ON DEFORMATIONS OF THE FILIFORM LIE SUPERALGEBRA $L_{n,m}$

M. GILG

Abstract

Many work was done for filiform Lie algebras defined by M. Vergne [8]. An interesting fact is that this algebras are obtained by deformations of the filiform Lie algebra $L_{n,m}$. This was used for classifications in [4]. Like filiform Lie algebras, filiform Lie superalgebras are obtained by nilpotent deformations of the Lie superalgebra $L_{n,m}$. In this paper, we recall this fact and we study even cocycles of the superalgebra $L_{n,m}$ which give this nilpotent deformations. A family of independent bilinear maps will help us to describe this cocycles. At the end an evaluation of the dimension of the space $Z_0^2(L_{n,m}, L_{n,m})$ is established. The description of this cocycles can help us to get some classifications which was done in [2, 3].

1. Deformation of Lie superalgebras

1.1. Nilpotent Lie superalgebras.

Definition 1.1. A $\mathbb{Z}_2$-graded vector space $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ over an algebraic closed field is a Lie superalgebra if there exists a bilinear product $[,]$ over $\mathcal{G}$ such that

$$[G_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta \mod 2},$$
$$[g_\alpha, g_\beta] = (-1)^{\alpha.\beta}[g_\beta, g_\alpha]$$

for all $g_\alpha \in \mathcal{G}_\alpha$ and $g_\beta \in \mathcal{G}_\beta$ and satisfying Jacobi identity:

$$(-1)^{\gamma.\alpha}[A,[B,C]] + (-1)^{\alpha.\beta}[B,[C,A]] + (-1)^{\beta.\gamma}[C,[A,B]] = 0$$

for all $A \in \mathcal{G}_\alpha$, $B \in \mathcal{G}_\beta$ and $C \in \mathcal{G}_\gamma$.

For such a Lie superalgebra we define the lower central series

$$\begin{cases} C^0(\mathcal{G}) = \mathcal{G}, \\ C^{i+1}(\mathcal{G}) = [\mathcal{G}, C^i(\mathcal{G})]. \end{cases}$$

Definition 1.2. A Lie superalgebra $\mathcal{G}$ is nilpotent if there exist an integer n such that $C^n(\mathcal{G}) = \{0\}$.

We define for a Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ two sequences :

$$C^0(\mathcal{G}_0) = \mathcal{G}_0, \qquad C^{i+1}(\mathcal{G}_0) = [\mathcal{G}_0, C^i(\mathcal{G}_0)]$$

and

$$C^0(\mathcal{G}_1) = \mathcal{G}_1, \qquad C^{i+1}(\mathcal{G}_1) = [\mathcal{G}_0, C^i(\mathcal{G}_1)]$$

Theorem 1.1. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a Lie superalgebras. Then $\mathcal{G}$ is nilpotent if and only if there exist $(p,q) \in \mathbb{N}^2$ such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$.*

Proof. If the Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is nilpotent the existence of (p, q) such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$ is obvious.

For the converse, assume that there exist (p, q) such that $C^p(\mathcal{G}_0) = \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$, then every operator $ad(X)$ with $X \in \mathcal{G}_0$ is nilpotent. Let $Y \in \mathcal{G}_1$, as

$$ad(Y) \circ ad(Y) = \frac{1}{2} ad([Y, Y])$$

$[Y, Y]$ is an element of $\mathcal{G}_0$, then $ad([Y, Y])$ is nilpotent. This implies that $ad(Y)$ is nilpotent for every $Y \in \mathcal{G}_1$. By Engel's theorem for Lie superalgebras [6], this implies that $\mathcal{G}$ is nilpotent Lie superalgebra. □

Definition 1.3. Let $\mathcal{G}$ be a nilpotent Lie superalgebra, the super-nilindex of $\mathcal{G}$ is the pair (p, q) such that : $C^p(\mathcal{G}_0) = \{0\}$, $C^{p-1}(\mathcal{G}_0) \neq \{0\}$ and $C^q(\mathcal{G}_1) = \{0\}$, $C^{q-1}(\mathcal{G}_1) \neq \{0\}$. It is and invariant up to isomorphism.

1.2. **Cohomology.** We recall some definition from [1].

By definition, the superspace of q-dimensional cocycles of the Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ with coefficient in the $\mathcal{G}$-module $A = A_0 \oplus A_1$ is given by

$$C^q(\mathcal{G}; A) = \bigoplus_{q_0+q_1=q} Hom(\bigwedge^{q_0} \mathcal{G}_0 \otimes \bigvee^{q_1} \mathcal{G}_1, A).$$

This space is graded by $C^q(\mathcal{G}; A) = C_0^q(\mathcal{G}; A) \oplus C_0^q(\mathcal{G}; A)$ with

$$C_p^q(\mathcal{G}; A) = \bigoplus_{\substack{q_0+q_1=q \\ q_1+r \equiv p \mod 2}} Hom(\bigwedge^{q_0} \mathcal{G}_0 \otimes \bigvee^{q_1} \mathcal{G}_1, A_r)$$

The differential

$$d : \quad C^q(\mathcal{G}; A) \longrightarrow C^{q+1}(\mathcal{G}; A)$$

defined by

$$\begin{aligned}
&d\ c(u_1, \ldots, u_{q_0}, v_1, \ldots, v_{q_1}) \\
&= \sum_{1 \leq s < t \leq q_0} (-1)^{s+t-1}\ c([u_s, u_t], u_1, \ldots, \hat{g}_s, \ldots, \hat{g}_t, \ldots, u_{q_0}, v_1, \ldots, v_{q_1}) \\
&\quad + \sum_{s=1}^{q_0} \sum_{t=1}^{q_1} (-1)^{s-1}\ c(u_1, \ldots, \hat{g}_s, \ldots u_{q_0}, [u_s, v_t], v_1, \ldots, \hat{h}_t, \ldots, v_{q_1}) \\
&\quad + \sum_{1 \leq s < t \leq q_1} c([v_s, v_t], u_1, \ldots, u_{q_0}, v_1, \ldots, \hat{h}_s, \ldots, \hat{h}_t, \ldots v_{q_1}) \\
&\quad + \sum_{s=1}^{q_0} (-1)^s\ u_s\ c(u_1, \ldots, \hat{g}_s, \ldots u_{q_0}, v_1, \ldots v_{q_1}) \\
&\quad + (-1)^{q_0-1} \sum_{s=1}^{q_1} v_s\ c(u_1, \ldots u_{q_0}, v_1, \ldots, \hat{h}_s, \ldots, v_{q_1})
\end{aligned}$$

where $c \in C^q(\mathcal{G}; A)$, $u_1, \ldots, u_{q_0} \in \mathcal{G}_0$ and $v_1, \ldots, v_{q_1} \in \mathcal{G}_1$ and satisfies

$$\begin{cases} d \circ d = 0 \\ d(C_p^q(\mathcal{G}; A)) \subset C_p^{q+1}(\mathcal{G}; A). \end{cases}$$

for $q = 0, 1, 2 \ldots$ and $p = 0, 1$.

