathfrak{g} \to End\,(\mathfrak{g} \oplus \mathbb{R})$$

given by

$$\rho(X) : (Y,t) \mapsto (\nabla(X,Y) + tX, 0)$$

It is easy to verify that ρ is a faithful representation of dimension $n+1$.

We can note that this representation gives also an affine representation of $\mathfrak{g}$:

$$\begin{array}{rcc} \psi : & \mathfrak{g} \to & aff\,(\mathbb{R}^n) \\ & X \mapsto & \begin{pmatrix} A\,(X) & X \\ 0 & 0 \end{pmatrix} \end{array}$$

where $A\,(X)$ is the matrix of the endomorphisms $\nabla_X : Y \to \nabla\,(X,Y)$ in a given basis.

We say that the representation ρ is nilpotent if the endomorphisms $\rho(X)$ are nilpotent for every X in $\mathfrak{g}$.

Suppose that $\mathfrak{g}$ is a complex non abelian indecomposable nilpotent Lie algebra and let ρ be a faithful representation of $\mathfrak{g}$. Then there exists a faithful nilpotent representation of same dimension.

Proof: Let us consider the $\mathfrak{g}$-module M associated to ρ. Then, as $\mathfrak{g}$ is nilpotent, M can be decomposed as

$$M = i = 1\overset{k}{\oplus} M_{\lambda_i}$$

where M_{λ_i} is a $\mathfrak{g}$-submodule, and the λ_i are linear forms on $\mathfrak{g}$. For all $X \in \mathfrak{g}$, the restriction of $\rho\,(X)$ to M_i as the following form

$$\begin{pmatrix} \lambda_i(X) & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_i(X) \end{pmatrix}$$

Let $\mathbb{K}_{\lambda_i}$ be the one dimentional $\mathfrak{g}$-module defined by

$$\mu : X \in \mathfrak{g} \to \mu\,(X) \in End\mathbb{K}$$

with

$$\mu\,(X)\,(a) = \rho(X)\,(a) = \lambda_i(X)\,a$$

The tensor product $M_{\lambda_i} \otimes \mathbb{K}_{-\lambda_i}$ is the $\mathfrak{g}$-module associated to

$$X \cdot (Y \otimes a) = \rho\,(X)\,(Y) \otimes a - Y \otimes \lambda_i(X)\,a$$

Then $\widetilde{M} = \oplus\,(M_{\lambda_i} \otimes K_{-\lambda_i})$ is a nilpotent $\mathfrak{g}$-module. Let us prove that $\widetilde{M}$ is faithful. Recall that a representation ρ of $\mathfrak{g}$ is faithful if and only if $\rho(Z) \neq 0$ for every $Z \neq 0 \in Z(\mathfrak{g})$. Consider $X \neq 0 \in Z(\mathfrak{g})$. If $\widetilde{\rho}(X) = 0$, then $\rho(X)$ is a diagonal endomorphism. By hypothesis $\mathfrak{g} \neq Z(\mathfrak{g})$ and $\exists i \geq 1$ s.t $X \in \mathcal{C}^i(\mathfrak{g})$. Thus

$$X = j\sum a_j\,[Y_j, Z_j]$$

with $Y_j \in \mathcal{C}^{i-1}(\mathfrak{g})$ and $Z_j \in \mathfrak{g}$. The endomorphisms $\rho(Y_j)\rho(Z_j) - \rho(Z_j)\rho(Y_j)$ are nilpotent and the eigenvalues of $\rho(X)$ are 0. Thus $\rho(X) = 0$ and ρ is not faithful. Then. $\widetilde{\rho}(X) \neq 0$ and $\widetilde{\rho}$ is a faithful representation.

2 Affine structure on Lie algebra of maximal class

2.1 Definition

A n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is called of maximal class if the smallest k such that $\mathcal{C}^k\mathfrak{g} = \{0\}$ is equal to $n-1$.

In this case the descending sequence is

$$\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \cdots \supset \mathcal{C}^{n-2}\mathfrak{g} \supset \{0\} = \mathcal{C}^{n-1}\mathfrak{g}$$

and we have

$$\begin{cases} \dim \mathcal{C}^1\mathfrak{g} = n-2, \\ \dim \mathcal{C}^i\mathfrak{g} = n-i-1, fori = 1, ..., n-1. \end{cases}$$

The n-dimensional nilpotent Lie algebra L_n defined by

$$[X_1, X_i] = X_{i+1} fori \in \{2, ..., n-1\}$$

is of maximal class.

We can note that any Lie algebra of maximal class is a linear deformation of L_n $[G.K]$.

