Optimal control approach for the fluid-structure interaction problems

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1 Introduction

A fluid-structure interaction problem is studied. We are interested by the displacement of the structure and by the velocity and the pressure of the fluid.

The contact surface between fluid and structure is unknown a priori, therefore it is a free boundary like problem.

In the classical approaches, the fluid and structure equations are coupled via two boundary conditions: the continuity of the velocity and of the constraint vector at the contact surface.

In our approach, the equality of the fluid and structure velocities at the contact surface will be relaxed and treated by the Least Squares Method.

We start with a guess for the contact forces. The displacement of the structure can be computed. We suppose that the fluid domain is completely determined by the displacement of the structure. Knowing the actual domain of the fluid and the contact forces, we can compute the velocity and the pressure of the fluid.

In this way, the equality of the fluid and structure forces at the contact surface is trivially accomplished.

The problem is to find the contact forces such that the equality of the fluid and structure velocities at the contact surface holds.

It's a exact controllability problem with Dirichlet boundary control and Dirichlet boundary observation.

In order to obtain some existence results, this exact controllability problem will be transformed in an optimal control problem using the Least Squares Method.

This mathematical model permits to solve numerically the coupled fluid-structure problem via partitioned procedures (i.e. in a decoupled way, more precisely the fluid and the structure equations are solved separately).

The aim of this paper is to present an optimal control approach for a fluid structure interaction problem and some numerical tests.

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2 Notations

We study the flow in the two-dimensional canal of breadth L_2

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2;\ 0 < x_1 < L_2,\ -H < x_2 < +H \right\}.$$

In the interior of the canal there exists a deformable beam fixed at the one of the his extremities (see the Figure 1).

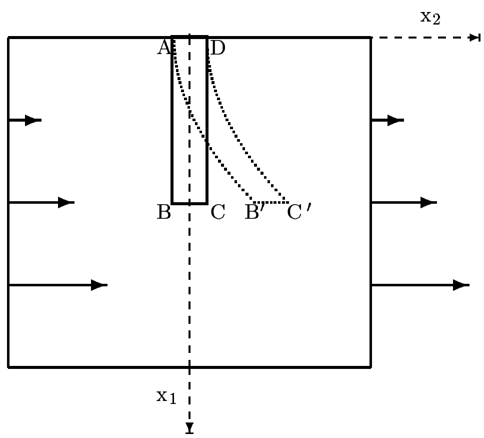

Figure 1: The flow around a deformable beam

In the absence of the fluid, the beam has the parallelepiped shape $[ABCD]$. The coordinates of the vertices are

$$A = (-r, 0)\,,\ B = (-r, L_1)\,,\ C = (r, L_1)\,,\ D = (r, 0)\,.$$

The beam is deformed under the action of the fluid and it will have the shape $[AB'C'D]$. The deformation of the beam is described using the displacement of the median thread

$$u = (u_1, u_2) : [0, L_1] \to \mathbb{R}^2.$$

which satisfy the compatibility condition $u_1(0) = 0, u_2(0) = 0$. For instant, we assume that $u_1 = 0$.

The domain occupied by the beam is

$$\Omega_u^S = \left\{ (x_1, x_2) \in \mathbb{R}^2;\ x_1 \in \left]0, L_1\right[,\ |x_2 - u_2(x_1)| < r \right\}.$$

Consequently, the domain occupied by the fluid is

$$\Omega_u^F = \Omega \backslash \overline{\Omega_u^S}.$$

The contact surface between fluid and beam is $\Gamma_u = \left]AB'\right[\cup [B'C'] \cup \left]C'D\right[$ where

$$\begin{aligned}
\left]AB'\right[&= \left\{ (x_1, x_2) \in \mathbb{R}^2;\ x_1 \in \left]0, L_1\right[,\ x_2 = u_2(x_1) - r \right\}, \\
[B'C'] &= \left\{ (x_1, x_2) \in \mathbb{R}^2;\ x_1 = L_1,\ x_2 \in [-r, r] \right\}, \\
\left]C'D\right[&= \left\{ (x_1, x_2) \in \mathbb{R}^2;\ x_1 \in \left]0, L_1\right[,\ x_2 = u_2(x_1) + r \right\}.
\end{aligned}$$

The other boundary of the fluid domain is noted Γ_1.

3 Beam equations

We suppose that the structure is governed by the beam equations without shearing stress (see [1]).