Let be $d_j : C_p^j(\mathcal{G};A) \longrightarrow C_p^{j+1}(\mathcal{G};A)$ with $p=0$ or $p=1$ the restriction of d to the space $C_p^j(\mathcal{G};A)$. This operator permit to define the spaces :

$$H_p^j(\mathcal{G},A) = \frac{Z_p^j(\mathcal{G};A)}{B_p^j(\mathcal{G};A)}$$

where $p=0$ or $p=1$. Therefore we have :

- $Z^j(\mathcal{G},A) = Z_0^j(\mathcal{G},A) \oplus Z_1^j(\mathcal{G},A)$.
- $B^j(\mathcal{G},A) = H_0^j(\mathcal{G},A) \oplus B_1^j(\mathcal{G},A)$.
- $H^j(\mathcal{G},A) = H_0^j(\mathcal{G},A) \oplus H_1^j(\mathcal{G},A)$.

1.3. **Algebraic variety of nilpotent Lie superalgebras.** We recall some facts from [5].

Let $\mathcal{L}_{p,q}^n$ be the set of Lie superalgebras law over $\mathbb{C}^n = \mathbb{C}^{p+1} \oplus \mathbb{C}^q$. Let $(X_1, X_2, \dots X_{p+1}, Y_1, Y_2, \dots Y_q)$ be a graded base of it. For $\mu \in \mathcal{L}_{p,q}^n$ we set :

$$\begin{cases} \mu(X_i,X_j) = \sum_{k=1}^{p+1} C_{i,j}^k \; X_k & 1 \le i < j \le p \\ \mu(X_i,Y_j) = \sum_{k=1}^{q} D_{i,j}^k \; Y_k & 1 \le i \le p+1,\ 1 \le j \le q \\ \mu(Y_i,Y_j) = \sum_{k=1}^{p+1} E_{i,j}^k \; X_k & 1 \le i \le j \le q \end{cases}$$

with $C_{j,i}^k = -C_{i,j}^k$ and $E_{j,i}^k = E_{i,j}^k$.

The elements $\{C_{i,j}^k, D_{i,j}^k, E_{i,j}^k\}_{i,j,k}$ are called *structure constants* of the Lie superalgebra with respect to basis $(X_1, \dots, X_{p+1}, Y_1, \dots, Y_q)$. The Jacobi identities show that $\mathcal{L}_{p,q}^n$ is an algebraic sub-variety of $\mathbb{C}^N$ with

$$N = (p+1)^2(\frac{p}{2}) + 2(p+1)q^2.$$

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space of dimension n with $\dim V_0 = p+1$ and $\dim V_1 = q$. Let $G(V)$ be the group of linear map of the type $g = g_0 + g_1$ where $g_0 \in GL(V_0)$ and $g_1 \in GL(V_1)$. This group is isomorphic to $GL(V_0) \times GL(V_1)$.

The algebraic group $G(V)$ acts on the variety $\mathcal{L}_{p,q}^n$ in the following way :

$$(g.\phi)(x,y) = g_{\alpha+\beta}(\phi(g_\alpha^{-1}(a), g_\beta^{-1}(b))) \quad \forall a \in V_\alpha,\ \forall b \in V_\beta,$$

with $g \in G(V)$ and $\phi \in \mathcal{L}_{p,q}^n$.

1.4. **Deformations of Lie superalgebras.** Let $\mathcal{G}$ be a Lie superalgebra over a field k, V be the underlying vector space and ν_0 the law of $\mathcal{G}$. Let $k[[t]]$ be the power series ring in one variable t. Let $V[[t]]$ be the $k[[t]]$-module $V[[t]] = V \otimes_k k[[t]]$. One can obtain an extension of V with a structure of vector space by extending the coefficient domain from k to $k((t))$, the quotient power series field of $k[[t]]$.Any bilinear map $f : V \times V \to V$ (in particular the multiplication in $\mathcal{G}$) can be extended to a bilinear map from $V[[t]] \times V[[t]]$ to $V[[t]]$.

Notation. Let $A_{p,q}^2$ be the set of bilinear forms

$$\phi :\ k^n \times k^n \to k^n$$

satisfying :

$$\begin{cases} \phi(V_i, V_j) \subset V_{i+j \mod 2}, \\ \phi(v_i, v_j) = (-1)^{d(v_i).d(v_j)}\ \phi(v_j, v_i). \end{cases}$$

where $k^n = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space , $\dim V_0 = p+1$, $\dim V_1 = q$ and $v_i \in V_{d(v_i)}$, $v_i \in V_{d(v_i)}$.

Definition 1.4. Let ν_0 be the law of the Lie superalgebra $\mathcal{G}$. A deformation of ν_0 is a one parameter family ν_t in $k\,[[t]] \otimes V$.

$$\nu_t = \nu_0 + t.\nu_1 + t^2.\nu_2 + \dots$$

where $\nu_i \in A^2_{p,q}$ for $i \geq 1$, ν_t satisfy the Jacobi formal identities :

$$(-1)^{\gamma.\alpha}\nu_t(A, \nu_t(B,C)) + (-1)^{\alpha.\beta}\nu_t(B, \nu_t(C,A)) + (-1)^{\beta.\gamma}\nu_t(C, \nu_t(A,B)) = 0,$$

For all $A \in \mathcal{G}_\alpha$, $B \in \mathcal{G}_\beta$ and $C \in \mathcal{G}_\gamma$.

