LOW-DIMENSIONAL FILIFORM LIE SUPERALGEBRAS

MARC GILG

ABSTRACT

The aim of this paper is to give a classification up to isomorphisme of low dimension filiform Lie superalgebras.

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1. INTRODUCTION

There exists a lot of work concerning Lie superalgebras. But less of them are interested in nilpotent Lie superalgebras, although, the case of nilpotent Lie algebras has been well studied, see for example [1]. This fact is a consequence of the development of deformation theory. In this paper, we will focus a particular class of nilpotent Lie superalgebras. We recall some definitions from [3]. We point out the definition of filiform Lie superalgebras. This definition is analogue to the definition of filiform Lie algebras given by Vranceanu [10] and Vergne [9]. The classification of this super-algebras is still a problem, as the classification of filiform Lie algebras over an algebraically closed field of characteristic zero is only done up to dimension 11 [4]. In this paper, we will give a first classification for filiform Lie superalgebras in low dimensions.

2. —FILIFORM LIE SUPERALGEBRAS

A $\mathbb{Z}_2$-graded vector space $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ over an algebraically closed field is a Lie superalgebra if there exists a bilinear product $[,]$ over $\mathcal{G}$ such that

$$[G_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta \mod 2},$$

$$[g_\alpha, g_\beta] = -(-1)^{\alpha.\beta}[g_\beta, g_\alpha]$$

for all $g_\alpha \in \mathcal{G}_\alpha$ and $g_\beta \in \mathcal{G}_\beta$ and satisfying Jacobi's super-relation :

$$(-1)^{\gamma.\alpha}[A,[B,C]] + (-1)^{\alpha.\beta}[B,[C,A]] + (-1)^{\beta.\gamma}[C,[A,B]] = 0$$

for all $A \in \mathcal{G}_\alpha$, $B \in \mathcal{G}_\beta$ and $C \in \mathcal{G}_\gamma$.

For such a Lie superalgebra we define the lower central series by

$$\begin{cases} C^0(\mathcal{G}) = \mathcal{G}, \\ C^{i+1}(\mathcal{G}) = [\mathcal{G}, C^i(\mathcal{G})] \end{cases}$$

A Lie superalgebra $\mathcal{G}$ is nilpotent if there exists an integer n such that $C^n(\mathcal{G}) = \{0\}$.

This definition is not easy to use. Therefore we define the following sequences for a Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$:

$$C^0(\mathcal{G}_0) = \mathcal{G}_0, \qquad C^{i+1}(\mathcal{G}_0) = [\mathcal{G}_0, C^i(\mathcal{G}_0)]$$

and

$$C^0(\mathcal{G}_1) = \mathcal{G}_1, \qquad C^{i+1}(\mathcal{G}_1) = [\mathcal{G}_0, C^i(\mathcal{G}_1)]$$

Theorem 2.1. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a Lie superalgebras. Then $\mathcal{G}$ is nilpotent if and only if there exist (p, q) such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$.*

Proof. If the Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is nilpotent it is easy to prove that there exist (p, q) such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$.

For the converse, we use the classical Engel's theorem. Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a Lie superalgebra. Assume that there exist (p, q) such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$, then $ad(X)$ with $X \in \mathcal{G}_0$ is nilpotent. We have for $Y \in \mathcal{G}_1$:

$$ad(Y) \circ ad(Y) = \frac{1}{2} ad([Y, Y])$$

As $[Y, Y]$ is an element of $\mathcal{G}_0$, then $ad([Y, Y])$ is nilpotent. This implies that $ad(Y)$ is nilpotent for every $Y \in \mathcal{G}_1$. Hence $ad(X)$ and $ad(Y)$ is nilpotent, using Engel's theorem given in [6], $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is a nilpotent Lie superalgebra. $\square$

Definition 2.1. Let $\mathcal{G}$ be a nilpotent Lie superalgebra, the super-nilindex of $\mathcal{G}$ is the pair (p, q) such that : $C^p(\mathcal{G}_0) = \{0\}$, $C^{p-1}(\mathcal{G}_0) \neq \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$, $C^{q-1}(\mathcal{G}_1) \neq \{0\}$. It is and invariant up to isomorphism.