In view of the Sobolev Embedding Theorem, we have

$$H^2\left(]0,L_1[\right) \hookrightarrow \mathcal{C}^1\left([0,L_1]\right)$$

and we denote

$$U = \left\{\phi \in H^2\left(]0,L_1[\right);\ \phi(0) = \phi'(0) = 0\right\}.$$

Let $D_2 \in \mathbb{R}_+^*$ be given by the formula

$$D_2 = E \int_S x_2^2\, dx_2 dx_3$$

where E is the Young's module and S is the cross section of the beam. We set

$$\left\{\begin{array}{c} a_S : U \times U \to \mathbb{R} \\ a_S(\phi,\psi) = D_2 \displaystyle\int_{]0,L_1[} \frac{d^2\phi}{dx_1^2}(x_1)\, \frac{d^2\psi}{dx_1^2}(x_1)\, dx_1. \end{array}\right. \tag{1}$$

Remark 1 *As a consequence of the Lax-Milgram Theorem, we have the following result: Let $f_2^S \in L^2\left(]0,L_1[\right)$ and $\eta_2 \in L^2\left(]0,L_1[\right)$. Then the problem:*
Find u_2 in U such that

$$a_S(u_2,\psi) = \int_{]0,L_1[} \eta_2(x_1)\, \psi(x_1)\, dx_1 + \int_{]0,L_1[} f_2^S(x_1)\, \psi(x_1)\, dx_1, \quad \forall \psi \in U \tag{2}$$

has a unique solution.

When the data and the solution are smooth enough the solution u_2 verifies the strong formulation given by:

$$u_2''''(x_1) = \frac{1}{D_2}\left(\eta_2(x_1) + f_2^S(x_1)\right), \quad \forall x_1 \in]0,L_1[$$
$$u_2(0) = u_2'(0) = u_2''(L_1) = u_2'''(L_1) = 0.$$

In the particular case $f_2^S = 0$ and $\eta_2(x_1) = \alpha + \beta x_1 + \gamma x_1^2$, we obtain

$$\begin{aligned} u_2(x_1) \;=\; & \frac{1}{360 D_2}\left[15\left(x_1^2 - 4x_1L_1 + 6L_1^2\right)\alpha + 3\left(x_1^3 - 10x_1L_1^2 + 20L_1^3\right)\beta \right. \\ & \left. + \left(x_1^4 - 20x_1L_1^3 + 45L_1^4\right)\gamma\right]. \end{aligned}$$

4 Fluid equations in moving domain

Let u_2 be the solution of the equation (2).

We have

$$H^2\left(]0,L_1[\right) \hookrightarrow \mathcal{C}^1\left([0,L_1]\right)$$

therefore the domain Ω_u^F has a Lipschitz boundary, so that we can define the spaces $H^1\left(\Omega_u^F\right)$, $H^{1/2}\left(\Gamma_1\right)$ and $H^{1/2}\left(\Gamma_u\right)$. We recall that $\partial\Omega_u^F = \overline{\Gamma}_u \cup \overline{\Gamma}_1$. We denote by $n = (n_1, n_2)$ the unit outward normal vector and by $\tau = (-n_2, n_1)$ an unit tangential vector to $\partial\Omega_u^F$.

We denote by $\cdot$ the scalar product.

Let us consider the following vectorial spaces

$$\begin{aligned} W &= \left\{ w = (w_1, w_2) \in H^1\left(\Omega_u^F\right)^2;\ w = 0 \text{ on } \Gamma_1 \text{ and } w \cdot n = 0 \text{ on } \Gamma_u \right\}, \\ Q &= \left\{ q \in H^1\left(\Omega_u^F\right);\ q = 0 \text{ on } \Gamma_u \right\}, \\ M &= \left\{ \omega \in H^1\left(\Omega_u^F\right);\ \omega = 0 \text{ on } \Gamma_1 \right\}. \end{aligned}$$

Let $g \in H_0^{1/2}\left(\Gamma_1\right)^2$ be given, such that $\int_{\Gamma_1} g \cdot n \, d\sigma = 0$. Then there exists $v_0 \in H^1\left(\Omega_u^F\right)^2$, such that $\operatorname{div} v_0 = 0$ in Ω_u^F, $v_0 = 0$ on Γ_u and $v_0 = g$ on Γ_1.

Let p_0 be given in $H^{1/2}\left(\Gamma_u\right)$. Then there exists a function in $H^1\left(\Omega_u^F\right)$, such that its trace is p_0. We denote this function by p_0, also.