The coefficient of t^k of the formal Jacobi identity is

$$\begin{cases} \displaystyle\sum_{i=0}^{k} (-1)^{\gamma.\alpha}\nu_i(A, \nu_{k-i}(B,C)) + (-1)^{\alpha.\beta}\nu_i(B, \nu_{k-i}(C,A)) + (-1)^{\beta.\gamma}\nu_i(C, \nu_{k-i}(A,B)) = 0 \\ k = 0, 1, 2, \dots \end{cases}$$

for all $A \in \mathcal{G}_\alpha$, $B \in \mathcal{G}_\beta$ and $C \in \mathcal{G}_\gamma$. This last relations are called the deformation equations.

For $k = 0$ we get the Jacobi identity of the Lie superalgebra ν_0. For $k = 1$ the condition on the coefficient t implies the next proposition :

Proposition 1.1. *Let ν_0 be a Lie superalgebra and ν_t of it :*

$$\nu_t = \nu_0 + t.\nu_1 + t^2.\nu_2 + \dots$$

then ν_1 is an even 2-cocycle of the Lie superalgebra ν_0 ($\nu_1 \in Z^2_0(\nu_0, \nu_0)$).

1.5. **Deformation in $\mathcal{N}^{p,q}_{n,m}$.** Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a nilpotent Lie superalgebra of $\mathcal{N}^{p,q}_{n,m}$ with multiplication ν_0 and ν_t be a deformation of it.
We write $\nu_0 = \mu_0 + \rho_0 + b_0$ where :

$$\mu_0 \in \hom(\mathcal{G}_0 \wedge \mathcal{G}_0, \mathcal{G}_0)$$
$$\rho_0 \in \hom(\mathcal{G}_0 \otimes \mathcal{G}_1, \mathcal{G}_1)$$
$$b_0 \in \hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0)$$

For ν_t to be a deformation in $\mathcal{N}^{p,q}_{n,m}$, we must have :

$$\text{(N)} \quad \begin{cases} \nu_t(x_1, \nu_t(x_1, \dots, \nu_t(x_p, x_0)\dots)) = 0 \\ \nu_t(x_1, \nu_t(x_1, \dots, \nu_t(x_q, y)\dots)) = 0 \end{cases}$$

for all x_i in $\mathcal{G}_0$ and y in $\mathcal{G}_1$. Proposition 1.1 implies that $\nu_1 \in Z^2_0(\nu_0, \nu_0)$. Let be $\nu_1 = \psi_1 + \rho_1 + b_1$ with :

$$\psi_1 \in \hom(\mathcal{G}_0 \wedge \mathcal{G}_0, \mathcal{G}_0)$$
$$\rho_1 \in \hom(\mathcal{G}_0 \otimes \mathcal{G}_1, \mathcal{G}_1)$$
$$b_1 \in \hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0)$$

2. Filiform Lie superalgebras

Definition 2.1. Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a nilpotent Lie superalgebra with $\dim \mathcal{G}_0 = n+1$ and $\dim \mathcal{G}_1 = m$. $\mathcal{G}$ is called filiform if it's super-nilindex is (n, m).

We will note $\mathcal{F}_{n,m}$ the set of filiform Lie superalgebras.

Remark. We can write the set of filiform Lie superalgebras as the complement of the closed set for the Zariski topology of the nilpotent superalgebras with super-nilindex (k, p) such that $k \leq n-1$ and $p \leq m-1$. Hence the set of filiform Lie superalgebras is an open set of the variety of nilpotent Lie superalgebras.

As for the filiform Lie algebras [8], there exists an adapted base of a filiform Lie superalgebra :

Theorem 2.1. *Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a filiform Lie superalgebra with $\dim \mathcal{G}_0 = n+1$ and $\dim \mathcal{G}_1 = m$. Then there exists a base $\{X_0, X_1, \dots X_n, Y_1, Y_2, \dots Y_m\}$ of $\mathcal{G}$ with $X_i \in \mathcal{G}_0$ and $Y_i \in \mathcal{G}_1$ such that :*

$$\begin{cases} [X_0, X_i] = X_{i+1} \quad 1 \leq i \leq n-1, \quad [X_0, X_n] = 0; \\ [X_1, X_2] \in \mathbb{C}.X_4 + \mathbb{C}.X_5 + \cdots + \mathbb{C}.X_n\,; \\ [X_0, Y_i] = Y_{i+1} \quad 1 \leq i \leq m-1, \quad [X_0, Y_m] = 0. \end{cases}$$

The proof is the same as for Lie algebras [8](see also [3]).

Example Define the superalgebra $L_{n,m} = L^0_{n,m} \oplus L^1_{n,m}$ by

$$\begin{cases} [X_0, X_i] = X_{i+1} \quad 1 \leq i \leq n-1, \\ [X_0, Y_i] = Y_{i+1} \quad 1 \leq i \leq m-1. \end{cases}$$

where the other brackets vanished, $\dim L^0_{n,m} = n+1$, $\dim L^1_{n,m} = m$ and $\{X_0, X_1, \dots X_n, Y_1, \dots, Y_m\}$ is an adapted base. The law of $L_{n,m}$ is written by $\mu = \mu_0 + \rho_0$ where μ_0 is the law of the Lie algebras $L^0_{n,m}$ and ρ_0 is the representation associated to the $L^0_{n,m}$-module $L^1_{n,m}$.

Proposition 2.1. *Every filiform Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ such that $\dim \mathcal{G}_0 = n+1$ and $\dim \mathcal{G}_1 = m$ can be written :*

$$[\bullet, \bullet] = \mu_0 + \rho_0 + \Phi$$

with Φ satisfying :

$$\begin{aligned} &\Phi \in Z^2_0(L_{n,m}, L_{n,m}) \\ &\Phi(X_0, Z) = 0 \quad \forall Z \in \mathcal{G} \\ &\Phi(S^0_i, S^0_j) \subset S^0_{i+j+1} \text{ if } i+j < n \\ &\Phi(X_i, X_{n-i}) = (-1)^i \alpha\ X_n \text{ where } \alpha = 0 \text{ if } n \text{ is even} \\ &\Phi(S^0_i, S^1_j) \subset S^1_{i+j} \end{aligned}$$

where $\mu_0 + \rho_0$ is the law of $L_{n,m}$, and $S^0_\cdot$, $S^1_\cdot$ the filtrations associated to the graduations of $\mathcal{G}_0$ and $\mathcal{G}_1$ by the sequences given in 1.1.

