# ON THE CLASSIFICATION OF RIGID LIE ALGEBRAS

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ABSTRACT. After having given the classification of solvable rigid Lie algebras of low dimensions, we study the general case concerning rigid Lie algebras whose nilradical is filiform and present their classification.
*Keywords : Rigid Lie Algebra, Filiform Lie algebra*

## 1. STRUCTURE OF RIGID LIE ALGEBRAS

1.1. **Definition.** Let $L^n$ be the algebraic variety of complex Lie algebra laws on $\mathbb{C}^n$. Consider the natural action of the algebraic group $GL(n,\mathbb{C})$ on $L^n$ given by

$$\begin{array}{rcl} GL(n,\mathbb{C}) \times L^n & \to & L^n \\ (f,\mu) & \to & f * \mu \end{array}$$

with $(f*\mu)(X,Y) = f^{-1}(\mu(f(X),f(Y))$ for all $X,Y \in \mathbb{C}^n$. We note by $\mathcal{O}(\mu)$ the orbit of $\mu$.

**Definition 1.1.** *The Lie algebra law $\mu$ (or the complex Lie algebra $\mathfrak{g}$ of law $\mu$) is called rigid if $\mathcal{O}(\mu)$ is a Zariski open set of $L^n$.*

Each open orbit of this action of $GL(n,\mathbb{C})$ on $L^n$ gives, considering its Zariski closure, an irreducible component of $L^n$ . Therefore, only a finite number of those orbits exist. The first results about rigid Lie algebras are due to Gerstenhaber [7], Nijenhuis and Richardson [10]. The last two authors have transformed the topological problems related to rigidity into cohomological problems, proving that an algebra is rigid if the second group of the Chevalley cohomology is trivial. This theorem allows the construction of examples of rigid Lie algebras and is used in proving that semi-simple algebras are rigid. However, the existence of rigid Lie algebras with non trivial second cohomology group shows that the cohomological viewpoint is not fully satisfactory in the study of rigidity [1]. In this paper, we recall some structural theorems related to rigid complex Lie algebras which allow a general classification of these Lie algebras.

All Lie algebras considered are finite dimensional complex Lie algebras.

1.2. **Decomposability of rigid Lie algebras.** By a result due to Carles [6], it follows that any rigid Lie algebra $\mathfrak{g}$ is algebraic, i. e., it is isomorphic to the Lie algebra of an algebraic group. As the algebraicity is equivalent to the decomposability of the algebra [6], it follows that for solvable rigid Lie algebras $\mathfrak{g}$ we have the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical and $\mathfrak{t}$ an exterior torus of derivations in the sense of Malcev; that is, $\mathfrak{t}$ an abelian subalgebra of $\mathfrak{g}$ such that $adX$ is semisimple for all $X \in \mathfrak{t}$.

In [2] we introduced the notion of roots for rigid solvable Lie algebras. We recall this approach briefly.

**Definition 1.2.** *We say that $X \in \mathfrak{t}$ is regular if the dimension of*

$$V_0(X) = \{Y \in \mathfrak{g} \quad | \quad [X,Y] = 0\}$$

*is minimal that is,* $\dim V_0(X) \leq \dim V_0(Z)$ *for all $Z$ such that $adZ$ belongs to $T$.*

Choose a regular vector $X$ and let be $p = \dim V_0(X)$. Consider a basis $(X, Y_1, ..., Y_{n-p}, X_1, ..., X_{p-1})$ of eigenvectors of $adX$ such that $(X, X_1, ..., X_{p-1})$ is a basis of $V_0(X)$, $(Y_1, ..., Y_{n-p}, X_1, ..., X_{k_0})$ is a basis of the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$, and $(X_{k_0+1}, ..., X_{p-1})$ are vectors such that $adX_i \in T$.

**Definition 1.3.** *Suppose that $\mathfrak{g}$ is not nilpotent. The root system of $\mathfrak{g}$ associated to $(X, Y_1, ..., Y_{n-p}, X_1, ..., _{p-1})$ is the linear system (S) defined by the following equations :*
*$x_i + x_j = x_k$ if the $X_k$-component of $[X_i, X_j]$ is non-zero.*
*$y_i + y_j = y_k$ if the $Y_k$-component of $[Y_i, Y_j]$ is non-zero.*
*$x_i + y_j = y_k$ if the $Y_k$-component of $[X_i, Y_j]$ is non-zero.*
*$y_i + y_j = x_k$ if the $X_k$ component of $[Y_i, Y_j]$ is non-zero.*

**Theorem 1.1.** *If rank $(S) \neq \dim(\mathfrak{n}) - 1$, then $\mathfrak{g}$ is not rigid.*

See [2] for a proof.

**Corollary 1.2.** *If $\mathfrak{g}$ is rigid then there is regular vector $X$ such that $ad_{\mathfrak{g}}X$ is diagonal and its eigenvalues are integers.*

These properties determine if a given Lie algebra is rigid or not. For example, let us suppose that all elements of $V_0(X)$ are semi-simple. If

$$\text{rank}(S) \neq \dim D^1(\mathfrak{g}) - 1$$

where $D^1(\mathfrak{g})$ is the derived subalgebra of $\mathfrak{g}$, then $\mathfrak{g}$ is not rigid.

**Remark 1.1.** *Even if the roots can be choosen in $\mathbb{Z}$, this does not in general imply that the Lie algebra is rational. In [4] various examples of this have been worked out.*

## 2. CONSTRUCTION OF RIGID LIE ALGEBRAS FROM THE NILRADICAL

The goal of this section is to prepare a classification method of rigid Lie algebras. We sketch here an approach based on the theorem 1.1. By fixing some properties of the nilradical we determine the corresponding rigid Lie algebras.

2.1. **Characteristic sequence of a nilpotent Lie algebra.** Let $\mathfrak{n}$ be a nilpotent complex finite dimensional Lie algebra. Let $Y \in \mathfrak{n} - D^1(\mathfrak{n})$ be a vector of $\mathfrak{n}$ . Consider the ordered sequence

$$c(Y) = (h_1, h_2, \cdots,)$$

$h_1 \geq h_2, ..., \geq h_p$, where $h_i$ is the dimension of the $i^{th}$ Jordan block of the nilpotent operator $adY$. As $Y$ is an eigenvector of $adY$, $h_p = 1$. Let $Y_1$ and $Y_2$ be in $\mathfrak{n} - D^1(\mathfrak{n})$. Let be $c(Y_1) = (h_1, ..., h_{p_1} = 1)$ and $c(Y_2) = (k_1, ..., k_{p_2} = 1)$ the corresponding sequences. We will say that $c(Y_1) \geq c(Y_2)$ if there is an $i$ such that $h_1 = k_1$, $h_2 = k_2$, ... , $h_{i-1} = k_{i-1}$, $h_i > k_i$. This defines a total order relation on the set of sequences $c(Y)$ (lexicografic order).

**Definition 2.1.** *The characteristic sequence of the nilpotent Lie algebra* $\mathfrak{n}$ *is :*

$$c(\mathfrak{n}) = Sup\{c(Y), Y \in \mathfrak{n} - D^1(\mathfrak{n})\}$$

It is an invariant of $\mathfrak{n}$ ( up to isomorphism). A vector $X \in \mathfrak{n}$ such that $c(X) = c(\mathfrak{n})$ is called a characteristic vector of $\mathfrak{n}$.

2.2. **Rigid Lie algebras whose radical is abelian.** We begin by studying the case where the nilradical has the smallest characteristic sequence : $(1, 1, ..., 1)$. If $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ is rigid, then

$$\text{rank} S(X) = dim \mathfrak{n} - 1$$

where $X \in \mathfrak{t}$ is a regular vector. As $\mathfrak{n}$ is abelian, the equations of $S(X)$ are the equations corresponding to the action of $\mathfrak{t}$ on $\mathfrak{n}$. Thus

$$dim \mathfrak{t} = dim \mathfrak{n}$$

and $\mathfrak{t}$ corresponds to the subalgebra of diagonal derivations of the abelian algebra $\mathfrak{n}$. Then $dim \mathfrak{g} = 2p$, where $\dim \mathfrak{n} = p$. Let $(X = X_1, ..., X_p)$ be a basis of $\mathfrak{t}$. There is a basis of $\mathfrak{n}$ satisfying

$$[X_i, Y_i] = Y_i, \ \ [X_i, Y_j] = 0 \text{ if } i \neq j$$

and there are no other nontrivial brackets because rank$S(X) = \dim \mathfrak{n}$-1. Now consider the 2-dimensional Lie algebra $\mathbf{r_2}$ defined by $[X, Y] = Y$. Then we have :

$$\mathfrak{g} = \mathbf{r_2} \oplus \mathbf{r_2} \oplus \cdots \oplus \mathbf{r_2}.$$

**Proposition 2.1.** *Every solvable rigid Lie algebra whose nilrdical is abelian is isomorphic to*

$$\mathfrak{g} = \mathbf{r_2} \oplus \mathbf{r_2} \oplus \cdots \oplus \mathbf{r_2} \ \textit{(direct sum)}$$

