DEGENERATION OF HOPF ALGEBRAS AND IRREDUCIBLE COMPONENTS

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The aim of this work is to discuss the concept of degeneration in the case of bialgebra and Hopf algebras. This paper is organized as follow, ørst I present the basic concepts and I give a necessary and suŒcient condition such that a degeneration with an inønitesimal family of endomorphisms of a given bialgebra or a Hopf algebra exists. The second part of this work concerns the geometric description of the algebraic variety of n-dimensional Hopf algebras with n$\leq$ 13. I show that they are unions of open orbits.

1. Structure of algebraic varieties

1.1. Preliminaries. Throughout this paper K will be an algebraically closed øeld of characteristic 0 and V be an n-dimensional vector space over the øeld K. Let $H = (V, \mu, \eta, \Delta, \varepsilon, S)$ be a ønite dimensionnal Hopf algebra. Recall that the triple (V, μ, η) deønes an algebra structure where $\mu : V \otimes V \to V$ and $\eta : K \to V$ are linear maps satisfying :

$$\begin{array}{c} (1)\ \ \text{(Associativity)} \quad \mu\,(\mu \otimes id_V) = \mu\,(id_V \otimes \mu) \\ (2)\ \ \text{(Unitality)} \quad \mu\,(\eta \otimes id_V) = \mu\,(id_V \otimes \eta) = id_V \end{array}$$

then $e_1 = \eta\,(1) \in V$ becomes the unit for the multiplication μ. The triple (V, Δ, ε) deønes a structure of coalgebra where $\Delta : V \to V \otimes V$ and $\varepsilon : V \to k$ are linear maps satisfying :

$$\begin{array}{c} (3)\ \ \text{(Coassociativity)} \quad (\Delta \otimes id_V)\,\Delta = (id_V \otimes \Delta)\,\Delta \\ (4)\ \ \text{(Counitality)} \quad (\varepsilon \otimes id_V)\,\Delta = (id_V \otimes \varepsilon)\,\Delta = id_V \end{array}$$

To get the structure of bialgebra the maps Δ and ε must be algebra morphisms which may be expressed by

$$(5) \quad \left\{ \begin{array}{l} \Delta\,(e_1) = e_1 \otimes e_1 \\ \Delta\,(\mu(x \otimes y)) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\ \qquad \text{where } \Delta\,(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \\ \varepsilon\,(e_1) = 1 \\ \varepsilon\,(\mu(x \otimes y)) = \varepsilon\,(x)\,\varepsilon\,(y) \end{array} \right.$$

A bialgebra becomes a Hopf algebra if there is an endomorphism S, called antipode, satisfying :

$$(6) \quad \mu \circ S \otimes Id \circ \Delta = \mu \circ Id \otimes S \circ \Delta = \eta \circ \varepsilon$$

1.2. The algebraic variety Bialg$_n$. Setting a basis $\{e_1, ..., e_n\}$ of V where $e_1 = \eta(1)$ (fixed), we identify the multiplication μ and the comultiplication Δ with their n^3 structure constants C_{ij}^k and $D_i^{jk} \in K$, where $\mu(e_i, e_j) = \sum_{k=1}^n C_{ij}^k e_k$ and $\Delta(e_i) = \sum_{j=1}^n \sum_{k=1}^n D_i^{jk} e_j \otimes e_k$. The counity ε is identified to its n structure constants ξ_i, where $\varepsilon(e_i) = \xi_i$. The collection $\left(C_{ij}^k, D_i^{jk}, \xi_i : i, j, k = 1, \cdots, n\right)$ represents a bialgebra if the underlying multiplication, comultiplication and counit satisfy the conditions (1) (2) (3) (4) (5). These conditions translate respectively to the following polynomial equations :

$$(7) \quad \begin{cases} \sum_{l=1}^n C_{ij}^l C_{lk}^s - C_{il}^s C_{jk}^l = 0 \\ C_{1i}^j = C_{i1}^j = \delta_{ij} \text{ the Kronecker symbol} \end{cases} \qquad i, j, k, s \in \{1, .., n\}$$

$$(8) \quad \begin{cases} \sum_{l=1}^n D_s^{lk} D_l^{ij} - D_s^{il} D_l^{jk} = 0 \\ \sum_{l=1}^n D_i^{il} \xi_l = \sum_{l=1}^n D_i^{li} \xi_l = 1 \\ \sum_{l=1}^n D_i^{jl} \xi_l = \sum_{l=1}^n D_i^{lj} \xi_l = 0 \ i \neq j \end{cases} \qquad i, j, k, s \in \{1, .., n\}$$

$$(9) \quad \begin{cases} \sum_{l=1}^n C_{ij}^l - \sum_{r,t,p,q=1}^n D_i^{rt} D_j^{pq} C_{rp}^k C_{tq}^s = 0 \\ D_1^{11} = 1, \quad D_1^{ij} = 0 \ (i, j) \neq (1, 1) \\ \xi_1 = 1, \quad \sum_{l=1}^n C_{ij}^l \xi_l = \xi_i \xi_j \end{cases} \qquad i, j, k, s \in \{1, .., n\}$$

The polynomial relations (7) (8) (9) endow the set of the n-dimensional bialgebras, denoted by $Bia\lg_n$, with a structure of an algebraic variety imbedded in K^{2n^3+n} which we do consider here together with its natural structure of an algebraic variety over K.