Proof. Using the theorem 2.1, for every filiform lie superalgebra we have an adapted base $\mathcal{B}$ $\{X_0, X_1, \dots, X_n, Y_1, Y_2, \dots Y_n\}$ such that the product of $\mathcal{G}$ is given by :

$$[\bullet, \bullet] = \mu_0 + \rho_0 + \Phi$$

where $\Phi[X_0, Z] = 0$ for every vector $Z \in \mathcal{G}$. This product satisfies the Jacobi identity

$$(\mu_0 + \rho_0) \circ \Phi + \Phi \circ (\mu_0 + \rho_0) + \Phi \circ \Phi = 0. \tag{1}$$

Let Z_i, Z_j be two vectors of the adapted base $\mathcal{B}$ of $\mathcal{G}$. We have $\Phi \circ \Phi(X_0, Z_i, Z_j) = 0$ because $\Phi(X_0, \bullet) = 0$. The relation (1) becomes :

$$((\mu_0 + \rho_0) \circ \Phi + \Phi \circ (\mu_0 + \rho_0))(X_0, Z_i, Z_j) = 0$$

Also we have for every $Z_i, Z_j, Z_k \in \mathcal{B} \setminus \{X_0\}$:

$$((\mu_0 + \rho_0) \circ \Phi + \Phi \circ (\mu_0 + \rho_0))(Z_i, Z_j, Z_k) = 0.$$

As the superalgebra is nilpotent, $X_0 \notin Im\Phi$ shows :

$$((\mu_0 + \rho_0) \circ \Phi + \Phi \circ (\mu_0 + \rho_0))(U, V, W) = 0.$$

for all $U, V, W \in \mathcal{G}$, where $\circ$ is the graded Nijenhuis-Richardson bracket. This implies that $\Phi \in Z_0^2(L_{n,m}, L_{n,m})$. The filtrations :

$$\begin{cases} [S_i^0, S_j^0] \subset S_{i+j}^0 \\ [S_i^0, S_j^1] \subset S_{i+j}^1 \end{cases}$$

associated to the graduations $C^i(\mathcal{G}_0)$ and $C^i(\mathcal{G}_1)$ shows that $\Phi(S_i^0, S_j^1) \subset S_{i+j}^1$. From [8], we also have

$$\begin{cases} \Phi(S_i^0, S_i^0) \subset S_{i+j+1}^0 \text{ if } i + j < n \\ \Phi(X_i, X_{n-i}) = (-1)^i \alpha \ X_n \text{ where } \alpha = 0 \text{ if } n \text{ is even} \end{cases}$$

□

This lead us to the study of the 2-cocycles of $L_{n,m}$.

Proposition 2.2. *Let be $\Psi \in Z_0^2(L_{n,m}, L_{n,m})$, such that $\mu_0 + \rho_0 + \Psi$ is a nilpotent Lie superalgebra, then Ψ admits the following decomposition $\Psi = \psi + \rho + b$ with*

$$\psi \in Hom(\mathcal{G}_0 \wedge \mathcal{G}_0, \mathcal{G}_0) \cap Z^2(L_n, L_n)$$
$$\rho \in Hom(\mathcal{G}_0 \otimes \mathcal{G}_1, \mathcal{G}_1) \cap Z^2(L_{n,m}, L_{n,m})$$
$$b \in Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \cap Z^2(L_{n,m}, L_{n,m})$$

Proof. It is clear that Ψ can be decomposed into a sum of three homogeneous maps :

$$\begin{cases} \psi \in Hom(\mathcal{G}_0 \wedge \mathcal{G}_0, \mathcal{G}_0) \\ \rho \in Hom(\mathcal{G}_0 \otimes \mathcal{G}_1, \mathcal{G}_1) \\ b \in Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \end{cases}$$

As $\Psi \in Z_0^2(L_{n,m}, L_{n,m})$, we have :

$$\begin{cases} \psi(\mu_0(g_i, g_j), g_k) - \psi(\mu_0(g_i, g_k), g_j) + \psi(\mu_0(g_j, g_k), g_i) - \\ \quad \mu_0(g_i, \psi(g_j, g_k)) + \mu_0(g_j, \psi(g_i, g_k)) - \mu_0(g_k, \psi(g_i, g_j)) = 0, \\ \\ \rho(\mu_0(g_i, g_j), h_t) + \rho(g_j, \rho_0(g_i, h_t)) - \rho(g_i, \rho_0(g_j, h_t)) - \\ \quad \rho_0(g_i, \rho(g_j, h_t)) + \rho_0(g_j, \rho(g_i, h_t)) - \rho_0(h_t, \psi(g_i, g_j)) = 0, \\ \\ b(\rho_0(g_i, h_t), h_r) + b(\rho_0(g_i, h_r), h_t) - \mu_0(g_i, b(h_t, h_r)) = 0, \\ \\ \rho_0(b(h_r, h_s), h_t) + \rho_0(b(h_t, h_s), h_r) + \rho_0(b(h_t, h_r), h_s) = 0. \end{cases}$$

where g_i is an even element and h_j an odd element of $L_{n,m}$. This prove that ψ is a cocycle of the filiform Lie algebra L_n. As $\mu_0 + \rho_0 + \Psi$ is nilpotent, $\psi(X_i, X_j)$ has no component on X_0. This implies that $\rho_0(h_t, \psi(g_i, g_j)) = 0$ and

$$\begin{cases} \rho(\mu_0(g_i,g_j),h_t) + \rho(g_j,\rho_0(g_i,h_t)) - \rho(g_i,\rho_0(g_j,h_t)) - \\ \quad \rho_0(g_i,\rho(g_j,h_t)) + \rho_0(g_j,\rho(g_i,h_t)) = 0 \\ \\ b(\rho_0(g_i,h_t),h_r) + b(\rho_0(g_i,h_r),h_t) - \mu_0(g_i,b(h_t,h_r)) = 0 \\ \\ \rho_0(b(h_r,h_s),h_t) + \rho_0(b(h_t,h_s),h_r) + \rho_0(b(h_t,h_r),h_s) = 0 \end{cases}$$

This prove that both maps ρ and b are cocycles. □

We are reduced to study each space associated to the decomposition of Ψ.