*where* $\mathbf{r_2}$ *is the 2-dimensional non abelian solvable Lie algebras. Therefore, we have*

$$\dim H^2(\mathfrak{g},\mathfrak{g}) = 0.$$

2.3. **Rigid Lie algebras whose nilradical is of type (2,1,...,1).** Consider a rigid solvable Lie algebra $\mathfrak{g}$= $\mathfrak{t}$⊕$\mathfrak{n}$ such that $c(\mathfrak{n}) = (2, 1, ..., 1)$. Then

$$\mathfrak{g} = \mathfrak{t} \oplus ( H_p \times \mathcal{A}_{n-2p-1}).$$

where $H_p$ is the $2p + 1-$dimensional Heisenberg algebra. As the abelian case has already been studied we can suppose that $\mathfrak{n} = H_p$ is the Heisenberg algebra of dimension $2p + 1$. Thus $rgS(X) = 2p$. The brackets of $H_p$ are given by :

$$[Y_{2i+1}, Y_{2i+2}] = Y_{2p+1} \quad i = 0, \cdots, p - 1.$$

The associated linear system is the following :

$$\begin{cases} a_1 + a_2 & = a_{2p+1} \\ a_3 + a_4 & = a_{2p+1} \\ \quad \vdots & \quad \vdots \\ a_{2p-1} + a_{2p} & = a_{2p+1} \end{cases}$$

and its rank is $p$. As rang$S(X) = 2p$, necessary $dim \mathfrak{t} = p + 1$. The corresponding decomposable Lie algebra

$$\mathfrak{g} = \mathfrak{t}_{p+1} \oplus H_p$$

defined in the adapted basis

$$(X_0, X_1, \cdots, X_p, Y_1, Y_2, \cdots, Y_{2p+1})$$

by

$$\begin{cases} [Y_{2i-1}, Y_{2i}] = Y_{2p+1} \quad i = 1, \cdots, p \\ [X_0, Y_{2i-1}] = iY_{2i-1} \quad i = 1, \cdots, p \\ [X_0, Y_{2i}] = (2p+1-i)Y_{2i} \quad i = 1, \cdots, p \\ [X_0, Y_{2p+1}] = (2p+1)Y_{2p+1} \\ [X_i, Y_{2i-1}] = Y_{2i-1} \quad i = 1, \cdots, p \\ [X_i, Y_{2i}] = -Y_{2i} \quad i = 1, \cdots, p \end{cases}$$

is rigid. In fact, it is sufficient to compute the dimension of the second cohomology group. We can deduce :

**Proposition 2.2.** *Every solvable rigid Lie algebra whose nilradical has dimension equal to $n$ and characteristic sequence $(2,1,1,...1)$ has dimension $2n-p$ and it is isomorphic to*

$$\mathfrak{g} = (\mathfrak{t}_{p+1} \oplus H_p) \oplus \mathbf{r}_2 \oplus \mathbf{r}_2 \cdots \oplus \mathbf{r}_2$$

*where $\mathbf{r_2}$ is the 2-dimensional non abelian solvable Lie algebra. Therefore, we have*

$$\dim H^2(\mathfrak{g},\mathfrak{g}) = 0.$$

2.4. **Rigid Lie algebras whose nilradical is of type (2,2,..,2,1,...,1).** The most important cases concerning these nilpotent Lie algebras are the algebras whose characteristic sequence is $(2,2,\cdots,2,1)$. Let $\mathfrak{n}$ be a nilpotent Lie algebra such that $c(\mathfrak{n}) = (2,2,...,2,1)$. There is a basis $(Y_1,\cdots,Y_p,Z_1,\cdots,Z_p,Y)$ of $\mathfrak{n}$ satisfying :

$$\begin{cases} [Y, Y_i] = Z_i \quad 1 \leq i \leq p \\ [Y_i, Y_j] = \sum a_{ij}^k Z_k \quad 1 \leq i < j \leq p \\ [Y_i, Z_j] = 0 \\ [Z_i, Z_j] = 0. \end{cases}$$

(the $a_{ij}^k$ being free parameters).

Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ be a decomposable solvable Lie algebra with $c(\mathfrak{n}) = (2,2,...,2,1)$. The determination of all rigid Lie algebras of this type is still an open problem. Nevertheless, we will begin this study considering the particular case $a_{ij}^k = 0$. If $X \in \mathfrak{t}$ is a regular vector, then $\text{rank}(S(X)) = p$. In fact the linear system $S(X)$ is given by the linear equations

$$y + y_i = z_i \ , \ i = 1,...,p.$$

Thus, if $\mathfrak{g}$ is rigid, $dim\mathfrak{t} = p+1$. In this case, the Lie algebra $\mathfrak{g}$ is given by :

$$\begin{cases} [X, Y_i] = iY_i \ ; \ [X, Z_i] = (1+p+i)Z_i \ ; \ [X, Y] = (p+1)Y \\ [X_i, Y] = 0, \ i = 1, \cdots, p \\ [X_i, Y_j] = 0 \ ; \ [X_i, Z_j] = 0, \quad i = 1, \cdots, p \ \ j = 1, \cdots, p \quad i \neq j \\ [X_i, Y_i] = Y_i \ ; \ [X_i, Z_i] = Z_i, \ i = 1, \cdots, p \\ [X_{p+1}, Y_i] = iY_i \ ; [X_{p+1}, Z_i] = (i+p+1)Z_i, \quad i = 1, \cdots, p \\ [X_{p+1}, Y] = (p+1)Y \\ [Y, Y_i] = Z_i, \quad i = 1, \cdots, p \end{cases}$$

This Lie algebra is rigid. We can verify this by considering a perturbation. In this perturbation, there is a vector $X$ such that $adX$ is diagonalizable with the same eigenvalues $1, ..., p, p+1, ..., 2p+1, 0$. The multiplicity of this last eigenvalue is always equal to $p+1$.

## 3. CLASSIFICATION OF SOLVABLE RIGID LIE ALGEBRAS WITH FILIFORM NILRADICAL

The goal of this section is to give the general classification of solvable rigid Lie algebras whose nilradical is filiform.

### 3.1. **Filiform Lie algebras.**

**Definition 3.1.** *A $n-$dimensional nilpotent Lie algebra $\mathfrak{n}$ whose characteristic sequence is $c(\mathfrak{n}) = (n-1, 1)$ is called filiform.*

Filiform Lie algebras are completely classified up to dimension eleven [9].

**example 3.1.** *Let $L_n$ and $Q_n$ the $n$-dimensional filiform Lie algebras defined by*

$$L_n : \{[Y_1, Y_j] = Y_{1+j}, \quad j = 2, ..., n-1$$

$$Q_n = \begin{cases} [Y_1, Y_j] = Y_{1+j}, & j = 2, ..., n-1 \\ [Y_i, Y_{n-i+1}] = (-1)^{i+1} Y_n, & i = 2, ..., p \end{cases} \qquad \text{where } n = 2p.$$

*These Lie algebras are filiform and naturally graded.*

Let us recall that the rank of a nilpotent Lie algebra is the dimension of a maximal exterior torus. If the Lie algebra $\mathfrak{g}$ is filiform, its rank $r(\mathfrak{g})$ satisfies

$$r(\mathfrak{g}) \leq 2.$$

For the proof, see [8].

#### 3.1.1. *Filiform Lie algebras of rank 2.*

**Proposition 3.2.** *([8]) Every filiform Lie algebras of rank 2 is isomorphic to $L_n$ or $Q_n$.*

For each Lie algebra, a maximal exterior torus is precisely determined.

If $\mathfrak{g} = L_n$, there exists a torus generated by the diagonal derivations :

$$f_1(Y_1) = 0, \quad f_1(Y_i) = Y_i \ , \ 2 \leq i \leq n$$

$$f_2(Y_1) = Y_1, \quad f_2(Y_i) = iY_i, \ 2 \leq i \leq n$$

the basis $\{Y_i\}$ being as above.

If $\mathfrak{g} = Q_n$, the basis $\{Y_i\}$ is not a basis of eigenvectors for a diagonalizable derivation. We can consider the new basis given by

$$Z_1 = Y_1 - Y_2, Z_2 = Y_2, ..., Z_n = Y_n$$

This basis satisfies

$$[Z_1, Z_j] = Z_{1+j}, \quad j = 2, ..., n-2,$$
$$[Z_i, Z_{n-i+1}] = (-1)^{i+1} Z_n, \quad i = 2, ..., p \quad , \text{ and } n = 2p$$

Then the diagonal derivations

$$f_1(Z_1) = 0, \quad f_1(Z_i) = Z_i \ , \ 2 \leq i \leq n-1 \ , \ f_1(Z_n) = 2Z_n$$

$$f_2(Z_1) = Z_1, \quad f_2(Z_i) = (i-2)Z_i, \ 2 \leq i \leq n-1, \quad f_2(Z_n) = (n-3)Z_n.$$

generates a maximal exterior torus of derivations.