As a subset of K^{2n^3+n}, $Bialg_n$ is provided with the Zariski topology.

1.3. The scheme structure of Bialg$_n$. The scheme structure of $Bia\lg_n$ is defined by the following. Let $K[x_1, \cdots, x_N]$ be the polynomial ring, where $N = 2n^3 + n$, and Let Ω be the ideal of $K[x_1, \cdots, x_N]$ generated by the polynomials of the relations (7) (8) (9) where $\left(C_{ij}^k, D_i^{jk}, \xi_i : i, j, k = 1, \cdots, n\right) = \left(x_r : r = 1, \cdots, 2n^3 + n\right)$. Let R be the quotient ring $K[x_1, \cdots, x_N] / \Omega$. To this ring is associated as a "geometrical object", the set of its prime ideals denoted by $spec(R)$. This set is provided with the spectral topology in which the sets $U(E) = \{p \in spec(R) : p \supset E\}$ are all the closed subsets of $spec(R)$. On this topological space we construct a sheaf defined by the regular functions on $spec(R)$. The scheme of the n-dimensional bialgebras is given by $spec(R)$ and its sheaf.

1.4. The Hopf algebras set, Hopf$_n$. The elements of Bialg$_n$ with antipodes define the set of Hopf algebras, denoted by $Hopf_n$. If the antipode $S = (S_{ij})$, with respect to the basis $\{e_1, ..., e_n\}$ of V, then the condition (6) translates to

$$(10) \begin{cases} \sum_{j,k,r=1}^n D_i^{jk} S_{rj} C_{rk}^1 = \sum_{j,k,r=1}^n D_i^{jk} S_{rk} C_{jr}^1 = \xi_i \\ \sum_{j,k,r=1}^n D_i^{jk} S_{rj} C_{rk}^t = \sum_{j,k,r=1}^n D_i^{jk} S_{rk} C_{jr}^t = 0 \end{cases} \qquad i \in \{1, .., n\}, t \in \{2, .., n\}$$

The set $Hopf_n$ is a Zariski open subset of $Bialg_n$. It carries also a structure of scheme.

1.5. The $\mathrm{GL_n}(\mathrm{K})$ action and coaction, orbits. In the following I will deøne $GL_n(K)$ action and coaction for Hopf algebras. Naturally, we have a similar deønition for bialgebras.

Geometrically, a point $\left(C_{ij}^k, D_i^{jk}, \xi_i : i,j,k = 1,\cdots,n\right)$ of K^{2n^3+n} satisfying (7) (8) (9) (10) where the matrix (S_{ij}) deønes the antipode, represents an n-dimensional Hopf algebra H, along with a particular choice of basis. A change of basis in H may give rise to a diœerent point of $Hopf_n$. Let $H = (V,\mu,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra, The ȷstructure transportȷ action is deøned on H by the following action of $GL_n(K)$

$$\begin{gathered} GL_n(K) \times Hopf_n \longrightarrow Hopf_n \\ (f,H) \longrightarrow f\cdot H \end{gathered}$$

where $\forall X,Y \in V$

$$\begin{gathered} (f\cdot\mu)(X,Y) = f^{-1}(\mu(f(X),f(Y))) \\ (f\cdot\Delta)(X) = f^{-1}\otimes f^{-1}(\Delta(f(X))) \\ (f\cdot\varepsilon)(X) = \varepsilon(f(X)) \end{gathered}$$

The orbit of a Hopf algebra H is given by $\vartheta(H) = \{H' = f\cdot H, \quad f\in GL_n(K)\}$. The orbits are in 1-1-correspondence with the isomorphism classes of n-dimensional Hopf algebras.
The stabilizer subgroup of H $(stab(H) = \{f \in GL_n(K) : H = f\cdot H\})$ is exactly $Aut(H)$, the automorphism group of H.

The orbit $\vartheta(H)$ is identiøed with $GL_n(K)/Aut(H)$. Then

$$\dim\vartheta(H) = n^2 - \dim Aut(H)$$

The orbit $\vartheta(H)$ is provided with the structure of a diœerentiable manifold. In fact, $\vartheta(H)$ is the image through the action of the Lie group $GL_n(K)$ of the point H, considered as a point of $Hom(V\otimes V,V)\times Hom(V,V\otimes V)$.

The Zariski open orbits have a special interest in the geometric study of $Hopf_n$. It corresponds to a so called rigid Hopf algebras. The orbit's closure of a rigid Hopf algebra determines an irreducible component of $Hopf_n$.

2. Degenerations

This concept ørst appeared in the physics literature. The question was to show in which sense a group can be a limiting case of other groups. The degenerations, called also contactions or specialisations, was introduced for Lie groups by Segal, Inonu and Wigner (1953) [8]. They showed that the Galilei group of classical mechanics is a limiting case of the Lorentz group corresponding to relativistic mechanics. Later, Saletan (1960) [15] generalized the notion and stated a general condition to the existence of degeneration (contraction) of Lie algebras. This notion, useful for geometric study of a variety, was used by several peoples in the studies of associative algebras varieties or Lie algebras varieties (Gabriel, Gerstenhaber, Goze, Happel, Mazzola, Makhlouf, Schaps...). It was also used by Celegheni, Giachetti, Sorace and Tarlini [2] to deøne Heisenberg and Euclidean quantum groups.

