NON EXISTENCE OF COMPLEX STRUCTURES ON FILIFORM LIE ALGEBRAS

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Abstract

The aim of this work is to prove the nonexistence of complex structures over nilpotent Lie algebras of maximal class (also called filiform).

1 Preliminaries

The study of invariant complex structures on real connected Lie Groups is reduced to the study of linear operators J on the corresponding Lie algebra which satisfies the Nijenhuis condition and $J^2 = -Id$. When the Lie algebra is even dimensional, real and reductive, the existence of such structures follows from the work of Morimoto [1]. On the other hand, there exist solvable and nilpotent Lie algebras which are not provided with a complex structure. Some recent works present classifications of nilpotent or solvable Lie algebras with complex structures in small dimension (dimension four for the solvable case [2] and six for the nilpotent case [3]). As every six dimensional nilpotent Lie algebra does not admit complex structure, we are conducted to determine the classes of nilpotent Lie algebra which are not provided with such a structure.

Let G be a real Lie group and $\mathfrak{g}$ its Lie algebra.

Definition 1 *An invariant complex structure on G is an endomorphism J of $\mathfrak{g}$ such that*

(1) $J^2 = -Id$

(2) $[JX, JY] = [X, Y] + J\,[JX, Y] + J\,[X, J(Y)]\,,\ \forall X, Y \in \mathfrak{g}$

(The second condition is called the Nijenhuis condition of integrability).

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For simplify, we will say that J is an invariant complex structure over the Lie algebra $\mathfrak{g}$.

The underlying real vector space V to the Lie algebra $\mathfrak{g}$ can be provided with a complex vector space structure by putting

$$(a+ib)\,.v = a.v + b.J(v)$$

$\forall v \in V, \forall a, b \in \mathbb{R}$. We note by V_J this complex vector space. We have

$$\dim_{\mathbb{C}} V_J = \frac{1}{2}\dim_{\mathbb{R}} V = \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{g}.$$

Definition 2 *The complex structure J is called bi-invariant if it satisfies*
(3) $[J, adX] = 0\ ,\ \forall X \in \mathfrak{g}$.

Let us remark that (3) implies (2).

If J is bi-invariant, then V_J is a complex Lie algebra, noted $\mathfrak{g}_J$, because in this case we have

$$[(a+ib)\,X, (c+id)\,Y] = (a+ib)\,(c+id)\,[X,Y]$$

2 Decomposition associated to a complex structure

Let J be an invariant complex structure on the real Lie algebra $\mathfrak{g}$. We can extend the endomorphism J in a natural way on the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{g}$. It induces on $\mathfrak{g}_{\mathbb{C}}$ a direct vectorial sum

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{i} \oplus \mathfrak{g}_{\mathbb{C}}^{-i}$$

where

$$\mathfrak{g}_{\mathbb{C}}^{\varepsilon i} = \{X \in \mathfrak{g}_{\mathbb{C}} \quad / \quad J(X) = \varepsilon i X\} \quad , \quad \varepsilon = \pm 1$$

The Nijenhuis condition (2) implies that $\mathfrak{g}_{\mathbb{C}}^{\varepsilon i}$ is a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. If σ denotes the conjugation on $\mathfrak{g}_{\mathbb{C}}$ given by $\sigma\,(X+iY) = X - iY$, then

$$\mathfrak{g}_{\mathbb{C}}^{-i} = \sigma(\mathfrak{g}_{\mathbb{C}}^{i}).$$

Proposition 1 *Let $\mathfrak{g}$ be a $2n$-dimensional real Lie algebra. It is provided with an invariant complex structure if and only if the complexification $\mathfrak{g}_{\mathbb{C}}$ admits a decomposition*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \sigma(\mathfrak{h})$$

where $\mathfrak{h}$ is a n-dimensional complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