Definition 2.2 (Filiform Lie superalgebras)**.** Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a nilpotent Lie superalgebra with $\dim \mathcal{G}_0 = n + 1$ and $\dim \mathcal{G}_1 = m$. $\mathcal{G}$ is called filiform if it's super-nilindex is (n, m). We will note $\mathcal{F}_{n,m}$ the set of filiform Lie superalgebras.

Let's define the set $\mathcal{N}_{n,m}^{p,q}$ of Lie superalgebras with $\dim \mathcal{G}_0 = n + 1$, $\dim \mathcal{G}_1 = m$ and with super-nilindex (k_0, k_1) such that $k_0 \leq p$ and $k_1 \leq q$. It is obvious that this set can be define by polynomial relations given by the Jacobi relations and the nilpotency relations. This prove that it is a Zariski-closed set of the algebraic variety of nilpotent Lie superalgebras denoted by $\mathcal{N}_{n,m}$.

The set $\mathcal{F}_{n,m}$ of filiform Lie superalgebras can be written as the complementary of a close set :

$$\mathcal{F}_{n,m} = \mathcal{N}_{n,m} \setminus (\mathcal{N}_{n,m}^{n-1,m} \mathcal{N}_{n,m}^{n,m-1})$$

Hence the set of filiform Lie superalgebras is an open set of the variety of nilpotent Lie superalgebras (see [5]).

3. Classifications of filiforms over $\mathbb{C}$ in low dimensions

3.1. **Adapted basis.** Like for the filiform Lie algebras [9], there exists an adapted bases of a filiform Lie superalgebra :

Proposition 3.1. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a filiform Lie superalgebra with $\dim \mathcal{G}_0 = n+1$ and $\dim \mathcal{G}_1 = m$. Then there exists a bases $\{X_0, X_1, \ldots X_n, Y_1, Y_2, \ldots Y_m\}$ of $\mathcal{G}$ with $X_i \in \mathcal{G}_0$ and $Y_i \in \mathcal{G}_1$ such that :*

$$\begin{cases} [X_0, X_i] = X_{i+1} \quad 1 \leq i \leq n-1, \quad [X_0, X_n] = 0; \\ [X_1, X_2] \in \mathbb{C}.X_4 + \mathbb{C}.X_5 + \cdots + \mathbb{C}.X_n; \\ [X_0, Y_i] = Y_{i+1} \quad 1 \leq i \leq m-1, \quad [X_0, Y_m] = 0. \end{cases}$$

Such basis are called adapted.

Proof. Let $gr\mathcal{G} \in \mathcal{F}_{n,m}$ be a graded filiform lie superalgebra. The lower central series implies the following graduations :

$$gr\mathcal{G}_0 = W_1^0 \oplus W_2^0 \oplus \cdots \oplus W_{n-1}^0$$
$$gr\mathcal{G}_1 = W_1^1 \oplus W_2^1 \oplus \cdots \oplus W_{m-1}^1$$

with $\dim W_1^0 = 2$, $\dim W_i^0 = 1$ for $2 \leq i \leq n-1$ and $\dim W_j^1 = 1$ for $1 \leq j \leq m-1$. Then we have $[W_1^0, W_i^0] = W_{i+1}^0$ and $[W_1^0, W_j^1] = W_{j+1}^1$.

Let $x_i \in W_i^0$ and $y_j \in W_j^1$ be non zero elements, then if $w \in W_1^0$,

$$[w, x_i] = \lambda_i^0(w)\ x_{i+1}$$
$$[w, y_j] = \lambda_j^1(w)\ y_{j+1}$$

the maps $w \to \lambda_i^0(w)$ and $w \to \lambda_j^1(w)$ are non zero linear forms on W_1^0, because $[W_1^0, W_i^0] = W_{i+1}^0$ and $[W_1^0, W_j^1] = W_{j+1}^1$.
As the field is $\mathbb{C}$, there exists x_0 in W_1^0 such that $\lambda_i^0(w) \neq 0$ for $1 \leq i \leq n-1$ and $\lambda_j^1(w) \neq 0$ for $1 \leq j \leq m-1$. Hence it exist a bases $X_0, X_1, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ with X_i (resp. Y_j) multiple of x_i (resp. y_j) such that :

$$[X_0, X_i] = X_{i+1} \text{ for } 1 \leq i \leq n-1$$
$$[X_0, Y_j] = Y_{j+1} \text{ for } 1 \leq j \leq m-1$$
$$[X_1, X_2] = a\ X_3 \text{ for } a \in \mathbb{C}$$