Deønition. A degeneration of a Hopf algebra $H = (V,\mu,\Delta,\eta,\varepsilon,S)$ over K is the limit of a sequence $f_t\cdot H$ when $t\to 0$ and where f_t is a family of linear maps on V over K.

$$H_0 = \lim_{t\to 0} f_t \cdot H$$

where the multiplication and the comultiplication μ_0 and Δ_0 of H_0 satisfy

$$\mu_0 = \lim_{t\to 0} f_t \cdot \mu = \lim_{t\to 0} f_t^{-1} \circ \mu \circ f_t \otimes f_t$$
$$\Delta_0 = \lim_{t\to 0} f_t \cdot \Delta = \lim_{t\to 0} f_t^{-1} \otimes f_t^{-1} \circ \Delta \circ f_t$$

1. A degeneration of Hopf algebra is a Hopf algebra. In fact, the multiplication $\mu_t = f_t^{-1} \circ \mu \circ f_t \otimes f_t$ and the comultiplication $\Delta_t = f_t^{-1} \otimes f_t^{-1} \circ \Delta \circ f_t$ satisfy the conditions of Hopf algebra, then when t tends to 0 the conditions remain satisøed.
2. Equivalent deønition : H_0 is a degeneration of H if $H_0 \in \overline{\theta(H)}$ (the Zariski closure of the orbit of H). Geometrically, this means that H_0 and H belong to the same irreducible component.
3. Conversely, every Hopf algebra in $\overline{\theta(H)}$ is a degeneration of H.
4. The images of the multiplication and the comultiplication of $f_t \cdot H$ are in general in the Laurent power series ring $V\left[\left[t, t^{-1}\right]\right]$ or $V\left[\left[t, t^{-1}\right]\right] \bigotimes V\left[\left[t, t^{-1}\right]\right]$. But when the degeneration exists, they are in power series ring $V[[t]]$ or $V[[t]] \bigotimes V[[t]]$
5. The same deønitions and remarks hold for bialgebras.

3. Degenerations with $f_t = v + t \cdot w$

Let $f_t = v + tw$ be a family of endomorphisms where v is a singular linear map and w is a regular linear map. The aim of this section is to ønd a necessary and suŒcient conditions on v and w such that a degeneration of a given Hopf algebra $H = (V, \mu, \Delta, \eta, \varepsilon, S)$ exists.

We can set $w = id$ because $f_t = v + tw = \left(v \circ w^{-1} + t\right) \circ w$ which is isomorphic to $v \circ w^{-1} + t$. Then with no loss of generality we can consider the family $f_t = \varphi + t \cdot id$ from V into V where φ is a singular map. The vector space V can be decomposed by φ under the form $V_R \oplus V_N$ where V_R and V_N are φ-invariant deøned in a canonical way such that φ is surjective on V_R and nilpotent on V_N. Let q be the smallest integer such that $\varphi^q(V_N) = 0$. The inverse of f_t exists on V_R and is equal to $\varphi^{-1}\left(t\varphi^{-1} + id\right)^{-1}$. But on V_N, since $\varphi^q = 0$, it is given by

$$\frac{1}{\varphi + t \cdot id} = \frac{1}{t} \cdot \frac{1}{\varphi/t + id} = \frac{1}{t} \cdot \sum_{i=0}^{\infty} \left(-\frac{\varphi}{t}\right)^i = \frac{1}{t} \cdot \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i$$

It follows that

$$f_t^{-1} = \begin{cases} \varphi^{-1}\left(t\varphi^{-1} + id\right)^{-1} & \text{on } V_R \\ \frac{1}{t} \cdot \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i & \text{on } V_N \end{cases}$$

3.1. Degeneration of an algebra. Let $f_t = \varphi + t \cdot id$ be a family of linear maps on V, where φ is a singular map. The action of f_t on μ is deøned by $f_t \cdot \mu = f_t^{-1} \circ \mu \circ f_t \otimes f_t$ then

$$f_t \cdot \mu(x \otimes y) = f_t^{-1} \circ \mu(f_t(x) \otimes f_t(y))$$
$$= f_t^{-1}\left(\mu(\varphi(x) \otimes \varphi(y)) + t(\mu(\varphi(x) \otimes y) + \mu(x \otimes \varphi(y))) + t^2\mu(x \otimes y)\right)$$

Since every element v of V decomposes in $v = v_R + v_N$, we set

$$A = \mu(x \otimes y) = A_R + A_N,$$
$$B = \mu(\varphi(x) \otimes y) + \mu(x \otimes \varphi(y)) = B_R + B_N$$
$$C = \mu(\varphi(x) \otimes \varphi(y)) = C_R + C_N$$

Then

$$f_t \cdot \mu(x \otimes y) = \varphi^{-1}\left(t\varphi^{-1} + id\right)^{-1}\left(t^2 A_R + tB_R + C_R\right) + \frac{1}{t} \cdot \sum_{i=0}^{q-1}\left(-\frac{\varphi}{t}\right)^i \left(t^2 A_N + tB_N + C_N\right)$$