If J is bi-invariant, condition (3) implies that $\mathfrak{g}_{\mathbb{C}}^{\varepsilon i}$ is an ideal of $\mathfrak{g}_{\mathbb{C}}$. In fact, if $X \in \mathfrak{g}_{\mathbb{C}}^{\varepsilon i}$ and $Y \in \mathfrak{g}_{\mathbb{C}}^{-\varepsilon i}$ we have

$$J[X,Y] = [JX,Y] = \varepsilon i\,[X,Y] = [X,JY] = -\varepsilon i\,[X,Y]$$

Then

$$[X,Y] = 0.$$

Proposition 2 *Let $\mathfrak{g}$ be an $2n$-dimensional real Lie algebra. Then $\mathfrak{g}$ admits a bi-invariant complex structure if and only if the complexification $\mathfrak{g}_{\mathbb{C}}$ is a direct sum of ideals I and $\sigma(I)$:*

$$\mathfrak{g}_{\mathbb{C}} = I \oplus \sigma(I)$$

Of course every n-dimensional complex Lie algebra $\mathfrak{h}$ comes from an $2n$-dimensional real Lie algebra endowed with a bi-invariant complex structure.

3 Bi-invariant complex structures

3.1 Nilpotent case

Let $\mathfrak{g}$ be a $2n$-dimensional real nilpotent Lie algebra with a bi-invariant complex structure. Then $\mathfrak{g}_{\mathbb{C}} = I \oplus \sigma(I)$ where I is a n-dimensional complex ideal of $\mathfrak{g}_{\mathbb{C}}$. This describes entirely the structure of the complexifications $\mathfrak{g}_{\mathbb{C}}$ of the real Lie algebras $\mathfrak{g}$ provided with bi-invariant complex structure.

Let $c(\mathfrak{n})$ the characteristic sequence of the complex nilpotent algebra $\mathfrak{n}$ ([4]). It is defined by

$$c(\mathfrak{n}) = \quad Max\{c(X) \quad | \quad X \in \mathfrak{n} - [\mathfrak{n},\mathfrak{n}]\}$$

where $c(X) = (c_1(X), ..., c_k(X), 1)$ is the sequence of similitude invariants of the nilpotent operator adX. We deduce that

$$c(\mathfrak{g}_{\mathbb{C}}) = (c_1, c_1, c_2, c_2, ..., 1, 1)$$

As the Jordan normal form of the nilpotent operators adX does not depend of the field of scalars, we have

Proposition 3 *If $\mathfrak{g}$ is an $2n$-dimensional real nilpotent Lie algebra which admits a bi-invariant complex structure, then its characteristic sequence is of type*

$$(c_1, c_1, c_2, c_2, ..., c_j, c_j, ..., 1, 1)\,.$$

A nilpotent Lie algebra is called of maximal class (or filiform in Vergne's terminology [5],[6]) if the descending sequence of derived ideals $\mathcal{C}^i\mathfrak{g}$ satisfies :

$$\begin{aligned}&\dim \mathcal{C}^1\mathfrak{g} = \dim \mathfrak{g} - 2\\ &\dim \mathcal{C}^i\mathfrak{g} = \dim \mathfrak{g} - i - 1 \quad \forall i \geq 2\end{aligned}$$

Corollary 1 *There is no bi-invariant complex structure over a filiform Lie algebra.*

In fact, the characteristic sequence of a filiform Lie algebra is $(2n-1,1)$. Thus, by proposition 3, a filiform Lie algebra cannot admit a bi-invariant complex structure.

Let be $\{U_j\}$ and $\{V_j\}$ the basis of the complex ideals I and $\sigma(I)$. Respect to $\mathfrak{g}$, we can write $U_l = X_l - iY_l$, $V_l = X_l + iY_l$, where $\{X_1, ..., X_n, Y, ..., Y_n\}$ is a real basis of $\mathfrak{g}$. As $[U_l, V_l] = 0$, we obtain

$$[X_l - iY_l, X_k + iY_k] = [X_l, X_k] + [Y_l, Y_k] + i\left([X_l, Y_k] - [Y_l, X_k]\right) = 0$$

Thus

$$\begin{array}{l} [X_l, X_k] = -[Y_l, Y_k] \\ [X_l, Y_k] = [Y_l, X_k] \qquad k = 1, ..., n \quad ; \quad l = 1, ..., n. \end{array}$$

Likewise $[U_l, U_k] \in I$ and $[V_l, V_k] \in \sigma(I)$ imply

$$\begin{array}{l} [X_l, X_k] - [Y_l, Y_k] \in \mathbb{C}\{X_1, ..., X_k\} \\ [X_l, Y_k] + [Y_l, Y_k] \in \mathbb{C}\{Y_1, ..., Y_k\} \end{array}$$