Substitute X_1 by $X_1 - a\ X_0$ we get the adapted bases of $gr\mathcal{G}$:

$$[X_0, X_i] = X_{i+1} \text{ for } 1 \leq i \leq n-1$$
$$[X_0, Y_j] = Y_{j+1} \text{ for } 1 \leq j \leq m-1$$
$$[X_1, X_2] = 0$$

If the Lie superalgebras is not graded, then we introduce a graded Lie superalgebra, as it was done for Lie algebras in [9]. □

3.2. **Adapted changes of basis.**

Definition 3.1. Let f be a graded change of bases of a filiform Lie superalgebra $\mathcal{G}$. f is an adapted changes of bases if f is Lie superalgebra homomorphism and if the image of an adapted bases of $\mathcal{G}$ is an adapted bases.

Proposition 3.2. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a filiform Lie superalgebra. Let $f = f_0 + f_1$ be an adapted change of basis of $\mathcal{G}$. Then f_0 is an adapted change of bases of the filiform Lie algebra $\mathcal{G}_0$ and f_1 satisfies :*

$$\begin{cases} f_1(Y_1) = d_1\ Y_1 + d_2\ Y_2 + \cdots + d_m\ Y_m \\ f_1(Y_i) = [f_0(X_0), f_1(Y_{i-1})] \quad 2 \leq i \leq m \end{cases}$$

with the condition :

$$d_1 \prod_{j=1}^{m-1} A_j \neq 0$$

where the A_j's are defined by :

$$A_j = a_0 + \sum_{k=0}^{j-1} (-1)^k C_{j-1}^k \left(\sum_{i=1}^{n-k} a_i . r_{i+k,2+k} \right)$$

and the product of $\mathcal{G}$ is given in the adapted bases by

$$[X_i, Y_1] = \sum_{s=2}^{m} r_{i,s}\ Y_s \quad 1 \leq i \leq n$$

Proof. The proof can be found in [3]. We give a sketch of it.

First, we etablish that f_0 is an adapted change of bases for Lie algebras as in [4]. The we set :

$$\begin{cases} f_0(X_0) = a_0\ X_0 + a_1\ X_1 + \ldots a_n\ X_n \\ f_1(Y_1) = d_1\ Y_1 + d_2\ Y_2 + \ldots d_m\ Y_m \end{cases}$$

and assuming $f(Y_t) = [f(X_0), f(Y_{t-1})]$ for $2 \leq t \leq m$ we prove by induction on t that :

$$f(Y_t) = d_1 \prod_{p=1}^{t-1} A_p.Y_t + \sum_{p \geq t+1} d_p.Y_p$$

This implies that $f_1(Y_m) = d_1 \prod_{p=1}^{m-1} A_p \ Y_m$. For this vector to be non zero, we must have $d_1 \prod_{p=1}^{m-1} \neq 0$ This prove that

$$f(Y_t) = d_1 \prod_{p=1}^{t-1} A_p.Y_t + \sum_{p \geq t+1} d_p.Y_p$$

start we a non zero componant on Y_t, and then the images of the vectors Y_t by f form a bases.

Conversely, let f_0 be an adapted change of bases of filiform Lie algebras. Then the map $f = f_0 + f_1$, with f_1 given by :

$$\begin{cases} f_1(Y_1) = d_1 \ Y_1 + d_2 \ Y_2 + \dots d_m \ Y_m \\ f_1(Y_i) = [f_0(X_0), f_1(Y_{i-1})] \quad 2 \leq i \leq m \end{cases}$$

and the condition :

$$d_1 \prod_{j=1}^{m-1} A_j \neq 0$$

is an adapted change of basis. □

3.3. **Classification in low dimensions.** We have a description of the products of the filiform Lie superalgebras on $\mathcal{F}_{1,m}$. There exists two types :

(1)
$$\begin{cases} [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1 \\ [Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}} \ X_1, \text{ if } i+j \text{ even and } 2 \leq i+j \leq m+1 \end{cases}$$

the other products vanish.