If t goes to 0, then $\varphi^{-1}\left(t\varphi^{-1} + id\right)^{-1}\left(t^2 A_R + tB_R + C_R\right)$ goes to $\varphi^{-1}(C_R)$. The limit of the second term is :

$$\lim_{t\to 0} \tfrac{1}{t} \cdot \textstyle\sum_{i=0}^{q-1}\left(-\frac{\varphi}{t}\right)^i \left(t^2 A_N + tB_N + C_N\right)$$
$$= \lim_{t\to 0}\left(\tfrac{id}{t} - \tfrac{\varphi}{t^2} + \tfrac{\varphi^2}{t^3} - \cdots (-1)^{q-1}\tfrac{\varphi^{q-1}}{t^q}\right)\left(t^2 A_N + tB_N + C_N\right)$$
$$= \lim_{t\to 0} tA_N - \varphi(A_N) + \left(\tfrac{id}{t} - \tfrac{\varphi}{t^2} + \tfrac{\varphi^2}{t^3} - \cdots (-1)^{q-1}\tfrac{\varphi^{q-3}}{t^{q-2}}\right)\varphi^2(A_N) +$$
$$B_N - \left(\tfrac{id}{t} - \tfrac{\varphi}{t^2} + \tfrac{\varphi^2}{t^3} - \cdots (-1)^{q-1}\tfrac{\varphi^{q-2}}{t^{q-1}}\right)\varphi(B_N) +$$
$$\left(\tfrac{id}{t} - \tfrac{\varphi}{t^2} + \tfrac{\varphi^2}{t^3} - \cdots (-1)^{q-1}\tfrac{\varphi^{q-1}}{t^q}\right) C_N$$
$$= \lim_{t\to 0} tA_N + B_N - \varphi(A_N) +$$
$$\left(\tfrac{id}{t} - \tfrac{\varphi}{t^2} + \tfrac{\varphi^2}{t^3} - \cdots (-1)^{q-1}\tfrac{\varphi^{q-1}}{t^q}\right)\left(\varphi^2(A_N) - \varphi(B_N) + C_N\right)$$

This limit exists if and only if

$$\varphi^2(A_N) - \varphi(B_N) + C_N = 0$$

which is equivalent to

$$(11) \quad \varphi^2\left(\mu(x \otimes y)_N\right) - \varphi\left(\mu(\varphi(x) \otimes y)_N\right) - \varphi\left(\mu(x \otimes \varphi(y))_N\right) + \mu(\varphi(x) \otimes \varphi(y))_N = 0$$

And the limit is $B_N - \varphi(A_N)$.

Proposition 3.1. The degeneration of the algebra μ exists if and only if the condition

$$(11) \quad \varphi^2 \circ \mu_N - \varphi \circ \mu_N \circ \varphi \otimes id - \varphi \circ \mu_N \circ id \otimes \varphi + \mu_N \circ \varphi \otimes \varphi = 0$$

where $\mu_N(x, y) = (\mu(x, y))_N$, holds. And it is deøned by

$$\mu_0 = \varphi^{-1} \circ \mu_R \circ \varphi \otimes \varphi + \mu_N \circ \varphi \otimes id + \mu_N \circ id \otimes \varphi - \varphi \circ \mu_N$$

3.2. Degeneration of a coalgebra. Let $f_t = \varphi + t \cdot id$ be a family of linear maps on V, where φ is a singular map. The action of f_t on Δ is deøned by $f_t \cdot \Delta = f_t^{-1} \otimes f_t^{-1} \circ \Delta \circ f_t$ then

$$f_t \cdot \Delta(x) = f_t^{-1} \otimes f_t^{-1} \circ \Delta(f_t(x)) = t \cdot f_t^{-1} \otimes f_t^{-1} \circ \Delta(x) + f_t^{-1} \otimes f_t^{-1}(\Delta(\varphi(x)))$$

Setting $\Delta(x) = x^{(1)} \otimes x^{(2)}$, $\Delta(\varphi(x)) = \varphi(x)^{(1)} \otimes \varphi(x)^{(2)}$,
and for $i = 1, 2$ $\quad x^{(i)} = x_R^{(i)} + x_N^{(i)}$, $\varphi(x)^{(i)} = \varphi(x)_R^{(i)} \otimes \varphi(x)_N^{(i)}$.
Then

$$f_t \cdot \Delta(x) = t \cdot \left(\left(f_t^{-1}\left(x_R^{(1)}\right) + f_t^{-1}\left(x_N^{(1)}\right)\right) \otimes \left(f_t^{-1}\left(x_R^{(2)}\right) + f_t^{-1}\left(x_N^{(2)}\right)\right)\right) +$$

$+\left(f_t^{-1}\left(\varphi\left(x\right)_R^{(1)}\right)+f_t^{-1}\left(\varphi\left(x\right)_N^{(1)}\right)\right)\otimes\left(f_t^{-1}\left(\varphi\left(x\right)_R^{(2)}\right)+f_t^{-1}\left(\varphi\left(x\right)_N^{(2)}\right)\right)$

Setting $\psi=\varphi^{-1}\left(t\varphi^{-1}+id\right)^{-1}$, then when $t\to 0$, $\psi=\varphi^{-1}$ and