Proposition 1 *For all u_2 in U, and f^F in $L^2\left(\Omega_u^F\right)^2$, the problem:*
Find $v - v_0 \in W$, $p - p_0 \in Q$, $\omega \in M$, such that

$$\begin{aligned} &\left(\frac{\partial p}{\partial x_1} + \mu\frac{\partial \omega}{\partial x_2}, \frac{\partial q}{\partial x_1} + \mu\frac{\partial \rho}{\partial x_2}\right) + \left(\frac{\partial p}{\partial x_2} - \mu\frac{\partial \omega}{\partial x_1}, \frac{\partial q}{\partial x_2} - \mu\frac{\partial \rho}{\partial x_1}\right) \\ &+ \left(\omega - \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}, \rho - \frac{\partial w_2}{\partial x_1} + \frac{\partial w_1}{\partial x_2}\right) + (\operatorname{div} v, \operatorname{div} w) \\ &= \left(f_1^F, \frac{\partial q}{\partial x_1} + \mu\frac{\partial \rho}{\partial x_2}\right) + \left(f_2^F, \frac{\partial q}{\partial x_2} - \mu\frac{\partial \rho}{\partial x_1}\right), \quad \forall w \in W, \forall q \in Q, \forall \rho \in M \end{aligned} \tag{3}$$

has a unique solution.

Here, $\mu > 0$ is the dynamic viscosity of the fluid, f^F is the external given force per unit volume and $(\cdot,\cdot)$ is the inner product of $L^2\left(\Omega_u^F\right)$.

Proof. We first prove that

$$\left\|\frac{\partial q}{\partial x_1} + \mu\frac{\partial \rho}{\partial x_2}\right\|_0^2 + \left\|\frac{\partial q}{\partial x_2} - \mu\frac{\partial \rho}{\partial x_1}\right\|_0^2 = \|\nabla q\|_0^2 + \mu^2\|\nabla\rho\|_0^2$$

where $\|\cdot\|_0$ is the standard norm of $L^2\left(\Omega_u^F\right)$.

Le us consider that $q \in \mathcal{C}^1\left(\Omega_u^F\right)$, $q = 0$ on Γ_u and $\rho \in \mathcal{C}^2\left(\Omega_u^F\right)$, $\rho = 0$ on Γ_1. We have

$$\left\|\frac{\partial q}{\partial x_1} + \mu\frac{\partial \rho}{\partial x_2}\right\|_0^2 + \left\|\frac{\partial q}{\partial x_2} - \mu\frac{\partial \rho}{\partial x_1}\right\|_0^2 = \|\nabla q\|_0^2 + \mu^2\|\nabla\rho\|_0^2 + 2\mu\left(\frac{\partial q}{\partial x_1}, \frac{\partial \rho}{\partial x_2}\right) - 2\mu\left(\frac{\partial q}{\partial x_2}, \frac{\partial \rho}{\partial x_1}\right).$$

But using Green's formula, we have

$$\begin{aligned}
&\left(\frac{\partial q}{\partial x_1}, \frac{\partial \rho}{\partial x_2}\right) - \left(\frac{\partial q}{\partial x_2}, \frac{\partial \rho}{\partial x_1}\right) \\
&= \int_{\partial\Omega_u^F} q \frac{\partial \rho}{\partial x_2} n_1 \, d\sigma - \int_{\partial\Omega_u^F} q \frac{\partial \rho}{\partial x_1} n_2 \, d\sigma - \left(q, \frac{\partial^2 \rho}{\partial x_2 \partial x_1}\right) + \left(q, \frac{\partial^2 \rho}{\partial x_1 \partial x_2}\right) \\
&= \int_{\Gamma_1} q \left(\nabla \rho \cdot \tau\right) \, d\sigma + \left(q, -\frac{\partial^2 \rho}{\partial x_2 \partial x_1} + \frac{\partial^2 \rho}{\partial x_1 \partial x_2}\right).
\end{aligned}$$

By assumption of the regularity of ρ, we have $\frac{\partial^2 \rho}{\partial x_2 \partial x_1} = \frac{\partial^2 \rho}{\partial x_1 \partial x_2}$.

Since $\rho = 0$ on Γ_1, we obtain that $\nabla \rho \cdot \tau = 0$ on Γ_1 and then, by a density argument, the equality from the beginning of the proof holds.