3.1.2. *Filiform Lie algebras of rank 1.*

**Theorem 3.3.** *[8] Every filiform Lie algebra of rank 1 and dimension n is isomorphic to one of the following Lie algebras*

i) $A_n^k(\lambda_1,...,\lambda_{t-1})$ , $t=\left[\frac{n-k+1}{2}\right]$ , $2\le k\le n-3$

$$\begin{cases} [Y_1,Y_i]=Y_{i+1},\ i=2,...,n-1\\ [Y_i,Y_{i+1}]=\lambda_{i-1}Y_{2i+k-1}\ ,\ 2\le i\le t\\ [Y_i,Y_j]=a_{ij}Y_{i+j+k-2}\ ,\ 2\le i\le j\ ,\ i+j+k-2\le n\end{cases}$$

ii) $B_n^k(\lambda_1,...,\lambda_{t-1})$ $n=2m$ , $t=\left[\frac{n-k}{2}\right]$ , $2\le k\le n-3$

$$\begin{cases} [Y_1,Y_i]=Y_{i+1}\ i=2,...,n-2\\ [Y_i,Y_{n-i+1}]=(-1)_n^{i+1}Y\ ,\ i=2,...,n-1\\ [Y_i,Y_{i+1}]=\lambda_{i-1}Y_{2i+k-1}\ ,\ i=2,...,t\\ [Y_i,Y_j]=a_{ij}Y_{i+j-k-2}\ ,\ 2\le i,j\le n-2\ ,\ i+j+k-2\le n-2\ ,\ j\ne i+1\end{cases}$$

iii) $C_n(\lambda_1,...,\lambda_t)$ , $n=2m+2$ , $t=m-1$

$$\begin{cases} [Y_1,Y_i]=Y_{i+1}\ i=2,...,n-2\\ [Y_i,Y_{n-i+1}]=(-1)_n^{i-1}Y_n\ ,\ i=2,...m+1\\ [Y_i,Y_{n-i-2k+1}]=(-1)^{i+1}\lambda_kY_n\ ,\ i=2,...,n-2-2k\ ,\ k=1,...,m-1\end{cases}$$

The non defined brackets are equal to zero. In this theorem, $[x]$ denotes the integer part of $x$ and $(\lambda_1,...,\lambda_t)$ are non simultaneously vanishing parameters satisfying polynomial equations associated to the Jacobi conditions. Moreover, the constants $a_{ij}$ satisfy

$$a_{ij}=a_{ij+1}+a_{i+1,j}$$

and $a_{ii+1}=\lambda_{i-1}$.

3.2. **Classification of rigid algebras with nilradical of rank 2.**

**Proposition 3.4.** *Every solvable nonsplit Lie algebra whose nilradical is isomorphic to the filiform Lie algebra $L_n$ has dimension $n+2$ and it is isomorphic to the Lie algebra given by :*

$$\begin{cases} [X_1,Y_i]=iY_i\ \ 1\le i\le n\\ [X_2,Y_i]=Y_i\ \ 2\le i\le n\\ [Y_1,Y_i]=Y_{i+1}\ \ 2\le i\le n-1\end{cases}$$

*Morever, these algebras satisfy $H^2(\mathfrak{g},\mathfrak{g})=0$.*

**Proof.** These algebras have been studied in [4]. The rank of the roots system is equal to $n-3$. Then only the brackets associated to the filiformity of the nilradical are nonzero. The determination of the dimension of the second cohomology space is easy, using the Hochschild-Serre spectral sequence. From this we deduce the rigidity of these Lie algebras.

**Proposition 3.5.** *Every Lie algebra whose nilradical is isomorphic to the filiform Lie algebra $Q_n$ $(n = 2m)$ is isomorphic to $\mathfrak{s} \oplus \mathfrak{g}$ where $\mathfrak{s}$ is semi-simple and $\mathfrak{g}$ solvable, of dimension $n+2$ and isomorphic to the Lie algebra given by :*

$$\begin{cases} [X_1, Y_1] = Y_1 \\ [X_1, Y_i] = (i-2)Y_i \ , \ 3 \le i \le n-1 \\ [X_1, Y_n] = (n-3)Y_n \\ [X_2, Y_n] = 2Y_n \\ [X_2, Y_i] = Y_i \ , \ i = 2, \cdots, n-1 \\ [Y_1, Y_i] = Y_{i+1} \ , \ 2 \le i \le n-2 \\ [Y_i, Y_{n+1-i}] = (-1)^i Y_n \ , \ i = 1, \cdots, m \end{cases}$$

**Proof.** In fact, if $\mathfrak{n} = Q_{2m}$, the torus $\mathfrak{t}$ is generated by the derivations :

$$f_1(Y_1) = Y_1 \ , \ f_1(Y_i) = (i-2)Y_i \ , \ \ i = 2, \cdots, 2m-1 \ , \ f_1(Y_{2m}) = (2m-3)Y_{2m}$$

$$f_2(Y_1) = 0 \ , \ f_2(Y_i) = Y_i \ , \ \ i = 2, \cdots, 2m-1, \ f_2(Y_{2m}) = 2Y_{2m}.$$

As $Q_{2m}$ is the only algebra admitting such a torus of derivations, the algebra $\bar{\mathfrak{g}} = \mathfrak{t} \oplus \mathfrak{n}$ is rigid.

### 3.3. **Rigid Lie algebra whose nilradical is filiform of rank 1.**

3.3.1. $\mathfrak{n} = A_n^k(\lambda_1, \cdots, \lambda_{t-1})$. Let $\{Y_1, \cdots, Y_n\}$ be a basis of $\mathfrak{n}$ defined as in theorem 5.3. As the rank of $A_n^k$ is 1, we have $dim\mathfrak{t} = 1$. A diagonal derivation of $\mathfrak{n}$ is given by :

$$f(Y_1) = Y_1 \ , \ f(Y_2) = kY_2 \ , \ \cdots \ , \ f(Y_n) = (n-2+k)Y_n.$$

The classification of rigid Lie algebras of the form $\mathfrak{g} = \mathfrak{t} \oplus A_n^k$ is a consequence of the resolution of polynomial equations satisfied by the parameters $\lambda_1, \cdots, \lambda_{t-1}$.

One defines the weight of the Jacobi equation $\sum [X_i, [X_j, X_l]] = 0$ by $\rho = i+j+l$. It is clear that $6 \le \rho \le 3n-3$.

**Lemma 3.6.** *The number of Jacobi equations of weight $\rho$ is :*

$$\begin{cases} N(\rho) = 3\rho'^2 - 3\rho' + 1 \quad if \quad \rho = 6\rho' \\ N(\rho) = 3\rho'^2 - 2\rho' \quad if \quad \rho = 6\rho' + 1 \\ N(\rho) = 3\rho'^2 - \rho' \quad if \quad \rho = 6\rho' + 2 \\ N(\rho) = 3\rho'^2 \quad if \quad \rho = 6\rho' + 3 \\ N(\rho) = 3\rho'^2 + \rho' \quad if \quad \rho = 6\rho' + 4 \\ N(\rho) = 3\rho'^2 + 2\rho' \quad if \quad \rho = 6\rho' + 5 \end{cases}$$

As a consequence, one deduces that the number of Jacobi equations concerning the parameters $\lambda_1, \cdots, \lambda_{t-1}$, and defining the Lie algebra $\mathfrak{g}$ (or $A_n^k$ ), is equal to $\sum_{\rho=6}^{3n-3} N(\rho)$. This system of polynomial equations was completely solved in [2] and [3] in the cases $k = 1$ and $k = 2$. We pretend to determine the algebraic set parametrized by $\lambda_1, \cdots, \lambda_{t-1}$ defined by the $N$ homogeneous algebraic equations of degree 1 or 2.

Let us choose an adapted basis. The equations of weight $\rho = 1+i+j$ are linear. They are the only linear equations, and they are written as :

$$a_{ij} = a_{ij+1} + a_{i+1j}$$

This implies

$$a_{ij} = \sum_{s=0}^{j+h} C^s_{j-i-h-1} \lambda_{i-1+s}.$$

Therefore we can study the equations of weight $\rho = i + j + l$ with $i \geq 2, j > i$. The first non trivial equation is the one of weight $\rho = 9 = 2 + 3 + 4$. This equation concerns the vector of weight $3k + 3$.

**Proposition 3.7.** 1. *If $k > n - 3$, the Lie algebra $\mathfrak{g}$ is not rigid .*

2. *If $k = n - 3$, there is only one ( up to isomorphism ) Lie algebra $\mathfrak{t} \oplus A_n^{n-3}$, which is defined by :*

$$\begin{cases} [X, Y_1] = Y_1 \ , \ [X, Y_2] = kY_2 \ , \ \cdots \ , \ [X, Y_{n-1}] = (n - 3 + k)Y_{n-1} \\ [X, Y_n] = (n - 2 + k)Y_n \\ [Y_1, Y_i] = Y_{i+1} \quad 2 \leq i \leq n - 1 \\ [Y_2, Y_3] = Y_n \end{cases}$$

3. *If $k = n - 4$, there is only one ( up to isomorphism ) Lie algebra of type $\mathfrak{t} \oplus A_n^{n-4}$, which is defined by*

$$\begin{cases} [X, Y_1] = Y_1 \ , \ [X, Y_2] = kY_2 \ , \ \cdots \ , \ [X, Y_{n-1}] = (n - 3 + k)Y_{n-1} \\ [X, Y_n] = (n - 2 + k)Y_n \\ [Y_1, Y_i] = Y_{i+1} \quad 2 \leq i \leq n - 1 \\ [Y_2, Y_3] = Y_{n-1} \ , \quad [Y_2, Y_4] = Y_n. \end{cases}$$

*Proof.* In fact, if $2k + 1 > n + k - 2$, $[Y_i, Y_j] = 0$ and the torus is 2-dimensional. Let us suppose $k \leq n - 5$.