$f_t\cdot\Delta\left(x\right)=t\cdot\psi\left(x_R^{(1)}\right)\otimes\psi\left(x_R^{(2)}\right)+\psi\left(\varphi\left(x\right)_R^{(1)}\right)\otimes\psi\left(\varphi\left(x\right)_R^{(2)}\right)+$
$+\sum_{i=0}^{q-1}\psi\left(x_R^{(1)}\right)\otimes\left(-\frac{\varphi}{t}\right)^i\left(x_N^{(2)}\right)+\left(-\frac{\varphi}{t}\right)^i\left(x_N^{(1)}\right)\otimes\psi\left(x_R^{(2)}\right)+$
$+\frac{1}{t}\sum_{i=0}^{q-1}\psi\left(\varphi\left(x\right)_R^{(1)}\right)\otimes\left(-\frac{\varphi}{t}\right)^i\left(\varphi\left(x\right)_N^{(2)}\right)+\left(-\frac{\varphi}{t}\right)^i\left(\varphi\left(x\right)_N^{(1)}\right)\otimes\psi\left(\varphi\left(x\right)_R^{(2)}\right)+$
$+\frac{1}{t}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\left(-\frac{\varphi}{t}\right)^j\left(x_N^{(1)}\right)\otimes\left(-\frac{\varphi}{t}\right)^i\left(x_N^{(2)}\right)+\frac{1}{t^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\left(-\frac{\varphi}{t}\right)^j\left(\varphi\left(x\right)_N^{(1)}\right)\otimes$
$\left(-\frac{\varphi}{t}\right)^i\left(\varphi\left(x\right)_N^{(2)}\right)$

Then

$f_t\cdot\Delta\left(x\right)=\psi\otimes\psi\left(t\cdot x_R^{(1)}\otimes x_R^{(2)}+\varphi\left(x\right)_R^{(1)}\otimes\varphi\left(x\right)_R^{(2)}\right)$
$+\sum_{i=0}^{q-1}\frac{1}{(-t)^i}\left(\psi\otimes\varphi^i\left(x_R^{(1)}\otimes x_N^{(2)}\right)+\varphi^i\otimes\psi\left(x_N^{(1)}\otimes x_R^{(2)}\right)\right)+$
$-\sum_{i=0}^{q-1}\frac{1}{(-t)^{i+1}}\left(\psi\otimes\varphi^i\left(\varphi\left(x\right)_R^{(1)}\otimes\varphi\left(x\right)_N^{(2)}\right)+\varphi^i\otimes\psi\left(\varphi\left(x\right)_N^{(1)}\otimes\varphi\left(x\right)_R^{(2)}\right)\right)+$
$+\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\frac{-1}{(-t)^{i+j+1}}\varphi^j\otimes\varphi^i\left(x_N^{(1)}\otimes x_N^{(2)}\right)+\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\frac{-1}{(-t)^{i+j+2}}\varphi^j\otimes\varphi^i\left(\varphi\left(x\right)_N^{(1)}\otimes\varphi\left(x\right)_N^{(2)}\right)$

By rewriting the sums one obtain

$f_t\cdot\Delta\left(x\right)=\psi\otimes\psi\left(t\cdot x_R^{(1)}\otimes x_R^{(2)}+\varphi\left(x\right)_R^{(1)}\otimes\varphi\left(x\right)_R^{(2)}\right)+$
$\psi\otimes id\left(x_R^{(1)}\otimes x_N^{(2)}\right)+id\otimes\psi\left(x_N^{(1)}\otimes x_R^{(2)}\right)+$
$\sum_{i=0}^{q-1}\frac{1}{(-t)^{i+1}}\psi\otimes\varphi^i\left(x_R^{(1)}\otimes\varphi\left(x_N^{(2)}\right)-\varphi\left(x\right)_R^1\otimes\varphi\left(x\right)_N^{(2)}\right)+$
$\sum_{i=0}^{q-1}\frac{1}{(-t)^{i+1}}\varphi^i\otimes\psi\left(\varphi\left(x_N^{(1)}\right)\otimes x_R^{(2)}-\varphi\left(x\right)_N^{(1)}\otimes\varphi\left(x\right)_R^{(2)}\right)+$
$\sum_{i=0}^{q-1}\frac{1}{(-t)^{i+1}}\varphi^i\otimes id\left(x_N^{(1)}\otimes x_N^{(2)}\right)+$
$\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\frac{-1}{(-t)^{i+j+2}}\varphi^j\otimes\varphi^i\left(\varphi\left(x\right)_N^{(1)}\otimes\varphi\left(x\right)_N^{(2)}-x_N^{(1)}\otimes\varphi\left(x_N^{(2)}\right)\right)$

This limit of $f_t\cdot\Delta\left(x\right)$ exists if and only if

$$(12)\quad\begin{cases}x_R^{(1)}\otimes\varphi\left(x_N^{(2)}\right)-\varphi\left(x\right)_R^1\otimes\varphi\left(x\right)_N^{(2)}=0\\ \varphi\left(x_N^{(1)}\right)\otimes x_R^{(2)}-\varphi\left(x\right)_N^{(1)}\otimes\varphi\left(x\right)_R^{(2)}=0\\ x_N^{(1)}\otimes x_N^{(2)}=0\\ \varphi\left(x\right)_N^{(1)}\otimes\varphi\left(x\right)_N^{(2)}-x_N^{(1)}\otimes\varphi\left(x_N^{(2)}\right)=0\end{cases}\qquad\forall x\in V$$

Proposition 3.2. The degeneration of the coalgebra Δ exists if and only if the condition (12) holds and it is deøned for all $x\in V$ by

$$\Delta_0\left(x\right)=\varphi^{-1}\left(\varphi\left(x\right)_R^{(1)}\right)\otimes\varphi^{-1}\left(\varphi\left(x\right)_R^{(2)}\right)+\varphi^{-1}\left(x_R^{(1)}\right)\otimes x_N^{(2)}+x_N^{(1)}\otimes\varphi^{-1}\left(x_R^{(2)}\right)$$