2.2 On non-nilpotent affine structure

Let be $\mathfrak{g}$ an n-dimensional Lie algebra of maximal class provided with an affine structure ∇. Let ρ be the $(n+1)$-dimensional faithful representation associated to ∇ and let us note $M = \mathfrak{g} \oplus \mathbb{C}$ the corresponding complex $\mathfrak{g}$-module. As $\mathfrak{g}$ is of maximal class, its decomposition has one of the following form

$$\begin{array}{l} M = M_0 \quad and M is irreducible, \\ or M = M_0 \oplus M_\lambda, \quad \lambda \neq 0 \end{array}$$

For a general faithful representation, let us call characteristic the ordered sequence of the dimensions of the irreducible submodules. In the case of maximal class we have $c(\rho) = (n+1)$ or $(n, 1)$ or $(n-1, 1, 1)$ or $(n-1, 2)$. In fact, the maximal class of $\mathfrak{g}$ implies that exists an irreducible submodule of dimension greater or equal to $n-1$. More generally, if the characteristic sequence of a nilpotent Lie algebra is equal to $(c_1, .., c_p, 1)$ (see [G.K]) then for every faithful represenation ρ we have $c(\rho) = (d_1, .., d_q)$ with $d_1 \geq c_1$.

Let $\mathfrak{g}$ be the Lie algebra of maximal class L_n. There are faithful $\mathfrak{g}$-modules which are not nilpotent.

Proof: Consider the following representation given by the matrices $\rho(X_i)$ where $\{X_1, .., X_n\}$ is a basis of $\mathfrak{g}$

$$\rho(X_1) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & & & 0 & 1 \\ a & a & 0 & & & & & \vdots & 0 \\ 0 & 0 & 0 & & & & & 0 & 0 \\ \vdots & \ddots & \frac{1}{2} & \ddots & & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots & 0 \\ \vdots & & & \ddots & \frac{i-3}{i-2} & \ddots & & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & \vdots & 0 \\ \alpha & \beta & 0 & \cdots & \cdots & 0 & \frac{n-3}{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(X_2) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & & \cdots & 0 & 0 \\ a & a & 0 & & & & & \vdots & 1 \\ -1 & 1 & 0 & & & & & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \ddots & & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots & 0 \\ \vdots & & & \ddots & \frac{1}{i-2} & \ddots & & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & \vdots & 0 \\ \beta & \alpha & 0 & \cdots & \cdots & \cdots & \frac{1}{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and for $j \geq 3$ the endomorphismes $\rho(X_j)$ satisfy :

$$\begin{cases} \rho(X_j)(e_1) = -\frac{1}{j-1} e_{j+1} \\ \rho(X_j)(e_2) = \frac{1}{j-1} e_{j+1} \\ \rho(X_j)(e_3) = \frac{1}{j(j-1)} e_{j+2} \\ \cdots \\ \rho(X_j)(e_{i-j+1}) = \frac{(j-2)!(i-j-1)!}{(i-2)!} e_i, \quad i = j-2, ..., n \\ \rho(X_j)(e_{i-j+1}) = 0, \quad i = n+1, .., n+j-1 \\ \rho(X_j)(e_{n+1}) = e_j \end{cases}$$

where $\{e_1, ..., e_n, e_{n+1}\}$ is the basis given by $e_i = (X_i, 0)$ and $e_{n+1} = (0,1)$. We easily verify that these matrices describe a non nilpotent faithful representation.

2.3 Non complete affine structure on L_n

The previous representation is associated to an affine strucure on the Lie algebra L_n given by

$$\nabla(X_i, Y) = \rho(X_i)(Y, 0)$$

where L_n is identified to the $n-$dimensional first factor of the $n+1$ dimensional faithful module. This affine strucure is complete if and only if the endomorphismes $R_X \in End(\mathfrak{g})$ defined by

$$R_X(Y) = \nabla(Y, X)$$

are nilpotent for all $X \in \mathfrak{g}$ ([H]). But the matrix of R_{X_1} has the form :

$$\begin{pmatrix} a & a & 0 & \cdots & 0 & \cdots & 0 & 0 \\ a & a & & & \vdots & & \vdots & 0 \\ 0 & -1 & & & \vdots & & \vdots & 0 \\ 0 & 0 & -\frac{1}{2} & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & 0 & \ddots & & \cdots & \vdots & 0 \\ 0 & 0 & \vdots & \ddots & -\frac{1}{j-1} & & \vdots & 0 \\ \alpha & \beta & \vdots & \cdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{n-2} & 0 \end{pmatrix}$$

Its trace is $2a$ and for $a \neq 0$ it is not nilpotent. We have proved :

There existe affine structures on the Lie algebra of maximal class L_n which are non complete.

The most simple example is on $\dim 3$ and concerns the Heisenberg algebra. We find a non nilpotent faithful representation associated to the non complete affine structure given by :

$$\nabla_{X_1} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \alpha & \beta & 0 \end{pmatrix}, \quad \nabla_{X_2} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \beta - 1 & \alpha + 1 & 0 \end{pmatrix}, \quad \nabla_{X_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\{X_1, X_2, X_3\}$ is a basis of H_3 satisfying $[X_1, X_2] = X_3$ and ∇_{X_i} the endomorphisms of $\mathfrak{g}$ given by

$$\nabla_{X_i}(X_j) = \nabla(X_i, X_j)$$

The affine representation is written

$$\begin{pmatrix} a(x_1+x_2) & a(x_1+x_2) & 0 & x_1 \\ a(x_1+x_2) & a(x_1+x_2) & 0 & x_2 \\ \alpha x_1 + (\beta-1)x_2 & \beta x_1 + (\alpha+1)x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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