**Lemma 3.8.** *If $n > 2k + 5$, then $\mathfrak{g}$ is not rigid .*

In fact, the Jacobi equation of weight $\rho = 9$ concerns the vector of weight $k + k + 1 + k + 2 = 3k + 3$. This equation is trivial if $3k + 3 > n + k - 2$ (that is the highest weight).
Therefore, we can suppose that

$$\frac{n-5}{2} < k < n - 4.$$

One looks for the independent Jacobi equations. We saw that the first nontrivial equation is the one of weight $\rho = 9$. It is written as :

$$-a_{2,s}a_{3,4} + a_{3,s-1}a_{2,4} - a_{4,s-2}a_{2,3} = 0 \ , \quad s = k + 5.$$

**Lemma 3.9.** *The equation $E_{10}$ corresponding to the weight $\rho = 10 = 2 + 3 + 5$ is related to the equation $E_9$.*

In fact, this equation $E_{10}$ is :

$$-a_{2,s+1}a_{3,5} + a_{3,s}a_{2,5} - a_{5,s-2}a_{2,3} = 0$$

As $a_{ij} = a_{ij+1} + a_{i+1j}$ then :

$$E_{10} = -(a_{2,s} - a_{3,s})a_{3,4} + (a_{4,s-2} - a_{4,s-1}a_{2,3} - (a_{3,s-1} - a_{4,s-1})(a_{2,4} - a_{2,3})$$

$$= E_9 + -a_{4,s_1}a_{3,4} - a_{4,s-1}(a_{2,4} - a_{2,3}) + a_{4,s-1}a_{3,4}$$

and

$$E_{10} = E_9.$$

**Consequences** ¿From the Jacobi conditions, we find dependance relations between the equations for $\rho \geq 10$ and $k = 2$. *So the scheme $L^n$ is not reduced at the point $\mathfrak{n} \oplus \mathfrak{t}$ as soon as $n \geq 11$.*

**Remark 3.1.**

1. The variables of the polynomial equation $E_9$ are $\lambda_1, \cdots, \lambda_{t_0}$ where $t_0 = \frac{4+k}{2}$ if $k$ is even, or $\frac{3+k}{2}$ if $k$ is odd.

2. There are two equations of weight $11 : \rho = 2+3+6$ and $\rho = 2+4+5$. The first equation is independent of $E_9$. We will denote these equations by $E_{11}^{236}$ and $E_{11}^{245}$.

**Lemma 3.10.**

$$E_{11}^{245} = -E_{11}^{236} + E_{10}^{235}.$$

In fact

$$\begin{aligned}
E_{11}^{245} &= -a_{2,s+2}a_{4,5} + a_{4,s}a_{2,5} - a_{5,s-1}a_{2,4} \\
&= -a_{2,s+2}(a_{3,5} - a_{3,6}) + (a_{3,s} - a_{3,s-1})a_{2,5} - (a_{5,s-2} - a_{6,s-2})a_{2,4} \\
&= a_{2,s+2}a_{3,6} - a_{3,s+1}a_{2,5} - a_{6,s-2}a_{2,4} - a_{2,s+2}a_{3,5} + a_{3,s}a_{2,5} - a_{5,s-2}a_{2,4} \\
&= a_{2,s+2}a_{3,6} - a_{3,s+1}(a_{2,6} + a_{3,5}) + a_{6,s-2}a_{2,3} - a_{2,s+2}a_{3,5} + a_{3,s}a_{2,5} \\
&\quad -a_{5,s-2}a_{2,4} \\
&= -E_{11}^{236} - a_{3,s+1}a_{3,5} - a_{2,s+2}a_{3,5} + a_{3,s}a_{2,5} - a_{5,s-2}a_{2,4} \\
&= -E_{11}^{236} - a_{2,s+1}a_{3,5} + a_{3,s}a_{2,5} - a_{5,s-2}a_{2,4} \\
&= -E_{11}^{236} + E_{10}.
\end{aligned}$$

To understand the dependance relations, we will describe the equations of weight $\rho = 12$.

**Lemma 3.11.**

$$E_{12}^{246} = -E_{12}^{237} + E_{11}^{236}.$$

$$E_{12}^{345} = E_{12}^{237} - 2E_{11}^{236} + E_9^{234}.$$

*The polynomial system $\{E_9, E_{11}^{236}, E_{12}^{237}\}$ is a reduced system with $t_0+1$ or $t_0+2$ parameters, depending on the parity of $n+k$.*

For $k = 2$, the reduced system is :

$$\begin{cases} -2\lambda_1\lambda_3 + 3\lambda_2^2 - \lambda_2\lambda_3 = 0 \\ 2\lambda_1\lambda_4 + \lambda_2(-4\lambda_3 - \lambda_4) + \lambda_3(6\lambda_3 - \lambda_4) = 0 \\ -4\lambda_3^2 + 3\lambda_3\lambda_4 + 3\lambda_2\lambda_4 = 0 \end{cases}$$

A resolution is given in [2].

Let us now consider, for a fixed weight $\rho$, the reduced system $S_\rho$ of Jacobi equations of weight less or equal than $\rho$. Let us determine the reduced system $S_{\rho+1}$. We put $\rho' = \rho - 5$ and we suppose that

$$\rho' < n - 2k - 1.$$

**Lemma 3.12.**

$$E^{ijk}_{\rho+1} = -E^{ij-1k}_{\rho+1} + E^{ijk-1}_{\rho} \quad (i,j,k) > (2,3,\rho'+1).$$

The proof is analogous to the ones of the previous lemmas.

Suppose now that $\rho < n-2k+4$, then the system $S_\rho$ is :

$$S_\rho = \{E^{234}_9, E^{236}_{11}, \cdots, E^{23\rho'}_{\rho}\}.$$

Thus, for a given weight $\rho$, the reduced system has $\rho-9$ equations. Let us suppose $n \geq 3k+7$ (or $3k+6$). There exist at least two nontrivial solutions of the Jacobi system for which the corresponding Lie algebras are rigid and nonisomorphic. Take, for example, $(\lambda_1 = 1, \lambda_i = 0, i \neq 0)$ and $(\lambda_i = 0, i \neq t, \lambda_t = 1)$.We can conclude :

**Theorem 3.13.** *Let* $\mathfrak{g} = \mathfrak{t} \oplus A^k_n(\lambda_1, .., \lambda_{t-1})$ *be a decomposable* $p = (n+1)-$*dimensional Lie algebra. Then*

*i) If* $n-k$ *is odd and* $2k+5 \leq n \leq 3k+7$ *or if* $n-k$ *is even and* $2k+5 \leq n \leq 3k+6$, *then* $\mathfrak{g}$ *is not rigid.*

*ii) If* $n \geq 3k+7$ *with* $n-k$ *odd or* $n \geq 3k+6$ *with* $n-k$ *even, then* $\mathfrak{g}$ *is rigid as soon as there exists an* $i$ *such that* $\lambda_i \neq 0$. *In this case, two rigid Lie algebras associated with the parameters* $(\lambda_1, ..., \lambda_t)$ *and* $(\lambda'_1, ..., \lambda'_t)$ *are isomorphic if and only if there is* $\alpha \neq 0$ *such that* $(\lambda_1, ..., \lambda_t) = \alpha(\lambda'_1, ..., \lambda'_t)$

3.4. $\mathfrak{n} = B^k_n(\lambda_1, \cdots, \lambda_t)$. Let us consider the Lie algebra $B^k_n(\lambda_1, .., \lambda_{t-1})$ defined by :

$$\begin{cases} [Y_1, Y_i] = Y_{i+1}\ , \quad 2 \leq i \leq n-2 \\ [Y_i, Y_{n-i+1}] = (-1)^{i+1} Y_n\ , \quad 2 \leq i \leq n-1 \\ [Y_i, Y_j] = a_{ij} Y_{k+i+j-2}\ , \ 2 \leq i,j \leq n-2\ , \ i+j+k-2 \leq n-1 \end{cases}$$

It is defined for even $n$. Let us put $n = 2p+4$. There is a derivation $f$ of this algebra $\mathfrak{n}$, which generates the maximal external torus. Its eigenvalues are :

$$1, k, k+1, k+2, \cdots, k+2p+1, 2k+2p+1$$

The basis $\{Y_i\}$ given in the definition of $B^k_n(\lambda_1, .., \lambda_{t-1})$ is the basis of corresponding eigenvectors. It is clear that if $k \geq 2p$, the Lie algebra $B^k_n(\lambda_1, .., \lambda_{t-1})$ is isomorphic to $Q_n$. This case has been studied. Let us suppose $k < 2p$.