(2)
$$\begin{cases} [X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1 \\ [X_1, Y_r] = \sum_{s=2}^{m} r_s Y_{s+r-1} \text{ with } s+r-1 \leq m \end{cases}$$

the other products vanish.

Using adapted chages of bases, like it was done for Lie algebras in [4], we can eliminate the parameters $a_{\frac{i+j}{2}}$.

Theorem 3.1. *Every filiform Lie superalgebra of $\mathcal{F}_{1,m}$ which is not a Lie algebra is isomorphic to one of the following filiform Lie superalgebras :*

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \leq i \leq m-1$$
$$[Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \leq i \leq k$$

with $1 \leq k \leq z+1$ if $m = 2z+1$ and $1 \leq k \leq z$ if $m = 2z$.

Proof. We can assume that every non trivial filiform Lie superalgebra with $1 \le p \le k-1$ is isomorphic to :

$$[X_0, Y_i] = Y_{i+1}, \quad 1 \le i \le m-1$$
$$[Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \le i \le k$$
$$[Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}}\ X_1, \text{ if } i+j \text{ even and } 2 \le i+j \le 2(k-p)$$

for $1 \le p \le k-1$. The other products vanish.

Using this change of adapted bases :

$$\begin{cases} X_0^1 = X_0 \\ X_1^1 = X_1 \\ Y_1^1 = Y_1 - (-1)^p \dfrac{a_{k-1}}{2} Y_{2p+1} \\ \dots \\ Y_{k-p}^1 = Y_{k-p} - (-1)^p \dfrac{a_{k-1}}{2} Y_{k+p} \\ Y_{k-p+s}^1 = Y_{k-p+s} - (-1)^p \dfrac{a_{k-1}}{2} Y_{k+p+s} \end{cases}$$

we can reduce to

$$\begin{cases} [X_0, Y_i] = Y_{i+1}, \quad 1 \le i \le m-1 \\ [Y_i, Y_{2k-i}] = (-1)^{k-i} X_1 \quad 1 \le i \le k \\ [Y_i, Y_j] = (-1)^{\frac{i-j}{2}} a_{\frac{i+j}{2}}\ X_1, \text{ if } i+j \text{ even and } 2 \le i+j \le 2(k-p-1) \end{cases}$$

The other products vanish.

By induction on p, we have complet classification. □

By using adapted changes of bases, we etablished the following classifications :
$\mathcal{F}_{1,2}$:

$$(1)\ \left\{ [X_0, Y_1] = Y_2 \right.$$

$$(2)\ \begin{cases} [X_0, Y_1] = Y_2 \\ [X_1, Y_1] = Y_2 \end{cases}$$

$$(3)\ \begin{cases} [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_1 \end{cases}$$

$\mathcal{F}_{1,3}$:

$$(1)\ \left\{ [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \right.$$

$$(2)\ \begin{cases} [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_1, Y_1] = X_1 \end{cases}$$

$$(3)\ \begin{cases} [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [Y_2, Y_2] = X_1, \quad [Y_1, Y_3] = -X_1 \end{cases}$$

$$(4)\ \begin{cases} [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_2, \quad [X_1, Y_2] = Y_3 \end{cases}$$

$$(5)\ \begin{cases} [X_0, Y_1] = Y_2, \quad [X_0, Y_2] = Y_3 \\ [X_1, Y_1] = Y_3 \end{cases}$$

$$(6)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2+Y_3, \quad [X_1,Y_2]=Y_3 \end{cases}$$