The rest of the proof runs as in [3, Sect. 8.2.2]. □

The problem (3) is the least squares variational formulation of the two dimensional Stokes equations in the velocity-pressure-vorticity formulation:
Find the velocity $v : \overline{\Omega}_u^F \to \mathbb{R}^2$, the pressure $p : \overline{\Omega}_u^F \to \mathbb{R}$ and the vorticity $\omega : \overline{\Omega}_u^F \to \mathbb{R}$, such that

$$\left\{\begin{array}{rcll}
\dfrac{\partial p}{\partial x_1} + \mu \dfrac{\partial \omega}{\partial x_2} & = & f_1^F & \text{in } \Omega_u^F \\
\dfrac{\partial p}{\partial x_2} - \mu \dfrac{\partial \omega}{\partial x_1} & = & f_2^F & \text{in } \Omega_u^F \\
\omega - \dfrac{\partial v_2}{\partial x_1} + \dfrac{\partial v_1}{\partial x_2} & = & 0 & \text{in } \Omega_u^F \\
\operatorname{div} v & = & 0 & \text{in } \Omega_u^F \\
v & = & g & \text{on } \Gamma_1 \\
v \cdot n & = & 0 & \text{on } \Gamma_u \\
p & = & p_0 & \text{on } \Gamma_u \\
\omega & = & 0 & \text{on } \Gamma_1.
\end{array}\right. \tag{4}$$

On Γ_u, we have the boundary conditions $v \cdot n = 0$ and $p = p_0$. The validity of these boundary conditions is difficult to prove using other variational formulation. See ([3, Chap. 8]) for more details.

The boundary conditions $p = p_0$ and $v \cdot \tau = 0$ were studied in [10] and the slip boundary conditions $v \cdot n = 0$ and $(\sigma n) \cdot \tau = 0$, where σ is the stress tensor, were studied in [11] and [12], but these boundary conditions aren't appropriate for our approach of the fluid structure interaction.

5 Optimal control approach of the fluid-structure interaction problem

In the classical approaches, the fluid and structure equations are coupled via two boundary conditions: continuity of the velocities and continuity of the forces on the contact surface.

We denote by $\lambda = (\lambda_1, \lambda_2)$ the forces induced by the beam on the contact surface. Consequently, $-\lambda$ represent the forces induced by the fluid acting to the beam.

We denote by $S : M \to U$ the application which computes the displacement of the beam knowing the forces on the contact surface. This application is linear and continuous.

We denote by $F : U \times M \to W \times Q$ the application which computes the velocity and the pressure of the fluid knowing the displacement of the beam (therefore the domain of the fluid) and the forces on the contact surface. This application is non-linear on $U \times M$.

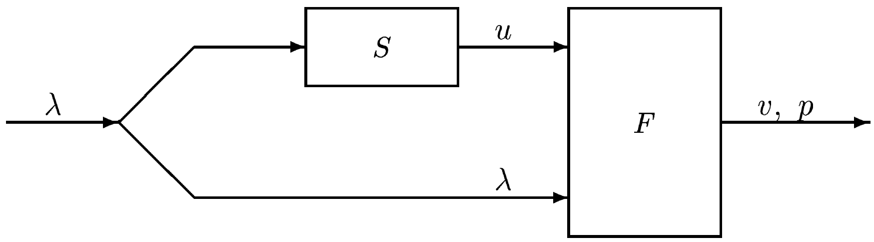

Figure 2: The computing scheme

We search to find out λ, such that $v_{|\Gamma_u} = 0$. This is a exact controllability problem. In our approach, the target condition will be relaxed. We assume that the forces on the contact surface have the form $\lambda = -p_0\, n$, where p_0 is the pressure of the fluid.

We consider the following optimal control problem:

$$\inf J(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) = \frac{1}{2} \|v \cdot \tau\|^2_{0,\Gamma_u} \tag{5}$$

subject to:

$$(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) \in K \subset \mathbb{R}^6 \tag{6}$$

$$\begin{aligned} u_2(x_1) \;=\; & \frac{1}{360 D_2} \left[15\left(x_1^2 - 4x_1 L_1 + 6L_1^2\right)(\alpha_1 - \alpha_2)\right. \\ & +3\left(x_1^3 - 10x_1 L_1^2 + 20L_1^3\right)(\beta_1 - \beta_2) \\ & \left.+\left(x_1^4 - 20x_1 L_1^3 + 45L_1^4\right)(\gamma_1 - \gamma_2)\right] \end{aligned} \tag{7}$$