*First case : k is even*

Let us put $k = 2p-2l$. This implies $n = k+2l+4$ and the eigenvalues of $f$ are :

$$1, k, k+1, k+2, \cdots, 2k, \cdots, 2k+2l+1, 3k+2l+1.$$

All the structural constants $a_{ij}$ are linear combinations of the constants $\lambda_i = a_{i+1,i+2}$ , $i = 1, \cdots, l+1$. Let us begin by examining the three particular cases : $l = 0, l = 1, l = 2$.

i) $l = 0$. The Lie algebra $\mathfrak{g}$ is defined by :

$$\begin{cases} [X, Y_1] = Y_1;\ [X, Y_i] = (k+i-2)Y_i,\ i = 2, \cdots, n-1;\ [X, Y_n] = (3k+1)Y_n \\ [Y_1, Y_i] = Y_{i+1},\ i = 2, \cdots, n-2; \\ [Y_i, Y_{n-i+1}] = (-1)^i Y_n,\ i = 2, \cdots, n-2; \\ [Y_2, Y_3] = Y_{n-1} \end{cases}$$

with $k = n-4 \neq 0$ and $n$ even. *This Lie algebra is rigid.*

ii) $l = 1$. There exists only one Lie algebra (up to isomorphism) verifying the hypothesis. It is given by :

$$\begin{cases} [X,Y_1]=Y_1;\ [X,Y_i]=(k+i-2)Y_i,\ i=2,\cdots,n-1;\ [X,Y_n]=(3k+3)Y_n \\ [Y_1,Y_i]=Y_{i+1},\ i=2,\cdots,n-2;\ [Y_i,Y_{n-i+1}]=(-1)^iY_n,\ i=2,\cdots,n-2; \\ [Y_2,Y_3]=Y_{n-3};\ [Y_2,Y_4]=Y_{n-2};\ [Y_2,Y_5]=9Y_{n-1};\ [Y_3,Y_4]=2Y_n-1 \end{cases}$$

with $k = n-6$ and $n$ even. *This Lie algebra is rigid.*

iii) $l = 2$. This case is different from the previous cases.

**Lemma 3.14.** *If $k = n-8$, then the corresponding Lie algebras is rigid if and only if $k = 2$ (and $n = 10$).*

In this case, there exist two non isomorphic Lie algebras (of dimension 11) : $\mathcal{G}_{11}^1$

$$\begin{cases} [X,Y_1]=Y_1;\ [X,Y_i]=iY_i,\ i=2,\cdots,9;\ [X,Y_{10}]=11Y_{10} \\ [Y_1,Y_i]=Y_{i+1},\ i=2,\cdots,8;\ [Y_i,Y_{11-i}]=(-1)^iY_{10},\ i=2,\cdots,5; \\ [Y_2,Y_i]=Y_{i+2},\ i=3,4;\ [Y_2,Y_i]=-Y_{i+2},\ i=6,7; \\ [Y_3,Y_i]=Y_{i+3},\ i=4,5;\ [Y_4,Y_5]=Y_9. \end{cases}$$

$\mathcal{G}_{11}^2$ :

$$\begin{cases} [X,Y_1]=Y_1;\ [X,Y_i]=iY_i,\ i=2,\cdots,9;\ [X,Y_{10}]=11Y_{10}; \\ [Y_1,Y_i]=Y_{i+1},\ i=2,\cdots,8;\ [Y_i,Y_{11-i}]=(-1)^iY_{10},\ i=2,\cdots,5; \\ [Y_2,Y_i]=Y_{i+2},\ i=3,4;\ [Y_2,Y_5]=2Y_7;\ [Y_2,Y_6]=3Y_8;\ [Y_2,Y_7]=7Y_9; \\ [Y_3,Y_i]=-Y_{i+3},\ i=4,5;\ [Y_3,Y_6]=-4Y_9;\ [Y_4,Y_5]=3Y_9. \end{cases}$$

Let us consider the general case $l \geq 3$.

**Proposition 3.15.** *If $2l < k+2$, then the decomposable algebra $\mathfrak{g}$ of nilradical $B_n^k(\lambda_1,..,\lambda_{t-1})$ is not rigid.*

This is a consequence of the previous section. We consider the parameters $\lambda_1 = a_{23}, \lambda_2 = a_{34}, \cdots, \lambda_{l+1} = a_{k+l+2,k+l+3}$. These parameters generate all the coefficients $a_{ij}$. From the previous section, the number of independent Jacobi equations is $2l-k-2$ if $2l-k-2 > 0$, otherwise we have zero equation. Then let us suppose $2l \geq k+2$.

**Proposition 3.16.** *Let $\mathfrak{g} = \mathfrak{t} \oplus B_n^k(\lambda_1,..,\lambda_{t-1})$ be a decomposable algebra of dimension $n+1 = k+2l+5$ with $\dim\mathfrak{t} = 1$ .*
*1) If $k > \left[\frac{l+2}{3}\right] + l - 2$, then $\mathfrak{g}$ is not rigid.*
*2) If $k \leq \left[\frac{l+2}{3}\right] + l - 2$, then $\mathfrak{g}$ is rigid.*

In fact the coefficients $a_{ij}$ are deduced from the parameters $\lambda_i$ by the rule

$$a_{r+2,s+2} = \sum_{i=1}^{\left[\frac{s-r+1}{2}\right]} (-1)^{i+1} S^i_{r-s-2i+2} \lambda_{s+i}, \quad s \geq r$$

where $S_i^j$ is determined by the relations

$$S_p^i = S_{p-1}^i + S_p^{i-1} \quad i=1,\cdots,l+1, \quad S_p^1 = S_1^p = 1 \ \ \forall p \in \mathbb{N}.$$

As the central quotient algebra $\frac{B_n^k(\lambda_1,..,\lambda_{t-1})}{\mathbb{C}Y_n}$ is isomorphic to $A_{n-1}^k(\lambda_1,..,\lambda_{t-1})$, the coefficients $a_{ij}$ verify the same Jacobi relations as the equations relative to the case $A_{n-1}^k(\lambda_1,..,\lambda_{t-1})$. Moreover, these coefficients satisfy linear equations corresponding to the Jacobi conditions related to the vectors $Y_2, Y_{2+h}, Y_{3+2l-h}$ for $2h < 2l+1$ and $h \geq 1$, $Y_{2+s}, Y_{2+s'}, Y_{3+2l-5s'}$ for $0 \leq s \leq s'$ and $s+2s' \leq 2l+1$.

**Lemma 3.17.** *The rank of the linear system defined by the parameters $\lambda_i$ is equal to $\left[\frac{l+2}{3}\right]$.*

In fact the Jacobi equations corresponding to the triples $(Y_2, Y_{2+h}, Y_{3+2l-h})$ are:

$$a_{h+2,2l+3-h} - (-1)^h a_{2,2l+3-h} + (-1)^{h-1} a_{2,h+2} = 0,$$

We replace the coefficients $a_{ij}$ by their expressions in the $\lambda_i$. We obtain:

$$\sum_{i=1}^{l-h+1} (-1)^{i+1} S^i_{2(l-h-i+1)+1} \lambda_{h+i} \; - \; (-1)^h \sum_{i=1}^{\left[\frac{l+2-h}{2}\right]} (-1)^{i+1} S^i_{2(l-i+1)+1-h} \lambda_i$$

$$+(-1)^{h-1} \sum_{i=1}^{\left[\frac{h+1}{2}\right]} (-1)^{i+1} S^i_{h-2(i-1)} \lambda_i = 0.$$

The equations corresponding to $(Y_{2+s}, Y_{2+s'}, Y_{2+2l-s-s'+1})$ are written as:

$$(-1)^{s+1} a_{2+s',2l+3-s-s'} - (-1)^{s'+1} a_{2+s,2l+3-s-s'} + (-1)^{4+2l-s-s'} a_{2+s,2+s'} = 0,$$

and this gives :

$$(-1)^{s+1} \sum_{i=1}^{2l-s-2s'+2} (-1)^{i+1} S^i_{2(l-s-s'-2i+3)+1} \lambda_{s'+i}$$

$$- \; (-1)^{s'} \sum_{i=1}^{\left[\frac{2l+2-2s-s'}{2}\right]} (-1)^{i+1} S^i_{2l-2i+3-2s-s'} \lambda_{s+i}$$

$$+(-1)^{4+2l-s-s'} \sum_{i=1}^{\left[\frac{s'-s+1}{2}\right]} (-1)^{i+1} S^i_{s'-s-2i+2)} \lambda_{s+i} = 0.$$

In particular the equations corresponding to $(Y_{2+2s}, Y_{3+2s}, Y_{2+2l-4s})$ with $6s \leq 2l-1$ have the following form:

$$-2\lambda_{2s+1} + (2l-6s-3)\lambda_{2s+2} + \cdots = 0,$$

so the system has rank less or equal to $\left[\frac{l-1}{3}\right]+1$. If we note by $E_{s,s'}$ the equation $(Y_{2+s}, Y_{2+s'}, Y_{3+2l-s-s'})$, we obtain the relations:

$$E_{s,s'} = -E_{s,s'-1} + E_{s-1,s'}.$$

This proves that every Jacobi equation is deduced from the equations $E_{s,s+1}$. As the equations $E_{2s+1,2s+2}$ coincide with the equations $E_{2s+2,2s+3}$, the Jacobi system is reduced to the system of independent equations $E_{2s,2s+1}$. This proves the lemma.