3.3. Degeneration of Hopf algebra. Let $H=(V,\mu,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra and $f_t=\varphi+t\cdot id$ be a family of linear maps of V, where φ is a singular map. We suppose also that φ decomposes the vector space V as $V=V_R+V_N$. Then the degeneration $H_0=\lim_{t\to 0}f_t\cdot H$ exists if and only if the conditions (11) (12) holds and the multiplication and the comultiplication deøned by

$$\mu_0(x,y)=\varphi^{-1}\left(\mu\left(\varphi\left(x\right)\otimes\varphi\left(y\right)\right)_R\right)+\mu\left(\varphi\left(x\right)\otimes y\right)_N+\mu\left(x\otimes\varphi\left(y\right)\right)_N-\varphi\left(\mu\left(x\otimes y\right)_N\right)$$
$$\Delta_0\left(x\right)=\varphi^{-1}\left(\varphi\left(x\right)_R^{(1)}\right)\otimes\varphi^{-1}\left(\varphi\left(x\right)_R^{(2)}\right)+\varphi^{-1}\left(x_R^{(1)}\right)\otimes x_N^{(2)}+x_N^{(1)}\otimes\varphi^{-1}\left(x_R^{(2)}\right)$$

satisfy the compatibility condition

$(\mu_0 \otimes \mu_0)(id \otimes \tau \otimes id)(\Delta_0 \otimes \Delta_0) = \Delta_0 \mu_0$ where τ is the twist map

4. Connection between deformation and degeneration

The notion of deformation is in some sense the dual notion of the degeneration.

Let $H = (V, \mu_0, \Delta_0, \eta_0, \varepsilon_0, S_0)$ be a Hopf algebra over a øeld K. Let $K[[t]]$ be the power series ring in one variable t. Let $V[[t]]$ be the extension of V by extending the coeŒcient domain from K to $K[[t]]$. Then $V[[t]]$ is a $K[[t]]$-module and $V[[t]] = V \otimes_K K[[t]]$.

A deformation of H is a one parameter family $H_t = (V[[t]], \mu_t, \Delta_t, \eta_t, \varepsilon_t, S_t)$. Since the unit, counit and the antipode are perserved by deformation [6], It follows that a deformation of $H = (V, \mu_0, \Delta_0, \eta_0, \varepsilon_0, S_0)$ can be considered as a pair of deformations (μ_t, Δ_t) which together give on $V[[t]]$ the structure of bialgebra over $k[[t]]$.

By $k[[t]]$-linearity the morphisms μ_t, Δ_t are determined by their restrictions to $V \otimes V$:

$$\mu_t : \begin{array}{l} V \otimes V \to V[[t]] \\ x \bigotimes y \to \mu_t(x \bigotimes y) = \sum_{m=0}^{\infty} \mu_m (x \bigotimes y) t^m \end{array} \quad \text{with } \mu_m \in Hom(V \otimes V, V)$$

$$\Delta_t : \begin{array}{l} V \to V[[t]] \otimes V[[t]] = (V \otimes V)[[t]] \\ x \to \Delta_t(x) = \sum_{m=0}^{\infty} \Delta_m(x) t^m \end{array} \quad \text{with } \Delta_m \in Hom(V, V \otimes V)$$

and they satisfy

- μ_t is associative
- Δ_t is coassociative
- $(\mu_t \otimes \mu_t)(id \otimes \tau \otimes id)(\Delta_t \otimes \Delta_t) = \Delta_t \mu_t$ where τ is the twist map

Remark 4.1. Since the neighbourhood of a Hopf algebra is formed by Hopf algebras in $Bialg_n$ then the set $Hopf_n$ is open for the metric topology in $Bialg_n$.

Proposition 4.1. Every degeneration corresponds to a deformation.

In fact, let $H_0 = \lim_{t \to 0} f_t \cdot H$ be a degeneration of H then $H_t = f_t \cdot H$ is a deformation of H_0.

Remark 4.2. The converse is in general false. For example, in the family of algebras $A_t = K\langle x, y\rangle / < x^2, y^2, yx - txy >$ where $K\langle x, y\rangle$ stand for the free algebra with unit. Two algebras A_t and A_s are not isomorphic if $t \cdot s \neq 1$. Thus A_t is a deformation of A_0 but the family A_t is not isomrphic to a given algebra.

Remark 4.3. The concepts of degeneration and deformation are useful in the geometric study of the irreducible component of $Hopf_n$. Two Hopf algebras are in the same irreducible component if one is a degeneration of the other or they belong to a parameter family of Hopf algebras. This means that a deformation of one is isomorphic the second.

5. Hopf algebras classifications,rigidity, irreducible components

5.1. Classiøcation. The general classiøcation, up to isomorphism, of Hopf algebras is not known. However, the complete classiøcation is known for dimension n, $n \leq 13$, see [21] [18] [12] [3]and [19]. There is also some results in the semisimple case, see [14]. I recall here the classiøcation of n-dimensional Hopf algebras with $n \leq 13$:

Notation

Let Z_n denotes the cyclic group, D_n the dihedral group, S_n the symmetric group, H_4 the quaternion group and Al the alternate group. Let KG be the Hopf algebra of the group G and $(KG)^*$ its dual. Let T_{n^2} be the Taft algebra, it is deøned by

$$\frac{K\langle x,y\rangle}{\langle x^n,\ y^n-1,\ xy-qyx\rangle}$$

Where q is the primitive root of unity of order n.