2.1. **Cocycles of** $Hom(\mathcal{G}_0 \wedge \mathcal{G}_0, \mathcal{G}_0)$**.**

Let ψ be a 2-cocycle of $L_{n,m}$ belonging to $Hom(\mathcal{G}_0 \wedge \mathcal{G}_0, \mathcal{G}_0)$. Then ψ is a 2-cocycle of the Lie algebras L_n. From [8], these cocycle are written as a linear sum of the following cocycles :

Let (k, s) be a pair of integers such that $1 \leq k \leq n-1$, $2k \leq s \leq n$. There exists an unique cocycle of L_n satisfying :

$$\Psi_{k,s}(X_i, X_{i+1}) = \begin{cases} X_s \text{ if } i = k \\ 0 \text{ otherwise} \end{cases}$$

and

$$\Psi_{k,s}(X_0, X_i) = 0 \quad 1 \leq i \leq n$$

is given for $i < j$ by

$$\Psi_{k,s}(X_i, X_j) = (-1)^{k-i} C_{j-k-1}^{k-j} (ad\ X_0)^{i+j-2k-1} X_s \text{ if } k - i \leq j - k - 1$$
$$\Psi_{k,s}(X_i, X_j) = 0 \text{ otherwise}$$

The cocycles $\psi_{i,j}$ give the nilpotent deformations of the filiform Lie algebra L_n.

2.2. **Cocycles of** $Hom(\mathcal{G}_0 \oplus \mathcal{G}_1, \mathcal{G}_1)$**.** These cocycles are described in the following proposition :

Proposition 2.3. *For $1 \leq k \leq n$ and $1 \leq s \leq m$, there exists an unique cocycle $\rho_{k,s}$ of $Hom(\mathcal{G}_0 \otimes \mathcal{G}_1, \mathcal{G}_1) \cap Z^2(L_{n,m}, L_{n,m})$ such that :*

$$\rho_{k,s}(X_i, Y_1) = \begin{cases} Y_s \text{ if } i = k \\ 0 \text{ otherwise} \end{cases}$$

and

$$\rho_{k,s}(X_0, Y_i) = 0 \quad 1 \leq i \leq m$$

It satisfies :

$$\rho_{k,s}(X_j, Y_r) = \begin{cases} (-1)^{k-j} C_{r-1}^{k-j}\ Y_{s+r-1-j+k} \text{ if } k - r + 1 \leq j \leq k \\ 0 \text{ otherwise} \end{cases}$$

for $1 \leq r \leq m$, where C_{r-1}^{k-j} are the binomial coefficients.

Proof. Let $\rho_{k,s}$ be a cocycle such that :

$$\rho_{k,s}(X_i, Y_1) = \begin{cases} Y_s \text{ if } i = k \\ 0 \text{ otherwise} \end{cases}$$

and

$$\rho_{k,s}(X_0, Y_i) = 0 \quad 1 \leq i \leq m$$

$\rho_{k,s}$ must satisfy

$$\rho_{k,s}([X_i, X_j], Y_r) + \rho_{k,s}(X_j, [X_i, Y_r]) - \rho_{k,s}(X_i, [X_j, Y_r]) - [X_i, \rho_{k,s}(X_j, Y_r)] + [X_j, \rho_{k,s}(X_i, Y_r)] = 0$$

then by induction on r and j we prove that

$$\rho_{k,s}(X_j, Y_r) = \begin{cases} (-1)^{k-j} C_{r-1}^{k-j} \ Y_{s+r-1-j+k} \text{ if } k - r + 1 \leq j \leq k \\ 0 \text{ otherwise} \end{cases}$$

for $1 \leq r \leq m$. □

Proposition 2.4. *Let $\{X_0, X_1, \ldots, X_n, Y_1, Y_2, \ldots, Y_m\}$ be an adapted base of $L_{n,m}$. The bilinear mapping $\varrho_{i,j}$ with $1 \leq i, j \leq m$ defined by :*

$$\begin{cases} \varrho_{i,j}(X_0, Y_i) = Y_j, \\ \varrho_{i,j}(X_p, Y_k) = \varrho_{i,j}(X_p, Y_k) = \varrho_{i,j}(Y_p, Y_k) = 0, \quad p \neq 0. \end{cases}$$

are cocycles.

The prove is obvious.

Theorem 2.2. *The family of cocycles $\varrho_{i,j}$ and $\rho_{k,s}$ with $1 \leq i \leq j \leq m$ and $1 \leq k \leq n$, $1 \leq s \leq m$ form a basis of $Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_0 \otimes \mathcal{G}_1, \mathcal{G}_1)$.*

Proof. Let ρ be a cocycle of $Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_0 \otimes \mathcal{G}_1, \mathcal{G}_1)$, such that $\rho(X_0, Y_j) = 0$ for $1 \leq j \leq m$. We can prove by induction on j that if $\rho(X_j, Y_1) = 0$ for $1 \leq j \leq n$ then $\rho \equiv 0$.

We can assume that $\rho(X_0, Y_j) = 0$, if not, we consider the cocycle $\rho_1 = \rho - \sum_{i,k} a_{i,k} \varrho_{i,k}$ such that $\rho_1(X_0, Y_j) = 0$. It is easy to see that there exists a linear combination of $\rho_{i,j}$ such that
$\rho' = \rho - \sum_{j=1}^{m} r_{i,j} \rho_{i,j}$ satisfies $\rho'(X_i, Y_1) = 0$. Using the previous paragraph, we have that $\rho' \equiv 0$ we deduce that $\rho = \sum_{j=1}^{m} r_{i,j} \rho_{i,j}$, and if $\rho(X_0, Y_j)$ was not zero, ρ will be

$$\rho = \sum_{i=1}^{n} \sum_{j=1}^{m} t_{i,j} \ \rho_{i,j} + \sum_{k=1}^{m} \sum_{r=1}^{m} s_{k,r} \varrho_{k,r}$$

this prove that $\rho_{i,j}$ and $\varrho_{k,r}$ are generator. As they are linearly independent, we have a base. □

2.3. Cocycles of $Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0)$.

In this case, we will not give a basis for this cocycles, but we will give the dimension of this space.