$\mathcal{F}_{1,4}$:

$$(1)\ \{\ [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4$$

$$(2)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [Y_1,Y_1]=X_1 \end{cases}$$

$$(3)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [Y_2,Y_2]=X_1, \quad [Y_1,Y_3]=-X_1 \end{cases}$$

$$(4)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_2, \quad [X_1,Y_2]=Y_3, \quad [X_1,Y_3]=Y_4 \end{cases}$$

$$(5)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_3, \quad [X_1,Y_2]=Y_4 \end{cases}$$

$$(6)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_4 \end{cases}$$

$$(7)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_2+Y_3, \quad [X_1,Y_2]=Y_3+Y_4, \quad [X_1,Y_3]=Y_4 \end{cases}$$

$$(8)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_2+Y_4, \quad [X_1,Y_2]=Y_3, \quad [X_1,Y_3]=Y_4 \end{cases}$$

$$(9)\ \begin{cases} [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3, \quad [X_0,Y_3]=Y_4 \\ [X_1,Y_1]=Y_2+Y_3+2\ Y_4, \quad [X_1,Y_2]=Y_3+Y_4, \quad [X_1,Y_3]=Y_4 \end{cases}$$

$\mathcal{F}_{2,2}$:

$$(1)\ \{\ [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2$$

$$(2)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\dfrac{1}{2}X_2 \end{cases}$$

$$(3)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_2 \end{cases}$$

$$(4)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(5)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_2 \end{cases}$$

$\mathcal{F}_{2,3}$:

$$(1)\ \{\ [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3$$

$$(2)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2, \quad [X_1,Y_2]=Y_3 \end{cases}$$

$$(3)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_3 \end{cases}$$

$$(4)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_2]=-Y_3, \quad [X_2,Y_1]=Y_3 \end{cases}$$

$$(5)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2+Y_3, \quad [X_1,Y_2]=Y_3 \end{cases}$$

$$(6)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2, \quad [X_2,Y_1]=Y_3 \end{cases}$$

$$(7)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=2Y_2, \quad [X_1,Y_2]=Y_3, \quad [X_2,Y_1]=Y_3 \end{cases}$$

$$(8)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [Y_1,Y_1]=X_2 \end{cases}$$

$$(9)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2, \quad [X_1,Y_2]=Y_3 \\ [Y_1,Y_1]=X_2 \end{cases}$$

$$(10)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_3 \\ [Y_1,Y_1]=X_2 \end{cases}$$

$$(11)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2+Y_3, \quad [X_1,Y_2]=Y_3 \\ [Y_1,Y_1]=X_2 \end{cases}$$

$$(12)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [Y_1,Y_3]=-X_2, \quad [Y_2,Y_2]=X_2 \end{cases}$$

$$(13)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_1]=Y_2, \quad [X_1,Y_2]=Y_3 \\ [Y_1,Y_3]=-X_2, \quad [Y_2,Y_2]=X_2 \end{cases}$$

$$(14)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\frac{1}{2}X_2 \end{cases}$$

$$(15)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [X_1,Y_2]=-Y_3, \quad [X_2,Y_1]=Y_3 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\frac{1}{2}X_2 \end{cases}$$

$$(16)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,Y_1]=Y_2, \quad [X_0,Y_2]=Y_3 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\frac{1}{2}X_2, \quad [Y_1,Y_3]=-X_2, \quad [Y_2,Y_2]=X_2 \end{cases}$$

$\mathcal{F}_{3,2}$:

$$(1)\quad \{ [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2$$

$$(2)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_1, \quad [Y_1,Y_2]=\frac{1}{2}X_2, \quad [Y_2,Y_2]=\frac{1}{2}X_3 \end{cases}$$

$$(3)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_2, \quad [Y_1,Y_2]=\frac{1}{2}X_3 \end{cases}$$

$$(4)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_3 \end{cases}$$

$$(5)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(6)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_3 \end{cases}$$