Suppose that I admits a real basis. In this case we have

$$\begin{array}{lll} [X_l, X_k] = \sum_{j=1}^{n} C_{lk}^{j} X_j & , & [Y_l, Y_k] = -\sum_{j=1}^{n} C_{lk}^{j} Y_j \\ [X_l, Y_k] = \sum_{j=1}^{n} D_{lk}^{j} Y_j & , & [Y_l, X_k] = \sum_{j=1}^{n} D_{lk}^{j} Y_j \end{array}$$

with C_{lk}^j and D_{lk}^j in $\mathbb{R}$. Let be $\mathfrak{g}_0 = \mathbb{R}\{X_1, ..., X_n\}$ and $\mathfrak{g}_1 = \mathbb{R}\{Y_1, ..., Y_n\}$. Then the previous relations show that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and

$$\begin{cases} [\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0 \\ [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0 \\ [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1 \end{cases}$$

which gives a structure of $\mathbb{Z}_2$-graded Lie algebra on $\mathfrak{g}$.

3.2 On the classification of nilpotent Lie algebras with bi-invariant complex structures

If $\mathfrak{g}$ is a $2n$-dimensional real nilpotent Lie algebra with a bi-invariant complex structure, then $\mathfrak{g}_{\mathbb{C}} = I \oplus \sigma(I)$. Then the classification of associated complexification $\mathfrak{g}_{\mathbb{C}}$ corresponds to the classification of complex nilpotent Lie algebra I. By this time, this classification is known only up to dimension 7, and for special classes up to dimension 8. To obtain a general list of these algebras is therefore a hopeless pretention. From [3] we can extract the list for the 6-dimensional case. From [4] we can present the classification of complexifications $\mathfrak{g}_{\mathbb{C}}$ of real nilpotent Lie algebras with bi-invariant complex structure up the dimension 14. Here we will present briefly the classification in the real case up dimension 8.

Dimension 2 : $\mathfrak{g}_2^1$ is abelian.

Dimension 4 : $\mathfrak{g}_4^1$ is abelian.
In fact $\mathfrak{g}_{\mathbb{C}} = I \oplus \sigma(I)$ and I is an abelian ideal.
Dimension 6 : - $\mathfrak{g}_6^1$ is abelian.

- $\mathfrak{g}_6^2 : \begin{cases} [X_2, X_3] = -[Y_2, Y_3] = X_1 \\ [X_2, Y_3] = [Y_2, X_3] = Y_1 \end{cases}$

Remark 1 *1. The complexification of $\mathfrak{g}_6^2$ is isomorphic to $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ where $\mathfrak{h}_3$ is the 3-dimensional Heisenberg Lie algebra.*

2. Let us note that the real nilpotent Lie algebra $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ doesn't have a bi-invariant complex structure. In fact if $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ is a basis of $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ with brackets

$$[X_1, X_2] = X_3, \quad [X_4, X_5] = X_6,$$

then every isomorphism J which commutes with adX for every X satisfies $J(X_3) = \alpha X_3$ and $J(X_6) = \beta X_6$. As $J^2 = -Id$, we will have $\alpha^2 = \beta^2 = -1$ and $\alpha, \beta \in \mathbb{R}$.

3. The Lie algebra $\mathfrak{g}_6^2$ is 2-step nilpotent and it is a Lie algebra of type H (see [7]).

Dimension 8 : - $\mathfrak{g}_8^1$ is abelian.

- $\mathfrak{g}_8^2 = \mathfrak{g}_6^2 \oplus \mathfrak{g}_2^1$
- $\mathfrak{g}_8^3 : \begin{cases} [X_2, X_4] = X_1 & ; \quad [Y_2, Y_4] = -X_1 \\ [X_2, Y_4] = +Y_1 & ; \quad [Y_2, X_4] = +Y_1 \\ [X_3, X_4] = X_2 & ; \quad [Y_3, Y_4] = -X_2 \\ [X_3, Y_4] = Y_2 & ; \quad [Y_3, X_4] = Y_2 \end{cases}$
- $\mathfrak{g}_8^4 \begin{cases} [X_2, X_3] = X_1 & ; \quad [Y_2, Y_3] = -X_1 \\ [X_2, Y_3] = Y_1 & ; \quad [Y_2, X_3] = Y_1 \\ [X_2, X_4] = Y_1 & ; \quad [Y_2, Y_4] = -Y_1 \\ [X_2, Y_4] = -X_1 & ; \quad [Y_2, X_4] = -X_1 \end{cases}$