(v, p, ω) solution of the Stokes problem (3) with

$$p_0(x_1, x_2) = \begin{cases} \left(\alpha_1 + \beta_1 x_1 + \gamma_1 x_1^2\right), & \text{if } (x_1, x_2) \in \,]A, B'] \\ \left(\alpha_2 + \beta_2 x_1 + \gamma_2 x_1^2\right), & \text{if } (x_1, x_2) \in \,]D, C'] \\ \left(\frac{1}{2} - \frac{x_2}{2r}\right) p_0(B') + \left(\frac{1}{2} + \frac{x_2}{2r}\right) p_0(C'), & \text{if } (x_1, x_2) \in \,]B', C'[\end{cases} \tag{8}$$

It's an optimal control problem with Dirichlet boundary control (p_0) and Dirichlet boundary observation ($v_{|\Gamma_u}$).

The relation (6) represents the control constraint.

The relation (7) represents the displacement of the beam under the cross forces $\lambda_2 = \left(\alpha_1 + \beta_1 x_1 + \gamma_1 x_1^2\right)$ on $]A, B'[$ and $\lambda_2 = \left(-\alpha_2 - \beta_2 x_1 - \gamma_2 x_1^2\right)$ on $]D, C'[$. We assume that the displacement of the beam under the longitudinal forces λ_1 is negligible.

This mathematical model permits to solve numerically the coupled fluid-cable problem via partitioned procedures (i.e. in a decupled way, more precisely the fluid and the cable equations are solved separately).

Remark 2 *The existence of an optimal control could be find in [5] for a related problem. In [7] it is proved the differentiability of the cost function and it is given the analytic formula for the gradient.*

Remark 3 *An open problem is to find additional conditions in order to obtain zero for the optimal value of the cost function. This is an approximate controllability problem. For a linear model (the domain of the fluid doesn't depend upon the displacement of the structure), we can find approximate controllability results in [4], [8] and [9].*

Remark 4 *If* $v \cdot \tau$ *is constant on* Γ_u*, then* v *is constant on* Γ_u*. Using [9, Prop. 3.1], we can prove that* $(\sigma n) \cdot n = -p_0$*, where* σ *is the stress tensor. Consequently, solving the beam equations under the action of the surface forces* $-\lambda = p_0 \, n$ *on* Γ_u *is reasonable.*

6 Numerical tests

The parameters for the simulation are listed below:

the geometry $L_1 = 0.5$, $L_2 = 1$, $H = 2$, $r = 0.05$,

the beam $D_2 = 5$,

the fluid $\mu = 1$, $f^F = 0$, $g = (0, Vx_1)$ on the left and right parts of Γ_1, $g = (0, V)$ on the bottom of Γ_1, $g = (0,0)$ on the top parts of Γ_1, $V = 0.5$.

For a guest $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$, we compute the displacement of the beam using the formula (7).

Now, we know the moving boundary of the fluid and we generate a mesh consisting of triangular elements. Then, we solve the fluid equations (3) with boundary condition (8). We have used the P_1 finite element for the velocity, the pressure and the vorticity.

The target is to minimize the cost function (5).

The numerical tests have been produced using *freefem+* (see [2]).

The boundary condition $v \cdot n = 0$ on Γ_u was replaced by $v_2 = 0$.

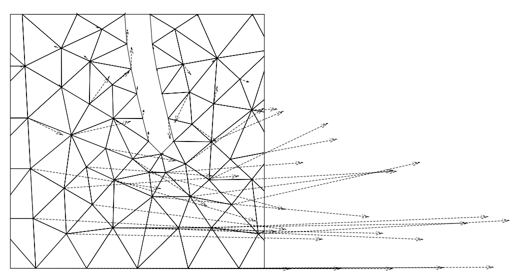

Figure 3: The computed velocity around the beam

The computed velocity isn't a divergence-free field. For a better approximation of the incompressibility condition, we can penalize the term $(\operatorname{div} v, \operatorname{div} w)$ in (3).

The optimal value of the cost function is J=4.08773e-04 and it was obtained for the penalizing factor 10^5. In this case $\|\mathrm{div}\, v\|_0^2$ is 5.19e-03. In the Figure 3, we can see the corresponding displacement of the beam and the velocity of the fluid. The velocity of the fluid was multiplied by 2 for a better visualization.

For the penalizing factor 10^2, the optimal value of the cost function is J=1.20132e-03 and $\|\mathrm{div}\, v\|_0^2$ is 7.91e-03.

We can avoid to generate a new mesh for each evaluation of the cost function by using the dynamic mesh like in [6].

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