*Second case : k is odd.*

It is clear that $k+1<2p$. Let us put $k=2p-2l-1$.This gives $n=k+2l+5$. The eigenvalues of the semisimple derivation $f$ are

$$1,k,k+1,\cdots,2k+2l+2,3k+2l+2.$$

The number of undetermined structural constants is equal to $l+1$. The corresponding cases to $l=0,1,2$ and 3 are particular, so we will study them firstly.
$l=0$ $(k=2p-1)$. There exists only one rigid law up to isomorphism :

$$\begin{cases} [X,Y_1]=Y_1,[X,Y_i]=(k+i-2)Y_i \ i=2,\cdots,n-1, \ [X,Y_n]=(2k+n-3)Y_n \\ [Y_1,Y_i]=Y_{i+1} \ i=2,\cdots,n-2 \ \ [Y_2,Y_3]=Y_{n-2}, \ [Y_2,Y_4]=Y_{n-1} \\ [Y_i,Y_{n-i+1}]=(-1)^iY_n. \end{cases}$$

$l=1$ $(k=2p-3)$. There does not exist a rigid law.
$l=2$ $(k=2p-5)$. There exist only two laws if $k=3$. Otherwise, the corresponding Lie algebras are not rigid.
$l=3$ $(k=2p-7)$. There exists 4 rigid laws as soon as $k=3$ $(dim\mathfrak{g}=15)$. These laws define non rational and non real rigid Lie algebras.
Now let us suppose $l\geq 4$. Then we have :

**Proposition 3.18.** *Let $\mathfrak{g}=\mathfrak{t}\oplus B_n^k(\lambda_1,..,\lambda_{t-1})$ be a decomposable Lie algebra of dimension $n+1=k+2l+6$ with $dim\mathfrak{t}=1$.*
*1) If $k>\left[\frac{l+1}{3}\right]+l-2$, then $\mathfrak{g}$ is not rigid.*
*2) If $k\leq\left[\frac{l+1}{3}\right]+l-2$, then $\mathfrak{g}$ is rigid.*

Let us note the equation $(Y_{2+s},Y_{2+s'},Y_{4+2l-s-s'})$ by $E'_{s,s'}$. The proof of this proposition is based on the proof of theorem 3.3 noting that the equation $E'_{s,s+1}$ is the same as $E_{s+1,s+2}$ in the lemma 3.17.
**Remark.**
The equation $E'_{s,s+1}$concerns only the parameters $\lambda_2,\cdots,\lambda_l$. Then the Lie algebras corresponding to $\lambda_1=1,\lambda_i=0,i\neq 1$ and $\lambda_{l+1}=1,\lambda_i=0,i\neq l+1$ and satisfying $k\leq\left[\frac{l+1}{3}\right]+l-2$ are rigid.

**Theorem 3.19.** *Let $\mathfrak{g}=\mathfrak{t}\oplus B_n^k(\lambda_1,...,\lambda_{t-1})$ $(n=2m)$ be a $n+1-$dimensional decomposable Lie algebra.*
*i) If $k$ is even, this Lie algebra is rigid if and only if $k\leq\left[\frac{l+2}{3}\right]+l-2$ where $l=\frac{1}{2}(n-k-4)$ or $k=n-4$, or $k=n-6$ or $k=2$ and $n=10$.*
*ii) If $k$ is odd, this Lie algebra is rigid if and only if $k\leq\left[\frac{l+1}{3}\right]+l-2$ where $l=\frac{1}{2}(n-k-5)$ or $k=n-5$, or $k=3$ and $n=12$ or $n=14$.*
*iii) two Lie algebras corresponding to the parameters $(\lambda_1,...,\lambda_{l+1})$ and $(\lambda'_1,...,\lambda'_{l+1})$ are isomorphic if and only if there exists an $\alpha\neq 0$ such that $(\lambda_1,...,\lambda_{l+1})=\alpha(\lambda'_1,...,\lambda'_{l+1})$.*

3.5. $\mathfrak{n}=C_n^k(\lambda_1,\cdots,\lambda_t)$.

**Theorem 3.20.** *Let $\mathfrak{g}=\mathfrak{t}\oplus C_n(\lambda_1,..,\lambda_t)$ be a $(n+1)$-dimensional decomposable Lie algebra. Then $\mathfrak{g}$ is not rigid.*

In fact if $(X,Y_1,...,Y_n)$ is a basis of $\mathfrak{g}$ with $X\in t$ and if $(Y_1,...,Y_n)$ is the basis of $C_n(\lambda_1,..,\lambda_t)$ given in 3.3, then the 2-cocycle defined by $\varphi(X,Y_2)=Y_{n-1}$ determines a deformation $\mu+\varphi$ non isomorphic to $\mu$, where $\mu$ is the law of $\mathfrak{g}$. Thus $\mathfrak{g}$ is not rigid.

## 4. Appendix : Rigid Lie algebras of dimension less than 8

In this section we recall the classification of solvable Lie algebras of dimension less or equal to 8. The proof can be found in [1]. In the following lists we note by $(X, X_0, X_0', X_0'', ...)$ the basis of $t$, $X$ being the characteristic vector and by $(Y_i)_{i\in I}$ a basis of $\mathfrak{n}$ satisfying $adX(Y_i) = iY_i$. If the eigenvalue $i$ is of multiplicity less than one, we note by $(Y_i, Y_i', Y_i'', ..)$ the corresponding basis of eigenvectors.

**Dimension 2** . We have, up an isomorphism, only one rigid law : $\mu_1^2\,(X, Y_1) = Y_1$.

**Dimension 3.** There are no solvable rigid laws in dimension 3.

**Dimension 4.** Any solvable rigid laws on $\mathbb{C}^4$ is isomorphic to the Lie algebra:

$$\{\mu_4^1\,(X, Y_1) = Y_1, \quad \mu_4^1\,(X_0, Y_2) = Y_2\,.$$

**Dimension 5**. Any solvable rigid law on $\mathbb{C}^5$ is isomorphic to:

$$\begin{cases} \mu_5^1\,(X, Y_i) = iY_i, \quad i = 1, 2, 3, \\ \mu_5^1\,(X_0, Y_i) = Y_i, \quad i = 2, 3, \quad \mu_5^1\,(Y_1, Y_2) = Y_3. \end{cases}$$

**Dimension 6.** Any solvable rigid law in $\mathbb{C}^6$ is isomorphic to one of the following :

| | |
|---|---|
| $\begin{cases} \mu_6^1\,(X, Y_i) = iY_i \quad \text{for } i = 1, 2, 3, 4, 5, \\ \mu_6^1\,(Y_1, Y_i) = Y_{i+1} \quad \text{for } i = 2, 3, 4, \\ \mu_6^1\,(Y_2, Y_3) = Y_5 \end{cases}$ | $\begin{cases} \mu_6^2\,(X, Y_i) = iY_i \quad \text{for } i = 1, 2, 3, 4, \\ \mu_6^2\,(X_0, Y_i) = Y_i \quad \text{for } i = 2, 3, 4, \\ \mu_6^2\,(Y_1, Y_i) = Y_{i+1} \quad \text{for } i = 2, 3. \end{cases}$ |
| $\begin{cases} \mu_6^3\,(X, Y_i) = iY_i \quad \text{for } i = 1, 2, 3, \\ \mu_6^3\,(X_0, Y_2) = Y_2, \\ \mu_6^3\,(X_0', Y_3) = Y_3. \end{cases}$ | |

**Dimension 7**. Any rigid solvable law in $\mathbb{C}^7$ is isomorphic to one of the following :

| | |
|---|---|
| $\begin{cases} \mu_7^1\,(X, Y_i) = iY_i, \quad i = 1, 2, 3, 4, 5, 6, \\ \mu_7^1\,(Y_1, Y_i) = Y_{i+1}, \quad i = 2, 3, 4, 5, \\ \mu_7^1\,(Y_2, Y_i) = Y_{i+2}, \quad i = 3, 4. \end{cases}$ | $\begin{cases} \mu_7^2\,(X, Y_i) = iY_i, \ i = 1, 2, 3, 4, 5, 7, \\ \mu_7^2\,(Y_1, Y_i) = Y_{i+1}, \quad i = 2, 3, 4, \\ \mu_7^2\,(Y_2, Y_i) = Y_{i+2}, \quad i = 3, 5, \\ \mu_7^2\,(Y_3, Y_4) = -Y_7. \end{cases}$ |
| $\begin{cases} \mu_7^3\,(X, Y_i) = iY_i, \quad i = 1, 3, 4, 5, 6, 7, \\ \mu_7^3\,(Y_1, Y_i) = Y_{i+1}, \quad i = 3, 4, 5, 6, \\ \mu_7^3\,(Y_3, Y_4) = Y_7. \end{cases}$ | $\begin{cases} \mu_7^4\,(X, Y_i) = iY_i, \quad i = 1, 2, 3, 4, 5, \\ \mu_7^4\,(X, Y_3') = 3Y_3', \\ \mu_7^4\,(Y_1, Y_i) = Y_{i+1}, \quad i = 2, 3, 4, \\ \mu_7^4\,(Y_2, Y_3) = \mu_4\,(Y_2, Y_3') = Y_5. \end{cases}$ |
| $\begin{cases} \mu_7^5\,(X, Y_i) = iY_i, \quad i = 1, 2, 3, 4, 5, \\ \mu_7^5\,(X_0, Y_i) = Y_i, \quad i = 2, 3, 4, 5, \\ \mu_7^5\,(Y_1, Y_i) = Y_{i+1}, \quad i = 2, 3, 4. \end{cases}$ | $\begin{cases} \mu_7^6\,(X, Y_i) = iY_i, \quad i = 1, 2, 3, 4, 5, \\ \mu_7^6\,(X_0, Y_i) = Y_i, \quad i = 2, 3, 4, \\ \mu_7^6\,(X_0, Y_5) = 2Y_5, \\ \mu_7^6\,(Y_1, Y_i) = Y_{i+1}, \quad i = 2, 3, \\ \mu_7^6\,(Y_2, Y_3) = Y_5. \end{cases}$ |
| $\begin{cases} \mu_7^7\,(X, Y_i) = iY_i, \quad i = 1, 2, 3, 4, 5, \\ \mu_7^7\,(X_0, Y_i) = Y_i, \quad i = 3, 4, 5, \\ \mu_7^7\,(Y_1, Y_i) = Y_{i+1}, \quad i = 3, 4, \\ \mu_7^7\,(Y_2, Y_3) = Y_5. \end{cases}$ | $\begin{cases} \mu_7^8\,(X, Y_i) = iY_i, \quad i = 1, 2, 3, 4, \\ \mu_7^8\,(X_0, Y_i) = Y_i, \quad i = 2, 3, \\ \mu_7^8\,(X_0', Y_4) = Y_4, \\ \mu_7^8\,(Y_1, Y_2) = Y_3. \end{cases}$ |