The coalgebra algebra structure and the antipode are detemined by

$\Delta(g)=g\otimes g,\quad \Delta(x)=x\otimes g+1\otimes x,$
$\varepsilon(x)=0,\quad \varepsilon(g)=1.$
$S(x)=-xg^{-1},\quad S(g)=g^{-1}.$

Theorem 5.1. If H is Hopf algebra of dimension $n \leq 13$, then H is isomorphic with one and only one of the following Hopf algebra

- $n \in \{2,3,5,7,11,13\}$

since the dimension is prime there is only the group algebra KZ_n.

- $n=4$

there is 3 isomorphic classes, the semisimple Hopf algebras KZ_4 and $K\left(Z_2\times Z_2\right)$, and the Taft-Sweedler algebras T_4.

- $n=6$

KZ_6, KS_3 and $(KS_3)^*$

- $n=8$

The semisimple Hopf algebras are : $K(Z_2\times Z_2\times Z_2)$, $K(Z_2\times Z_4)$, KZ_8, KD_4, $(KD_4)^*$, KH_4, $(KH_4)^*$ and A_8. Where A_8 denotes the unique $(n=8)$ semisimple Hopf algebra that is not a group algebra, it is deøned by

$$\frac{K\langle x,y,z\rangle}{\left\langle x^2-1,\ y^2-1,\ z^2-\frac{1}{2}(1+x+y-xy),\ xy-yx,\ zx-yz,\ zy-xz\right\rangle};$$

the coalgebra structure Δ, ε and the antipode S are detemined by

$\Delta(x)=x\otimes x,\quad \Delta(y)=y\otimes y,\quad \Delta(z)=\frac{1}{2}(1\otimes 1+1\otimes x+y\otimes 1-y\otimes x)(z\otimes z),$
$\varepsilon(x)=\varepsilon(y)=\varepsilon(z)=1$
$S(x)=x,\quad S(y)=y,\quad S(z)=z.$

The nonsemisimple Hopf algebras are :

In the following the subscript denote the set of grouplike elements

1.

$$A_{C_2}=\frac{K\langle x,y,g\rangle}{\langle g^2-1,x^2,y^2,gx+xg,yg+gy,xy+yx\rangle}$$

The coalgebra structure and the antipode are detemined by

$\Delta(g)=g\otimes g,\quad \Delta(x)=x\otimes g+1\otimes x,\quad \Delta(y)=y\otimes g+1\otimes y,$
$\varepsilon(x)=\varepsilon(y)=0,\quad \varepsilon(g)=1.$
$S(x)=-gx,\quad S(y)=-gy,\quad S(g)=g.$

2.

$$A'_{C_4} = \frac{K\langle x, g\rangle}{\langle g^4-1, x^2, gx+xg\rangle}$$

The coalgebra structure and the antipode are detemined by

$\Delta(g) = g\otimes g, \quad \Delta(x) = x\otimes g + 1\otimes x,$
$\varepsilon(x) = 0, \quad \varepsilon(g) = 1.$
$S(x) = -xg^3, \quad S(g) = g^3.$

3.

$$A''_{C_4} = \frac{K\langle x, g\rangle}{\langle g^4-1, x^2-g^2+1, gx+xg\rangle}$$

The coalgebra structure and the antipode are detemined by

$\Delta(g) = g\otimes g, \quad \Delta(x) = x\otimes g + 1\otimes x,$
$\varepsilon(x) = 0, \quad \varepsilon(g) = 1.$
$S(x) = -xg^3, \quad S(g) = g^3.$

4.

$$A'''_{C_4,q} = \frac{K\langle x, g\rangle}{\langle g^4-1, x^2, gx-qxg\rangle}$$

Where q is the primitive root of unity of order 4.
The coalgebra structure and the antipode are detemined by

$\Delta(g) = g\otimes g, \quad \Delta(x) = x\otimes g^2 + 1\otimes x,$
$\varepsilon(x) = 0, \quad \varepsilon(g) = 1.$
$S(x) = -xg^3, \quad S(g) = g^3.$

5. $\left(A''_{C_4}\right)^*$

6.

$$A_{C_2\times C_2} = \frac{K\langle g, h, x\rangle}{\langle g^2-1, h^2-1, x^2, gx+xg, hx+xh, gh-hg\rangle}$$

The coalgebra structure and the antipode are detemined by

$\Delta(g) = g\otimes g, \quad \Delta(h) = h\otimes h, \quad \Delta(x) = x\otimes g + 1\otimes x,$
$\varepsilon(x) = 0, \quad \varepsilon(g) = \varepsilon(h) = 1.$
$S(g) = g, \quad S(h) = h, \quad S(x) = -xg.$

- $n = 9$

KZ_9, $K(Z_3\times Z_3)$ and the Taft algebra T_9.

- $n = 10$

KZ_{10}, KD_5 and $(KD_5)^*$.