Let b be a cocycle in $Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \cap Z^2(L_{n,m}, L_{n,m})$. Then b has to verify the two conditions :

$$b(\rho_0(g_i, h_t), h_r) + b(\rho_0(g_i, h_r), h_t) - \mu_0(g_i, b(h_t, h_r)) = 0 \tag{2}$$

and

$$\rho_0(b(h_r, h_s), h_t) + \rho_0(b(h_t, h_s), h_r) + \rho_0(b(h_t, h_r), h_s) = 0 \tag{3}$$

where $g_i \in L_{n,m}^0$ and $h_t, h_r, h_s \in L_{n,m}^1$.

Now we will focus or work on relation (2). Note that if g_i is linearly independent of X_0, then (2) is satisfied. We suppose that $g_i = X_0$. Consider the adapted basis $\{X_0, X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$ of $L_{n,m}$ then (2) is written :

$$\mu_0(X_0, b(Y_t, Y_r)) = b(Y_{t+1}, Y_r) + b(Y_{r+1}, Y_t) \tag{4}$$

for $1 \leq r, t \leq m-1$.

Lemma 2.1. *Let b be a symmetric bilinear mapping satisfying (4), such that*

$$b(Y_i, Y_i) = 0 \text{ for } 1 \leq i \leq m;$$

then b is null.

Proof. Let us prove that $b(Y_i, Y_{i+k}) = 0$ for every k. For $k = 0$ we have :

$$b(Y_i, Y_i) = 0$$

Suppose that the relation is true up to k. For $k+1$ we have :

$$\mu_0(X_0, b(Y_i, Y_{i+k})) = b(Y_{i+1}, Y_{(i+1)+(k-1)}) + b(Y_i, Y_{i+k+1})$$
$$0 = 0 + b(Y_i, Y_{i+k+1}).$$

then $b(Y_i, Y_{i+k}) = 0$ for all integer k and i. □

This lemma shows that a symmetric bilinear map b satisfying (4) can be defined only by the value $b(Y_i, Y_i)$ with $1 \leq i \leq m$. Relation (3) implies that $\rho_0(Y_i, b(Y_i, Y_i)) = 0$, if $b(Y_i, Y_i) = a_i\ X_0 + \ldots$, we have that $a_i.\rho(Y_i, X_0) = 0$. This implies that $a_i = 0$, $1 \leq i < m$. Suppose that $a_m \neq 0$, then the relation (3) implies that $\rho_0(Y_1, b(Y_m, Y_m)) = 0$, then $a_m\ Y_2 = 0$ and $a_m = 0$. This prove that $b(Y_i, Y_i)$ does not have a component on X_0 for every i.

We define the vector space E of the symmetric bilinear maps satisfying the relation (4) such that $\forall b \in E, b(Y_m, Y_m) \in Vect(X_n)$.

Proposition 2.5. *The symmetric bilinear maps $f_{p,s}$ with $1 \leq s \leq n$ and $1 \leq p \leq m-1$ defined for $1 \leq i \leq p \leq j \leq m$ by :*

$$f_{p,s}(Y_i, Y_j) = \frac{(-1)^{p-i}}{2}\left(C_{j-p}^{p-i} + C_{j-p-1}^{p-i-1}\right) X_{s-2p+i+j}$$

with convention $C_{-1}^{-1} = 1$ and 0 otherwise; and

$$f_{m,n}(Y_i, Y_j) = \begin{cases} X_s \text{ if } i = j = m \\ 0 \text{ otherwise} \end{cases}$$

form a basis of E.

Proof. Let $b \in E$ be a symmetric bilinear map. It is easy to see that there exists coefficient $a_{p,s} \in \mathbb{C}$ such that

$$b(Y_i, Y_i) - \sum_{p=1}^{m}\sum_{s=1}^{n} a_{p,s}\ f_{p,s}(Y_i, Y_i) = 0$$

for $1 \leq i \leq m$, where $f_{m,i} = 0$ for $1 \leq i \leq n-1$. Using lemma 2.1 we deduce that this equality vanishes for every pair (Y_i, Y_j). This proves that

$$b = \sum_{p=1}^{m}\sum_{s=1}^{n} a_{p,s}\ f_{p,s}$$

Using the fact that $f_{p,s}(Y_p, Y_p) = X_s$ and $f_{p,s}(Y_i, Y_i) = 0$ if $i \neq p$, the family $\{f_{p,s}\}$ is free. □

Proposition 2.6. *The space* $Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0)$ *is the subspace of* E *defined by*

$$(5) \left\{ \begin{array}{l} f \in E, \\ \mu_0(X_0, f(Y_i, Y_m)) = f(Y_{i+1}, Y_m) \\ \text{for } 1 \leq i \leq m-1 \end{array} \right\}$$

Proof. A cocycle f satisfies the two relations (2) and (3). A consequence of this is that $f(Y_m, Y_m) = a_{m,n}\ X_n$. We deduce that $f \in E$. To satisfy relation (3), f has to satisfy the relation

$$\mu_0(X_0, f(Y_i, Y_m)) = f(Y_{i+1}, Y_m)$$

for $1 \leq i \leq m-1$.

Such a map f does not have a component on X_0 in its image, hence $[Y_i, b(Y_j, Y_k)] = 0$ for $1 \leq i, j, k \leq m$. This prove that relation (2) is satisfied and that every map f satisfying (3) is a cocycle. □

Consequence :

$$\dim Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \leq \dim E = n.m - n + 1$$

The maps $f_{p,s}$ are not always cocycles. Let $b_{p,s}^{(\alpha)} \in E$ be

$$b_{p,s}^{(\alpha)} = f_{p,s} + \sum_{k=1}^{m-p} \alpha_{p,s}^k f_{p+k,s+2k}$$

where $\alpha_{p,s}^k \in \mathbb{C}$. Let $A_{p,s}$ be the set of sequences $(\alpha) = (\alpha_{p,s}^1, \alpha_{p,s}^2, \dots \alpha_{p,s}^{m-p})$ such that $b_{p,s}^{(\alpha)}$ is a cocycle. Then $(\alpha_{p,s}^1, \alpha_{p,s}^2, \dots \alpha_{p,s}^{m-p})$ is a solution of equation (5). Remark that for some pair (p, s), $A_{p,s}$ can be empty.