**Dimension 8.** Any rigid solvable law in $\mathbb{C}^8$ is isomorphic to one of the following :

| | |
|---|---|
| $\begin{cases} \mu_8^1(X,Y_i)=iY_i,\ i=1,2,3,5,6,7,8 \\ \mu_8^1(Y_1,Y_i)=Y_{i+1},\ i=2,6,7, \\ \mu_8^1(Y_2,Y_3)=Y_5,\ \mu_8^1(Y_2,Y_5)=Y_7, \\ \mu_8^1(Y_2,Y_6)=Y_8,\ \mu_8^1(Y_3,Y_5)=Y_8. \end{cases}$ | $\begin{cases} \mu_8^2(X,Y_i)=iY_i,\ i=1,3,4,5,6,7,8, \\ \mu_8^2(Y_1,Y_i)=Y_{i+1},\ i=3,4,5,6,7, \\ \mu_8^2(Y_3,Y_i)=Y_{i+3},\quad i=4,5. \end{cases}$ |
| $\begin{cases} \mu_8^3(X,Y_i)=iY_i,\ i=1,3,4,5,6,7,9, \\ \mu_8^3(Y_1,Y_i)=Y_{i+1},\ i=3,4,5,6, \\ \mu_8^3(Y_3,Y_i)=Y_{i+3},\quad i=4,6, \\ \mu_8^3(Y_4,Y_5)=-Y_9. \end{cases}$ | $\begin{cases} \mu_8^4(X,Y_i)=iY_i,\ i=1,4,5,6,7,8,9, \\ \mu_8^4(Y_1,Y_i)=Y_{i+1},\ i=4,5,6,7,8, \\ \mu_8^4(Y_4,Y_5)=Y_9. \end{cases}$ |
| $\begin{cases} \mu_8^5(X,Y_i)=iY_i,\ i=2,3,4,5,6,7,8, \\ \mu_8^5(Y_2,Y_i)=Y_{i+2},\ i=3,4,5,6, \\ \mu_8^5(Y_3,Y_i)=Y_{i+3},\quad i=4,5. \end{cases}$ | $\begin{cases} \mu_8^6(X,Y_i)=iY_i,\ i=2,3,4,6,7,8,10, \\ \mu_8^6(Y_2,Y_i)=Y_{2+i},\quad i=4,6,8, \\ \mu_8^6(Y_3,Y_i)=Y_{i+3},\quad i=4,7, \\ \mu_8^6(Y_4,Y_6)=Y_{10}. \end{cases}$ |
| $\begin{cases} \mu_8^7(X,Y_i)=iY_i,\ i=2,3,5,6,7,8,9, \\ \mu_8^7(Y_2,Y_i)=Y_{i+2},\ i=3,5,6,7, \\ \mu_8^7(Y_3,Y_i)=Y_{i+3},\quad i=5,6. \end{cases}$ | $\begin{cases} \mu_8^8(X,Y_i)=iY_i,\ i=2,3,5,7,8,9,11, \\ \mu_8^8(Y_2,Y_i)=Y_{i+2}, i=3,5,7,9, \\ \mu_8^8(Y_3,Y_i)=Y_{i+3},\quad i=5,8. \end{cases}$ |
| $\begin{cases} \mu_8^9(X,Y_i)=iY_i,\ i=1,2,3,4,5,6, \\ \mu_8^9(X,Y_3')=3Y_3', \\ \mu_8^9(Y_1,Y_i)=Y_{i+1},\quad i=2,3,4,5, \\ \mu_8^9(Y_2,Y_i)=Y_{i+2}, i=3,4, \\ \mu_8^9(Y_2,Y_3')=Y_5,\ \mu_8^9(Y_3,Y_3')=Y_6. \end{cases}$ | $\begin{cases} \mu_8^{10}(X,Y_i)=iY_i,\quad i=1,2,3,4,5,6, \\ \mu_8^{10}(X,Y_4')=4Y_4', \\ \mu_8^{10}(Y_1,Y_i)=Y_{i+1},\quad i=2,3,4,5, \\ \mu_8^{10}(Y_2,Y_i)=Y_{i+2},\quad i=3,4, \\ \mu_8^{10}(Y_2,Y_4')=Y_6. \end{cases}$ |
| $\begin{cases} \mu_8^{11}(X,Y_i)=iY_i,\ i=1,2,3,4,5,6, \\ \mu_8^{11}(X,Y_5')=5Y_5', \\ \mu_8^{11}(Y_1,Y_i)=Y_{i+1},\ i=2,3,4,5, \\ \mu_8^{11}(Y_1,Y_5')=Y_6, \\ \mu_8^{11}(Y_2,Y_3)=Y_5', \\ \mu_8^{11}(Y_2,Y_4)=Y_6. \end{cases}$ | $\begin{cases} \mu_8^{12}(X,Y_i)=iY_i,\quad i=1,2,3,4,5,7, \\ \mu_8^{12}(X,Y_3')=3Y_3', \\ \mu_8^{12}(Y_1,Y_2)=Y_3^{'}, \\ \mu_8^{12}\left(Y_1,Y_3^{'}\right)=Y_4 \\ \mu_8^{12}(Y_1,Y_4)=Y_5, \\ \mu_8^{12}(Y_2,Y_3)=Y_5, \\ \mu_8^{12}\left(Y_2,Y_3^{'}\right)=Y_5 \\ \mu_8^{12}(Y_2,Y_5)=\mu_8^{12}\left(Y_3^{'},Y_4\right)=Y_7. \end{cases}$ |
| $\begin{cases} \mu_8^{13}(X,Y_i)=iY_i,\quad i=1,2,3,4,5,7 \\ \mu_8^{13}(X,Y_5')=5Y_5', \\ \mu_8^{13}(Y_1,Y_i)=Y_{i+1},\quad i=2,3,4, \\ \mu_8^{13}(Y_2,Y_3)=Y_5', \\ \mu_8^{13}(Y_2,Y_5)=\mu_8^{13}(Y_2,Y_5')=Y_7, \\ \mu_8^{13}(Y_3,Y_4)=-Y_7. \end{cases}$ | $\begin{cases} \mu_8^{14}(X,Y_i)=iY_i,\ i=1,3,4,5,6,7, \\ \mu_8^{14}(X,Y_4')=4Y_4', \\ \mu_8^{14}(Y_1,Y_i)=Y_{i+1},\ i=3,4,5,6, \\ \mu_8^{14}(Y_3,Y_4)=\mu_8^{14}(Y_3,Y_4')=Y_7. \end{cases}$ |
| $\begin{cases} \mu_8^{15}(X,Y_i)=iY_i,\quad i=1,2,3,4,5, \\ \mu_8^{15}(X,Y_3')=3Y_3', \\ \mu_8^{15}(X,Y_4')=4Y_4', \\ \mu_8^{15}(Y_1,Y_i)=Y_{i+1},\quad i=2,3,4, \\ \mu_8^{15}(Y_1,Y_3')=Y_4', \\ \mu_8^{15}(Y_1,Y_4')=\mu_8^{15}(Y_2,Y_3)=Y_5. \end{cases}$ | $\begin{cases} \mu_8^{16}(X,Y_i)=iY_i,\quad i=1,2,3,4,5, \\ \mu_8^{16}(X,Y_1')=Y_1', \\ \mu_8^{16}(X,Y_3')=3Y_3', \\ \mu_8^{16}(Y_1,Y_i)=Y_{i+1},\quad i=2,4, \\ \mu_8^{16}(Y_1,Y_i')=Y_{i+1},\quad i=1,3, \\ \mu_8^{16}(Y_1',Y_i)=Y_{i+1},\quad i=3,4, \\ \mu_8^{16}(Y_1',Y_2)=Y_3', \\ \mu_8^{16}(Y_2,Y_3)=-\mu_8^{16}(Y_2,Y_3')=Y_5 \end{cases}$ |
| $\begin{cases} \mu_8^{17}(X,Y_i)=iY_i,\quad i=1,2,3,4,5, \\ \mu_8^{17}(X,Y_i')=iY_i',\quad i=3,5, \\ \mu_8^{17}(Y_1,Y_i)=Y_{i+1},\quad i=2,3,4, \\ \mu_8^{17}(Y_1,Y_3')=Y_4, \\ \mu_8^{17}(Y_2,Y_3)=Y_5', \\ \mu_8^{17}(Y_2,Y_3')=Y_5+2Y_5'. \end{cases}$ | $\begin{cases} \mu_8^{18}(X,Y_i)=iY_i, \\ \mu_8^{18}(X,Y_i')=iY_i',\quad i=1,2,3, \\ \mu_8^{18}\left(X,Y_1^{''}\right)=Y_1^{''} \\ \mu_8^{18}\left(Y_1,Y_1^{'}\right)=Y_2, \\ \mu_8^{18}\left(Y_1,Y_1^{''}\right)=Y_2^{'}, \\ \mu_8^{18}\left(Y_1^{''},Y_2^{'}\right)=Y_3^{'}, \\ \mu_8^{18}(Y_1,Y_2')=Y_3^{'}, \\ \mu_8^{18}(Y_1',Y_2)=Y_3^{'}, \\ \mu_8^{18}(Y_1',Y_2')=\mu_8^{18}\left(Y_1^{''},Y_2\right)=Y_3. \end{cases}$ |