$\mathcal{F}_{4,2}$:

$$(1)\ \{\ [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2$$

$$(2)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_2, \quad [Y_1,Y_2]=\frac{1}{2}X_3, \quad [Y_2,Y_2]=\frac{1}{2}X_4 \end{cases}$$

$$(3)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_3, \quad [Y_1,Y_2]=\frac{1}{2}X_4 \end{cases}$$

$$(4)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_4 \end{cases}$$

$$(5)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(6)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_4 \end{cases}$$

$$(7)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4 \end{cases}$$

$$(8)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [Y_1,Y_1]=X_3, \quad [Y_1,Y_2]=\frac{1}{2}X_4 \end{cases}$$

$$(9)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [Y_1,Y_1]=X_4 \end{cases}$$

$$(10)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,Y_1]=Y_2 \end{cases}$$

$$(11)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_4 \end{cases}$$

$\mathcal{F}_{5,2}$:

$$(1)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \end{cases}$$

$$(2)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_3, \quad [Y_1,Y_2]=\frac{1}{2}X_4, \quad [Y_2,Y_2]=\frac{1}{2}X_5 \end{cases}$$

$$(3)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\frac{1}{2}X_5 \end{cases}$$

$$(4)\ \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [Y_1,Y_1]=X_5 \end{cases}$$

$$(5)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(6)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(7)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5 \end{cases}$$

$$(8)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [Y_1,Y_1]=X_3, \quad [Y_1,Y_2]=\frac{1}{2}X_4, \quad [Y_2,Y_2]=\frac{1}{2}X_5 \end{cases}$$

$$(9)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\frac{1}{2}X_5 \end{cases}$$

$$(10)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(11)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,Y_1]=Y_2 \end{cases}$$

$$(12)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(13)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \end{cases}$$

$$(14)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [Y_1,Y_1]=X_5 \end{cases}$$

$$(15)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(16)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(17)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \end{cases}$$

$$(18)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(19)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [X_1,Y_1]=Y_2 \end{cases}$$

$$(20)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(21)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [X_1,Y_1]=-Y_2 \end{cases}$$

$$(22)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2, \quad [X_1,X_4]=-X_5 \\ [X_2,X_3]=X_5, \quad [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\dfrac{1}{2}X_5 \end{cases}$$

$$(23)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5, \quad [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(24)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \end{cases}$$

$$(25)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [Y_1,Y_1]=X_5 \end{cases}$$

$$(26)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=Y_2 \end{cases}$$

$$(27)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=Y_2, \quad [Y_1,Y_1]=X_5 \end{cases}$$

$$(28)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=-Y_2 \end{cases}$$

$$(29)\quad \begin{cases} [X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5 \\ [X_0,Y_1]=Y_2 \\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5 \\ [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\dfrac{1}{2}X_5 \end{cases}$$

$$(30)\begin{cases}[X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5\\ [X_0,Y_1]=Y_2\\ [X_1,X_2]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5\\ [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_5\end{cases}$$

$$(31)\begin{cases}[X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5\\ [X_0,Y_1]=Y_2\\ [X_1,X_2]=X_4+X_5, \quad [X_1,X_3]=X_5, \quad [X_1,Y_1]=-2Y_2\end{cases}$$

$$(32)\begin{cases}[X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5\\ [X_0,Y_1]=Y_2\\ [X_1,X_2]=X_4+X_5, \quad [X_1,X_3]=X_5, \quad [X_1,Y_1]=-2Y_2, \quad [Y_1,Y_1]=X_5\end{cases}$$

$$(33)\begin{cases}[X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5\\ [X_0,Y_1]=Y_2\\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5\\ [X_1,Y_1]=-Y_2\end{cases}$$

$$(34)\begin{cases}[X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5\\ [X_0,Y_1]=Y_2\\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5\\ [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_4, \quad [Y_1,Y_2]=\frac{1}{2}X_5\end{cases}$$

$$(35)\begin{cases}[X_0,X_1]=X_2, \quad [X_0,X_2]=X_3, \quad [X_0,X_3]=X_4, \quad [X_0,X_4]=X_5\\ [X_0,Y_1]=Y_2\\ [X_1,X_2]=X_4, \quad [X_1,X_3]=X_5, \quad [X_1,X_4]=-X_5, \quad [X_2,X_3]=X_5\\ [X_1,Y_1]=-Y_2, \quad [Y_1,Y_1]=X_5\end{cases}$$

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Marc GILG, Université de Haute-Alsace, Laboratoire de Mathématiques, 4 rue des Frères Lumière, 68 093 Mulhouse Cedex, France

E-mail address: `M.Gilg@univ-mulhouse.fr`