Lemma 2.2. *The family of cocycles* $b_{p,s}^{(\alpha)}$ *with* $(\alpha) \in \cup_{p,s} A_{p,s}$ *spans*

$$Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \cap Z^2(L_{n,m}, L_{n,m})$$

Proof. Using theorem 2.6 every cocycle of $Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0)$ is given by :

$$f = \sum_{p=p_0}^{m} \sum_{s=s_0}^{n} a_{p,s}\ f_{p,s}$$

with $a_{p_0,s_0} \neq 0$.

We can write f like :

$$f = a_{s_0,p_0}\ b_{s_0,p_0}^{(\alpha)} + R$$

where $b_{s_0,p_0}(Y_i, Y_j)$ has a component on $X_{s_0-2p_0+i+j}$ and $R(Y_i, Y_j)$ does not have any component on $X_{s_0-2p_0+i+j}$.

As f is a cocycle, we have :

$$[X_0, f(Y_i, Y_m)] = f(Y_{i+1}, Y_m)$$

$$a_{s_0,p_0}\ [X_0, b_{s_0,p_0}(Y_i, Y_m)] + [X_0, R(Y_i, Y_j)] = a_{s_0,p_0}\ b_{s_0,p_0}(Y_{i+1}, Y_m) + R(Y_{i+1}, Y_m)$$

Let us consider the component on $X_{s_0-2p_0+i+1+m}$, we have :

$$a_{s_0,p_0}\ [X_0, b_{s_0,p_0}(Y_i, Y_m)] = a_{s_0,p_0}\ b_{s_0,p_0}(Y_{i+1}, Y_m)$$

$$[X_0, b_{s_0,p_0}(Y_i, Y_m)] = b_{s_0,p_0}(Y_{i+1}, Y_m) \text{ because } a_{s_0,p_0} \neq 0$$

This proves that b_{s_0,p_0} is a cocycle, as $f - a_{s_0,p_0}\ b_{s_0,p_0}$. Using the cocycle $f - a_{s_0,p_0}\ b_{s_0,p_0}$, we prove by induction that f is given by a linear combination of the cocycles $b_{p,s}^{(\alpha)}$.

□

The cocycles $b^{(\alpha)}_{p,s}$ are not linearly independent. Therefore, for a non empty set $A_{p,s}$, we consider the smallest cocycle $b^0_{p,s}$ in the sense that $b^0_{p,s}$ cannot be written

$$b^0_{p,s} = b^{(\alpha_1)}_{p,s} + a\ b^{(\alpha_2)}_{k,s}$$

with $b^{(\alpha_2)}_{k,s}$ non zero, $a \in \mathbb{C}^*$ and $k > p$.

Lemma 2.3. *For any non empty set $A_{p,s}$ there exists an unique cocycle $b^0_{p,s}$.*

Proof. Let be $A_{p,s} \neq \emptyset$ and $b^{(\alpha)}_{p,s}$ be a non zero cocycle. If we can decompose $b^{(\alpha)}_{p,s}$, we have the smallest cocycle, if not we choose the smallest integer k_0, $p < k_0 + p \leq m$ such that

$$b^{(\alpha)}_{p,s} = b^{(\alpha_0)}_{p,s} + \gamma^{k_0}_{p,s}\ b^{(\alpha_k)}_{p+k_0,s+2k_0}$$

If $b^{(\alpha_0)}_{p,s}$ is indecomposable, we stop. If not, we have

$$b^{(\alpha_0)}_{p,s} = b^{(\alpha_1)}_{p,s} + \gamma^{k_1}_{p,s}\ b^{(\alpha_2)}_{p+k_1,s+2k_1}$$

with $k_1 > k_0$, this sequence is increasing and has an upper bound, therefore it exists k_r such that $b^{(\alpha_r)}_{p,s}$ is indecomposable.

To proof the uniqueness, suppose that $b^{(\alpha_1)}_{p,s}$ and $b^{(\alpha_2)}_{p,s}$ are smallest. Then $b^1_{p,s} - b^2_{p,s} = b$ is a cocycle. We have $b^{(\alpha_1)}_{p,s} = b^{(\alpha_2)}_{p,s} + a\ b^{(\alpha_3)}_{k_0,r}$ with k_0 the smallest integer k such that $f_{k,r}$ is in b. As $b^{(\alpha_1)}_{p,s}$ is the smallest, we must have $a\ b_{k_0,r} = 0$ and then $b^{(\alpha_1)}_{p,s} = b^{(\alpha_2)}_{p,s}$. This proves the uniqueness. □

Theorem 2.3. *The cocycles' family $b^0_{p,s}$ with (p,s) such that $A_{p,s} \neq \emptyset$ is a basis of*

$$Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \cap Z^2(L_{n,m}, L_{n,m})$$

Proof. Let (p,s) be such that $A_{p,s} \neq \emptyset$. The cocycles $b^0_{p,s}$ span $Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \cap Z^2(L_{n,m}, L_{n,m})$ because every cocycle $b_{p,s}$ can be written as a linear sum of $b^0_{p+k,s+2k}$, $k \geq 0$.

Let be $a_{p,s} \in \mathbb{C}$ such that :

$$\sum_{p=1}^{m}\sum_{s=1}^{n} a_{p,s}\ b^0_{p,s} \equiv 0$$

Note that :

$$b^0_{p,s}(Y_i, Y_i) = \begin{cases} 0 \text{ if } i < p \\ X_s \text{ if } i = p \\ \alpha^k_{p,s} X_s \text{ if } i = p + k \end{cases}$$

We have for Y_1 :

$$\sum_{p=1}^{m}\sum_{s=1}^{n} a_{p,s}\ b^0_{p,s}(Y_1, Y_1) = \sum_{s=1}^{n} a_{1,s}\ X_s$$

then $a_{1,s} = 0$ for $1 \leq s \leq n$. By induction on $p = 1, 2, \ldots m$, every coefficient vanishes, and the $b^0_{p,s}$ are linearly independent.