- $n = 12$

The semisimple Hopf algebras are : KZ_{12}, $K(Z_6\times Z_2)$, $K(Z_4\times Z_3)$, KD_6, $(KD_6)^*$, Al_4, $(Al_4)^*$, A_+ and A_-. Where A_+ and A_- denote the semisimples Hopf algebras that are not a group algebra. They are self dual and may be deøned as the KS_3-rings generated by v with relations :

$$v^2 = v, \quad av = va \quad (a\in KS_3)$$

the coalgebra structure Δ, ε and the antipode S of A_+ (resp. A_-) are detemined by

$\Delta(\sigma) = \sigma v\otimes\sigma + \sigma(1-v)\otimes\sigma^2,$
$\Delta(\tau) = \tau\otimes\tau \qquad$ (resp. $\Delta(\tau) = \tau v\otimes\tau + \tau(1-v)\otimes\tau(2v-1)$)

$\Delta(v) = v \otimes v + (1-v) \otimes (1-v)$
$\varepsilon(\sigma) = \varepsilon(\tau) = \varepsilon(v) = 1$
$S(\sigma) = \sigma(1-v) + \sigma^2 v, \quad S(\tau) = \tau \qquad (\text{resp.} \quad S(\tau) = \tau(2v-1)), \quad S(v) = v.$
The nonsemisimple Hopf algebras are :
1.

$$A_0 = \frac{K\langle x, g\rangle}{\langle g^6 - 1, x^2, gx + xg\rangle}$$

The coalgebra structure and the antipode are detemined by
$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x,$
$\varepsilon(x) = 0, \quad \varepsilon(g) = 1.$
$S(g) = g^{-1}, \quad S(x) = -xg.$
2.

$$A_1 = \frac{K\langle x, g\rangle}{\langle g^6 - 1, x^2 + g^2 - 1, gx + xg\rangle}$$

The coalgebra structure and the antipode are detemined by
$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x,$
$\varepsilon(x) = 0, \quad \varepsilon(g) = 1.$
$S(g) = g^{-1}, \quad S(x) = -xg.$
3.

$$B_0 = \frac{K\langle x, g\rangle}{\langle g^6 - 1, x^2, gx + xg\rangle}$$

The coalgebra structure and the antipode are detemined by
$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g^3 \otimes x,$
$\varepsilon(x) = 0, \quad \varepsilon(g) = 1.$
$S(g) = g^{-1}, \quad S(x) = -xg.$
4.

$$B_1 = \frac{K\langle x, g\rangle}{\langle g^6 - 1, x^2, gx - qxg\rangle}$$

Where q is the primitive root of unity of order 6.
The coalgebra structure and the antipode are detemined by
$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g^3 \otimes x,$
$\varepsilon(x) = 0, \quad \varepsilon(g) = 1.$
$S(g) = g^{-1}, \quad S(x) = -xg.$

5.2. Rigid Hopf algebras and irreducible components.

Deønition 5.1. A Hopf algebra H is rigid if and only if every deformation of H is isomorphic to H.

Remark 5.1.
- The deønition may be rephrased by ȷThe orbit of H is Zariski openedȷ.
- If the second cohomological group of H vanishes then H is rigid.
- Every group algebra is rigid because it is semisimple and its second cohomological groups vanishes, see [6][17]
- The Zariski open orbits have a special interest in the geometric study of $Hopf_n$, Their closure determines an irreducible component.
- Two non isomorphic rigid Hopf algebras cannot belong to the same irreducible component

- Stefan has proved that the irreducible component containig a given semisimple Hopf algebra is n^2-dimensional[17].
- There is a ønite number of open orbits because the algebraic variety $Hopf_n$ decomposes in a ønite number of irreducible components.
- Many but not all components of $Hopf_n$ are orbits closures of rigid Hopf algebras, for example there are inønitely many isomorphisms classes for $\dim H = p^4$ (p odd and prime) [1].

Theorem 5.2. Every Hopf algebras of dimension $n \leq 13$ is rigid.

Proof. For $n \in \{2,3,5,7,11,13\}$, The Hopf algebras are all group algebra then rigid.

For $n = 4$, The group algebra KZ_4 and $K(Z_2 \times Z_2)$ are rigid and the Taft-Sweedler Hopf algebra cannot be deformed in a commutative algebra, then the Taft-Sweedler Hopf algebra is also rigid. In fact, as algebra, the group algebras belong to the irreducible component of commutative algebras and the Taft Sweedler algebra belongs to the irreducible component of continuous series ([3])

For $n = 6$, all algebras are semi-simple, then rigid.

For $n = 8$, the group algebras are rigid and the pointed nonsemisimple Hopf algebras cannot be deformed because there is no family deøning a deformation of the given Hopf algebra.

For $n = 9$, the group algebra are rigid and the Taft algebra cannot be deformed in a commutative algebra, then it is also rigid.

For $n = 10$, all the algebras are semisimple, then rigid.

For $n = 12$, the group algebras are rigid and the pointed non semisimple Hopf algebras cannot be deformed.

Remark 5.2. It is interesting to see whether for the nonsemisimple rigid Hopf algebras the second cohomological group vanishes.

∎

Corollary 5.3. The algebraic varieties $Hopf_n$ $n \leq 13$ are unions of open orbits.

Corollary 5.4. The following table gives the number of irreducible components of $Hopf_n$ for $n \leq 13$

dimension	number of irreducibe components of $Hopf_n$
$n \in \{2,3,5,7,11,13\}$	1
$n = 4$	3
$n = 6$	3
$n = 8$	14
$n = 9$	3
$n = 10$	3
$n = 12$	14

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