3.3 On the classification of solvable Lie algebras with bi-invariant complex structures

As known, there is, up an isomorphism, only one 2-dimensional non nilpotent solvable Lie algebra, denoted by $\mathfrak{r}_2^2$ and defined by $[X_1, X_2] = X_1$ over the basis $\{X_1, X_2\}$. This Lie algebra does not admit a bi-invariant complex structure. Thus, a non abelian solvable Lie algebra admitting it is at least four dimensional.

If we want classify all the 4-dimensional solvable Lie algebras admitting bi-invariant complex structures we can use the Dozias'classification, which can be found in [8] (see also [9]).

From this classification, we can affirm :

Theorem 1 : *Every 4-dimensional real solvable Lie algebras which admits a bi-invariant complex structure is isomorphic to one of the following :*
- $\mathfrak{r}_4^1 = \mathfrak{g}_4^1$ *the abelian Lie algebra.*

- $\mathfrak{r}_4^2 = \begin{cases} [X_1, X_3] = X_3 \\ [X_1, X_4] = X_4 \\ [X_2, X_3] = -X_4 \\ [X_2, X_4] = X_3 \end{cases}$

In the last case, any bi-invariant complexe structures satisfy

$$\begin{array}{rcl} J(X_1) & = & aX_1 + bX_2, \\ J(X_2) & = & -bX_1 + aX_2 \\ J(X_3) & = & aX_3 + bX_4, \\ J(X_4) & = & -bX_3 + aX_4. \end{array}$$

Its complexification $\mathfrak{r}_{4\mathbb{C}}^2$ is isomorphic to $\mathfrak{r}_2^2 \times \mathfrak{r}_2^2$. Let us note that the real Lie algebra $\mathfrak{r}_2^2 \times \mathfrak{r}_2^2$ which has the same complexification as $\mathfrak{r}_4^2$ is not provided with a bi-invariant complex structure.

4 Non Existence of invariant complex structures over nilpotent Lie algebras of maximal class

In section 3, we have given the definition of nilpotent Lie algebras of maximal class (or filiform Lie algebra).The simplest example is given by the following n-dimensional Lie algebra, denoted L_n :

$$[X_1, X_i] = X_{i+1}, \quad i = 2, ..., n-1$$

where the nondefined brackets are zero or obtained by antisymmetry.

Let $\mathfrak{n}$ be a n-dimensional filiform Lie algebra. Then there exists a basis $\{X_1, X_2, ..., X_n\}$ which is adapted to the flag

$$\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \mathcal{C}^2\mathfrak{g} \supset ... \supset \mathcal{C}^{n-1}\mathfrak{g} = \{0\}$$

with $\dim \mathcal{C}^1\mathfrak{g} = n-2$, $\dim \frac{\mathcal{C}^i\mathfrak{g}}{\mathcal{C}^{i-1}\mathfrak{g}}$ =1, $i \geq 1$, and which satisfies :

$$\begin{cases} [X_1, X_i] = X_{i+1}, \quad i = 2, ..., n-1 \\ [X_i, X_j] = \sum_{k \geq i+j} C_{ij}^k X_k \end{cases}$$

The change of basis $Y_1 = X_1$, $Y_i = tX_i$, $i \geq 2$, $t \neq 0$, shows that this Lie algebra is isomorphic to the following :

$$\begin{cases} [X_1, X_i] = X_{i+1}, \quad i = 2, ..., n-1 \\ [X_i, X_j] = t \sum C_{ij}^k X_k \end{cases}$$

4.1 Invariant complex structures and filiform Lie algebras

Proposition 4 *The real nilpotent filiform Lie algebra L_{2n} does not admit an invariant complex structure.*