□

The theorem shows that the determination of a basis of

$$Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \cap Z^2(L_{n,m}, L_{n,m})$$

is reduced to the case $A_{p,s} \neq \emptyset$.

Proposition 2.7. *The only pairs (p,s) such that $f_{p,s}$ is a cocycle, that is $(0,0,\dots)\in A_{p,s}$, are*

- *if m is odd : $f_{\frac{m-1}{2},n}$ and $f_{p,s}$ with $2p=m-k$, $n-k-1\leq s\leq n$ for $1\leq k\leq m-2$ and k odd.*
- *if m is even : $f_{p,s}$ with $2p=m-k$, $n-k-1\leq s\leq n$ for $0\leq k\leq m-2$ and k even.*

Proof. Let $f_{p,s}$ be a cocycle from the proposition. If $f_{p,s}\in Z^2(L_{n,m},L_{n,m})$ then

$$[X_0,f_{p,s}(Y_i,Y_m)]=f_{p,s}(Y_{i+1},Y_m)$$

Let (p,s) be such that $2p=m-k$ and $1\leq s\leq n-k-2$, we will proof that $f_{p,s}$ is not a cocycle. We have

$$f_{p,s}(Y_1,Y_m)=\frac{(-1)^{p-1}}{2}(C_{m-p}^{p-1}+C_{m-p-1}^{p-2})X_{s+k+1}$$

$$f_{p,s}(Y_2,Y_m)=\frac{(-1)^{p-2}}{2}(C_{m-p}^{p-2}+C_{m-p-1}^{p-3})X_{s+k+2}$$

If $f_{p,s}$ is a cocycle, we have

$$(C_{m-p}^{p-1}+C_{m-p-1}^{p-2})=-(C_{m-p}^{p-2}+C_{m-p-1}^{p-3})$$

This implies

$$C_{m-p}^{p-1}+C_{m-p-1}^{p-2}=0$$
$$C_{m-p}^{p-2}+C_{m-p-1}^{p-3}=0$$

Thus $C_{m-p}^{p-1}=0$, and $2p>m+1$. We have $f_{p,s}(Y_1,Y_m)=0$ and $f_{p,s}(Y_{2p-m},Y_m)=(-1)^{m-1}\ X_s\neq 0$. This proofs that $f_{p,s}$ cannot be a cocycle.

□

Proposition 2.8. *Let q be such that $1\leq q\leq \min\{m-1,n-2\}$. If p satisfies $2+m+q-n\leq 2p\leq m-q+1$ then $A_{p,s}$, with $s=n-m-q-1+2p$, is not empty.*

Proof. Let q be such that $1\leq q\leq \min\{m-1,n-2\}$, $s=n-m-q-1+2p$ and p such that $2+m+q-n\leq 2p\leq m-q+1$.

Let's proof that there exists $(\alpha_{p,s}^1,\dots,\alpha_{p,s}^q)\in A_{p,s}$ such that

$$b_{p,s}=f_{p,s}+\sum_{k=1}^{q}\alpha_{p,s}^k\ f_{p+k,s+2k}$$

is a cocycle.

We have $b_{p,s}(Y_1,Y_m)\in\mathbb{C}.X_{n-q}$, then for $b_{p,s}$ to be a cocycle, it must satisfy q equations given by (5). Suppose $1\leq i\leq q$, if

$$f_{p+k,s+2k}(Y_i,Y_m)=\frac{(-1)^{p+k-i}}{2}(C_{m-p-k}^{p+k-i}+C_{m-p-k-1}^{p+k-i-1})X_{n-q+i-1}$$

is vanishing then $C_{m-p-k}^{p+k-i}+C_{m-p-k-1}^{p+k-i-1}=0$ as $n-q+i-1\leq n$. This is possible only if $p+k-i-1\leq m-p-k-1$. As $2+m+q-n\leq 2p\leq m-q+1$, there exists a value of p and k such that $C_{m-p-k}^{p+k-i}+C_{m-p-k-1}^{p+k-i-1}\neq 0$. This proofs that in the q linear equations given by (5), every coefficient $\alpha_{p,s}^1,\dots,\alpha_{p,s}^q$ appear. This proof that this system admits a solution. □

Corollary 2.1. *Suppose that $m\geq n$, $m=2t$. Then if*

- *$n=4s$: $\dim Z_0^2(L_{n,m},L_{n,m})\cap Hom(\mathcal{G}_1\vee\mathcal{G}_1,\mathcal{G}_0)\geq t.n-2s^2+s$*

- $n = 4s+1$: $\dim Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \geq t.n - 2s^2$
- $n = 4s+2$: $\dim Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \geq t.n - 2s^2 - s$
- $n = 4s+3$: $\dim Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \geq t.n - 2s^2 - s$

For $m \geq n$, $m = 2t+1$ then if

- $n = 4s$: $\dim Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \geq (t+1).n - 2s^2 - s$
- $n = 4s+1$: $\dim Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \geq t.n - 2s^2 + 2s + 1$
- $n = 4s+2$: $\dim Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \geq (t+1).n - 2s^2 - 4s - 1$
- $n = 4s+3$: $\dim Z_0^2(L_{n,m}, L_{n,m}) \cap Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \geq t.n - 2s^2 + 2$

Proof. Using propositions 2.7 and 2.8, we can compute a lower bound of the number of non empty sets $A_{p,s}$. For each of this sets, there exists a unique cocycle $b^0_{p,s}$ (see lemma 2.3) which is a vector of the base of

$$Hom(\mathcal{G}_1 \vee \mathcal{G}_1, \mathcal{G}_0) \cap Z^2(L_{n,m}, L_{n,m})$$

this is established in the theorem 2.3. □

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Marc GILG, Université de Haute-Alsace, Laboratoire de Mathématiques, 4 rue des Frères Lumière, 68 093 Mulhouse Cedex, France

E-mail address: M.Gilg@univ-mulhouse.fr