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| $\begin{cases} \mu_8^{19}(X,Y_i) = iY_i, \ i=1,2,3,4,5,6, \\ \mu_8^{19}(X_0,Y_i) = Y_i, \quad i=2,3,4,5,6, \\ \mu_8^{19}(Y_1,Y_i) = Y_{i+1}, \quad i=2,3,4,5. \end{cases}$ | $\begin{cases} \mu_8^{20}(X,Y_i) = iY_i, \quad i=1,2,3,4,5,6, \\ \mu_8^{20}(X_0,Y_i) = Y_i, \quad i=3,4,5,6, \\ \mu_8^{20}(Y_1,Y_i) = Y_{i+1}, \quad i=3,4,5, \\ \mu_8^{20}(Y_2,Y_i) = Y_{i+2}, \quad i=3,4. \end{cases}$ |
| $\begin{cases} \mu_8^{21}(X,Y_i) = iY_i, \quad i=1,2,3,4,5,6, \\ \mu_8^{21}(X_0,Y_i) = Y_i, \quad i=4,5,6, \\ \mu_8^{21}(Y_1,Y_i) = Y_{i+1}, \quad i=2,4,5, \\ \mu_8^{21}(Y_2,Y_4) = Y_6. \end{cases}$ | $\begin{cases} \mu_8^{22}(X,Y_i) = iY_i, \quad i=1,2,3,4,5,6, \\ \mu_8^{22}(X_0,Y_i) = Y_i, \quad i=2,3,4, \\ \mu_8^{22}(X_0,Y_i) = 2Y_i, \quad i=5,6, \\ \mu_8^{22}(Y_1,Y_i) = Y_{i+1}, \quad i=2,3,5, \\ \mu_8^{22}(Y_2,Y_i) = Y_{i+2}, i=3,4 \end{cases}$ |
| $\begin{cases} \mu_8^{23}(X,Y_i) = iY_i, \quad i=1,2,3,4,5,6, \\ \mu_8^{23}(X_0,Y_6) = Y_6, \\ \mu_8^{23}(Y_1,Y_i) = Y_{i+1}, \quad i=2,3,4, \\ \mu_8^{23}(Y_2,Y_3) = Y_5. \end{cases}$ | $\begin{cases} \mu_8^{24}(X,Y_i) = iY_i, \quad i=1,2,3,4,5,7, \\ \mu_8^{24}(X_0,Y_i) = Y_i, \quad i=3,4,5, \\ \mu_8^{24}(X_0,Y_7) = 2Y_7, \\ \mu_8^{24}(Y_1,Y_i) = Y_{i+1}, \quad i=3,4, \\ \mu_8^{24}(Y_2,Y_3) = Y_5, \\ \mu_8^{24}(Y_3,Y_4) = Y_7. \end{cases}$ |
| $\begin{cases} \mu_8^{25}(X,Y_i) = iY_i, \quad i=1,2,3,4,5,7, \\ \mu_8^{25}(X_0,Y_i) = Y_i, \quad i=2,3,4, \\ \mu_8^{25}(X_0,Y_5) = 2Y_5, \\ \mu_8^{25}(X_0,Y_7) = 3Y_7, \\ \mu_8^{25}(Y_1,Y_i) = Y_{i+1}, \quad i=2,3, \\ \mu_8^{25}(Y_2,Y_i) = Y_{i+2}, \quad i=3,5. \end{cases}$ | $\begin{cases} \mu_8^{26}(X,Y_i) = iY_i, \quad i=1,2,3,4,5,7, \\ \mu_8^{26}(X_0,Y_i) = Y_i, \quad i=2,3,4,5, \\ \mu_8^{26}(Y_0,Y_7) = 2Y_7, \\ \mu_8^{26}(Y_1,Y_i) = Y_{i+1}, \quad i=2,3,4, \\ \mu_8^{26}(Y_2,Y_5) = Y_7, \\ \mu_8^{26}(Y_3,Y_4) = -Y_7. \end{cases}$ |
| $\begin{cases} \mu_8^{27}(X,Y_i) = iY_i, \quad i=1,2,3,5,6,7, \\ \mu_8^{27}(X_0,Y_i) = Y_i, \quad i=2,3, \\ \mu_8^{27}(X_0,Y_5) = 2Y_5, \\ \mu_8^{27}(X_0,Y_i) = 3Y_i, \quad i=6,7, \\ \mu_8^{27}(Y_1,Y_i) = Y_{i+1}, \quad i=2,6, \\ \mu_8^{27}(Y_2,Y_i) = Y_{i+2}, \quad i=3,5. \end{cases}$ | $\begin{cases} \mu_8^{28}(X,Y_i) = iY_i, \quad i=1,2,3,4,5,7, \\ \mu_8^{28}(X_0,Y_2) = Y_2, \\ \mu_8^{28}(X_0,Y_i) = 2Y_i, \quad i=3,4, \\ \mu_8^{28}(X_0,Y_5) = 3Y_5, \\ \mu_8^{28}(X_0,Y_7) = 4Y_7, \\ \mu_8^{28}(Y_1,Y_3) = Y_4, \\ \mu_8^{28}(Y_2,Y_i) = Y_{i+2}, \quad i=3,5, \\ \mu_8^{28}(Y_3,Y_4) = -Y_7 \end{cases}$ |
| $\begin{cases} \mu_8^{29}(X,Y_i) = iY_i, \quad i=1,2,3,4,5, \\ \mu_8^{29}(X,Y_3') = 3Y_3', \\ \mu_8^{29}(X_0,Y_i) = Y_i, \ i=2,3,4, \\ \mu_8^{29}(X_0,Y_3') = Y_3', \ \mu_8^{29}(X_0,Y_5) = 2Y_5 \\ \mu_8^{29}(Y_1,Y_i) = Y_{i+1}, \quad i=2,3, \\ \mu_8^{29}(Y_1,Y_3') = Y_4, \\ \mu_8^{29}(Y_2,Y_3) = Y_5. \end{cases}$ | $\begin{cases} \mu_8^{30}(X,Y_i) = iY_i, \ i=1,2,3,4,5, \\ \mu_8^{30}(X_0,Y_5) = Y_5, \\ \mu_8^{30}(X_0',Y_i) = Y_i, \quad i=2,3,4, \\ \mu_8^{30}(Y_1,Y_i) = Y_{i+1}, \quad i=2,3. \end{cases}$ |
| $\begin{cases} \mu_8^{31}(X,Y_i) = iY_i, \quad i=1,2,3,4,5, \\ \mu_8^{31}(X_0,Y_i) = Y_i, \quad i=2,3, \\ \mu_8^{31}(X_0',Y_i) = Y_i, \quad i=4,5, \\ \mu_8^{31}(Y_1,Y_i) = Y_{i+1}, \quad i=2,4. \end{cases}$ | $\begin{cases} \mu_8^{32}(X,Y_i) = iY_i, \quad i=1,2,3,4,5, \\ \mu_8^{32}(X_0,Y_i) = Y_i, \quad i=2,4,5, \\ \mu_8^{32}(X_0',Y_i) = Y_i, \quad i=3,4,5, \\ \mu_8^{32}(Y_1,Y_4) = Y_5, \\ \mu_8^{32}(Y_2,Y_3) = Y_5. \end{cases}$ |
| $\begin{cases} \mu_8^{33}(X,Y_i) = iY_i, \quad i=1,2,3,4, \\ \mu_8^{33}(X_0,Y_2) = Y_2, \\ \mu_8^{33}(X_0',Y_3) = Y_3, \\ \mu_8^{33}(X_0'',Y_4) = Y_4. \end{cases}$ | |

Moreover the laws are pairwise non isomorphic.

Remark : The first classification of these algebras was presented in [3]. It was based on the survey of the eigenvalues of the regular operator $adX$. The theorem of rank simplifies the problem considerably and allows to complete this list. Indeed two algebras were forgotten in the first paper. Later Carles established a list based on the classification of the nilpotent Lie algebras of dimension seven [11].

4.1. **Remark : Dimensions greater than 8.** The problem of classifying the rigid algebras of dimension greater than 8 lies in the existence of a too big number of laws. For example, in [1] we have given an approach of dimension 9, and in a particular case we have found 49 classes of pairwise non isomorphic rigid Lie algebras.

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