Proof . Let T be a linear isomorphism of the real vector space L_{2n} satisfying the Nijenhuis condition :

$$[T(X),T(Y)]=[X,Y]+T\,[X,T(Y)]+T\,[T(X),Y]$$

where $[,]$ is the bracket of L_{2n}. Consider the basis $\{X_1,...,X_{2n}\}$ of L_{2n} satisfying :

$$\begin{cases} [X_1,X_i]=X_{i+1}, & i=2,...,2n-1 \\ [X_i,X_j]=0, & i,j\neq 1. \end{cases}$$

We have

$$\begin{aligned} [T(X_{2n-1}),T(X_{2n})] &= [X_{2n-1},X_{2n}]+T\,[X_{2n-1},T(X_{2n})]+T\,[T(X_{2n-1}),X_{2n}] \\ &= T\,[X_{2n-1},T(X_{2n})] \end{aligned}$$

As

$$[X_{2n-1},X_1]=-X_{2n}$$

we obtain

$$[T(X_{2n-1}),T(X_{2n})]=T\,[X_{2n-1},T(X_{2n})]=-aT(X_{2n})$$

where

$$T(X_{2n})=aX_1+\sum_{i\geq 2}a_iX_i.$$

The nilpotency of L_{2n} implies that the constant a which appears as an eigenvalue of $ad(T(X_{2n-1}))$ is zero. Then

$$[T(X_{2n-1}),T(X_{2n})]=0.$$

This implies that

$$[T(X_i),T(X_{2n})]=T\,[X_i,T(X_{2n})]=0$$

for $i=2,...,2n$. If $T(X_{2n})\notin Z(L_{2n})=\mathbb{R}\{X_{2n}\}$, then $T(X_i)=\sum_{j\geq 2}a_{ij}X_j$ for $j=2,...,2n$. As T is non singular, necessarily we have

$$T(X_1)=a_{11}X_1+\sum_{i\geq 2}a_{i1}X_i$$

with $a_{11}\neq 0$. The condition $T^2=-Id$ implies $a_{11}^2=-1$. As $a_{11}\in\mathbb{R}$, this constitutes a contradiction. Thus $T(X_{2n})\in Z(L_{2n})=\mathbb{R}\{X_{2n}\}$, and it follows that $T(X_{2n})=\alpha X_{2n}$. As above, we can conclude $\alpha^2=-1$. This case is also excluded and the proposition is proved.

We will show that the non existence of invariant complexe structure on the model filiform L_{2n} is always true in any deformation of this algebra.

Theorem 2 *There does not exist invariant complex structure over a real filiform Lie algebra.*

Proof. Let $\mathfrak{g}$ be a $2n$-dimensional real filiform Lie algebra and let $\mathfrak{g}_{\mathbb{C}}$ be its complexification. If there a complex structure J on $\mathfrak{g}$, then $\mathfrak{g}_{\mathbb{C}}$ admits the following decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \sigma(\mathfrak{g}_1)$$

where $\mathfrak{g}_1$ is a n-dimensional sub-algebra of $\mathfrak{g}_{\mathbb{C}}$ and σ the conjugated automorphism. As the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is filiform, there is an adapted basis $\{X_1, .., X_{2n}\}$ satisfying

$$(*) \begin{cases} [X_1, X_i] = X_{i+1}, \quad 2 \leq i \leq 2n-1 \\ [X_2, X_3] = \sum_{k \geq 5} C_{23}^k X_k \\ \mathcal{C}^i(\mathfrak{g}) = \mathbb{R}\{X_{i+2}, ..., X_n\} \end{cases}$$

In particular we have

$$\dim \frac{\mathfrak{g}_{\mathbb{C}}}{\mathcal{C}^1(\mathfrak{g}_{\mathbb{C}})} = 2, \quad \dim \frac{\mathcal{C}^i(\mathfrak{g}_{\mathbb{C}})}{\mathcal{C}^{i+1}(\mathfrak{g}_{\mathbb{C}})} = 1 \quad , \quad i \geq 1$$

The ordered sequence of the dimension of Jordan blocks of the nilpotent operator adX_1 is $(n-1, 1)$. Such a vector is called characteristic vector.

Lemma 1 *Every characteristic vector can be written as* $Y = \alpha X_1 + U$ *where* U *is in the complex vector space generated by* $\{X_2, ..., X_{2n}\}$ *and* $\alpha \neq 0$.

It follows that the set of characteristic vectors of $\mathfrak{g}_{\mathbb{C}}$ is the open set $\mathfrak{g}_{\mathbb{C}} - \mathbb{C}\{X_2, .., X_{2n}\}$.

Lemma 2 *Either* $\mathfrak{g}_1$ *or* $\sigma(\mathfrak{g}_1)$ *contains a characteristic vector of* $\mathfrak{g}_{\mathbb{C}}$.

Observe that otherwise we would have $\mathfrak{g}_1 \subset \mathbb{C}\{X_2, .., X_{2n}\}$ and $\sigma(\mathfrak{g}_1) \subset \mathbb{C}\{X_2, .., X_{2n}\}$, which contradicts the previous decomposition.

Thus $\mathfrak{g}_1$ or $\sigma(\mathfrak{g}_1)$ is a $n-$dimensional complex filiform Lie algebra. But if $Y \in \mathfrak{g}_1$ is a characteristic vector of $\mathfrak{g}_1$, then $\sigma(Y)$ is a characteristic vector of $\sigma(\mathfrak{g}_1)$ with the same characteristic sequence. Then every $2n-$dimensional filiform Lie algebra appears as a direct vectorial sum of two n-dimensional filiform Lie algebras. We shall prove that it is impossible. More precisely we have

Proposition 5 *Let* $n \geq 3$. *Then no* $2n-$*dimensional complex filiform Lie algebra is a vectorial direct sum of two* $n-$*dimension filiform sub-algebras.*

Proof. Let $\mathfrak{g}_{\mathbb{C}}$ be a filiform Lie algebra of dimension $2n$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. From the previous lemma, one of them for example $\mathfrak{g}_1$, contains a characteristic vector. Let be X_1 this vector an $\{X_1, X_2, .., X_{2n}\}$ the corresponding basis. This implies that $\mathfrak{g}_1 \cap \mathbb{C}\{X_2, .., X_{2n}\} = \mathfrak{g}_1 \cap \mathbb{C}\{X_{n+1}, .., X_{2n}\}$. But $\mathfrak{g}_2$ cannot contain characteristic vector of $\mathfrak{g}$, if not $\mathfrak{g}_2 \cap \mathbb{C}\{X_2, .., X_{2n}\} = \mathfrak{g}_2 \cap \mathbb{C}\{X_{n+1}, .., X_{2n}\}$ and this is at variance with $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then $\mathfrak{g}_2 \subset \mathbb{C}\{X_2, .., X_{2n}\}$. But from the brackets in $(*)$ this is impossible as soon as $n > 2$.

The four dimensional case can be treated directly. Up to isomorphism, there exists only one 4 dimensional filiform Lie algebra, L_4, for which we have proven the nonexistence of invariant complex structures. Note that, respect to the basis used in $(*)$, this algebra admits the decomposition $L_4 = \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$,where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are the abelian subalgebras generated respectively by $\{X_1, X_4\}$ and $\{X_2, X_3\}$.

4.2 Consequence.

Let J be an invariant complex structure on a Lie algebra $\mathfrak{g}$. Let us note by μ the law (the bracket) of $\mathfrak{g}$ and let us consider the Chevalley cohomology of $\mathfrak{g}$. The coboundary operator is denoted by δ_μ.

Proposition 6 *We have*

$$\delta_\mu J = \mu_J$$

where μ_J is the law of Lie algebra, isomorphic to μ, defined by

$$\mu_J(X,Y) = J^{-1}(\mu(J(X),J(Y)).$$

In fact the Nijenhuis condition is written as:

$$\mu(JX,JY) = \mu(X,Y) + J\mu(JX,Y) + J\mu(X,J(Y))$$

Then

$$J^{-1}\mu(JX,JY) = J^{-1}\mu(X,Y) + \mu(JX,Y) + \mu(X,J(Y))$$

that is, as $J^2 = -Id$

$$\begin{aligned} J^{-1}\mu(JX,JY) &= -J\mu(X,Y) + \mu(JX,Y) + \mu(X,J(Y)) \\ &= \delta_\mu J(X,Y) \end{aligned}$$

Corollary 2 *If $\mathfrak{g}$ is a filiform Lie algebra,then there does not exist 2-coboundaries for the Chevalley cohomology such that*

$$\delta_\mu(J) = \mu_J$$

where $J^2 = -Id$

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