ON A FAMILY OF DIFFERENTIAL EQUATIONS FOR BOUNDARY LAYER APPROXIMATIONS IN POROUS MEDIA

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Free convection along a vertical flat plate embedded in a porous medium is considered, within the framework of boundary layer approximations. In some cases, similarity solutions can be obtained by solving a boundary value problem involving an autonomous third-order nonlinear equation, depending on a parameter related to the temperature on the wall. The paper deals with existence and uniqueness questions to this problem, for every value of the parameter.

1 Introduction

Natural convection problems arise when dealing with a heated impermeable flat plate embedded in an unbounded porous medium. In fact, at high Rayleigh numbers, the most important part of the convection takes place in a thin layer around the heated source. In this case, thermal boundary layer approximations can be derived (in analogy with Prandtl theory for flow at high Reynolds numbers) and similarity solutions can be obtained by solving the following the boundary value problem $(\mathcal{P}_\alpha)$ involving the autonomous third order nonlinear differential equation

$$f''' + \frac{\alpha+1}{2} f f'' - \alpha f'^2 = 0 \quad \text{on} \quad (0,\infty) \tag{1.1}$$

and the boundary conditions

$$f(0) = 0, \qquad f'(0) = 1, \tag{1.2}$$

$$f'(\infty) := \lim_{t\to\infty} f'(t) = 0. \tag{1.3}$$

The parameter α describes the temperature distribution prescribed on the wall, and the range for which the problem has some physical meaning is $-\frac{1}{3} \le \alpha \le 1$. See Ene & Poliševski 1987 and Cheng & Minkowicz 1977. However, for the mathematical analysis, we will be concerned with every value of α.

For physical reasons (see (2.3) and (2.7) below) we will also consider the following constraint:

$$\forall t \in [0,\infty) \quad 0 \le f'(t) \le 1. \tag{1.4}$$

The question of existence or nonexistence does not necessarily depend on the condition (1.4). For example, in the case $\alpha = -1$, if f is a solution of (1.1), then f' is concave and we cannot have $f'(0) = 1$ and $f'(\infty) = 0$ together. So, we see that, even without the restriction (1.4), the problem may not have a solution. Furthermore, the question of uniqueness for the problem (1.1)-(1.3) is not really clear. In fact, we will show that there is an infinite number of solutions to (1.1)-(1.3) when $\alpha = -\frac{1}{3}$, whereas uniqueness holds for $\alpha \in [0, \frac{1}{3}]$. But, in any other case (except, when we prove nonexistence), we are unable to say if uniqueness holds or not.

On the other hand, if we look at the full problem (1.1)-(1.4), we prove nonexistence for $\alpha < -\frac{1}{3}$, existence for $\alpha \geq -\frac{1}{3}$, and uniqueness for $\alpha = -\frac{1}{3}$ and $\alpha \geq 0$. We also compute explicitly the solution for $\alpha = -\frac{1}{3}$ and $\alpha = 1$.

Let us note that for $\alpha = 0$, the equation (1.1) reduces to the Blasius equation

$$f''' + \frac{1}{2} f f'' = 0.$$

Numerical results about (1.1)-(1.4) for some values of α between $-\frac{1}{3}$ and 1 can be found in Cheng & Minkowicz 1977. Further numerical investigations for α close to $-\frac{1}{2}$ and $\alpha > 1$ are in Banks 1983 and Ingham & Brown 1986.

On the other hand, the explicit solutions for $\alpha = -\frac{1}{3}$ and $\alpha = 1$ were already given by Stuart 1966. See also Crane 1970.

Nonexistence of solutions for $\alpha = -\frac{1}{2}$ was noted by Banks 1983. In Ingham & Brown 1986, the argument used to get nonexistence for $\alpha < -\frac{1}{2}$ is only valid if the possible solution f is assumed to satisfy $f'f^2 \to 0$ at infinity. We will see that it is unnecessary to add this assumption.

The equation (1.1) with the boundary conditions $f(0) = a$, $f'(0) = 1$ and $f'(\infty) = 0$ is considered by Magyari & Keller 2000, in the particular cases $\alpha = -\frac{1}{2}$, $\alpha = -\frac{1}{3}$ and $\alpha = 1$. Such situations correspond to lateral suction ($f(0) > 0$) or injection ($f(0) < 0$) of the fluid through a permeable stretching wall and have been considered by Banks & Zaturska 1986, Chaudary, Merkin & Pop 1995a,b. See also Merkin & Zhang 1990.

Finally, let us mention that, according to the boundary layer technique, a steady two-dimensional flow of a slighty viscous incompressible fluid past a wedge, can be described in terms of the solution of the following problem:

$$f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) = 0 \tag{1.4}$$

with the boundary conditions $f(0) = f'(0) = 0$ and $f'(\infty) = 1$ (see Rosenhead 1963). If $m > -1$, which is true for the values of m having physical significance, setting

$$\beta = \frac{2m}{m+1}, \qquad \lambda = \sqrt{\frac{2}{m+1}} \qquad \text{and} \qquad u(t) = \lambda^{-1} f(\lambda t),$$

we get

$$u''' + uu'' + \beta(1 - u'^2) = 0. \tag{1.5}$$

with $u(0) = u'(0) = 0$ and $u'(\infty) = 1$. The equations (1.4) and (1.5) are called Falkner-Skan equations. In the special case $\beta = \frac{1}{2}$, corresponding to $m = \frac{1}{3}$, then (1.5) is called the Homann equation and every solution of it also satisfies the equation

$$u'''' + uu''' = 0,$$

obtained from (1.5) by differentiation. See Coppel 1960, Falkner & Skan 1931, Hartman 1964, Hasting & Troy 1987, 1988, Kays & Crawford 1993, Walter 1970 and Weyl 1942. However, because of the non-homogeneity of (1.4) or (1.5), it is not clear how to relate the Falkner-Skan problem to ours.

The paper is organized as follows : in the second section we explain how to derive the problem (1.1)-(1.3) and the condition (1.4). In the third section, we show how the value $\alpha = -\frac{1}{3}$ appears in the computations, and recall the explicit solution to (1.1)-(1.4) for this case and also for $\alpha = 1$. Next, we give properties of solutions to the problem (1.1)-(1.3), which will be of great importance in obtaining nonexistence and uniqueness results. The fourth section is devoted to nonexistence results for some values of the parameter α. In the section 5, we prove existence of solution to the problem (1.1)-(1.4), for $\alpha \geq -\frac{1}{3}$. In section 6 we discuss the question of uniqueness, and show that for $\alpha \geq 0$ and $\alpha = -\frac{1}{3}$, the problem (1.1)-(1.4) has one and only one solution, and that for $\alpha = -\frac{1}{3}$, the problem (1.1)-(1.3) has an infinite number of solutions, whereas for $\alpha \in [0, \frac{1}{3}]$ we have no solution other than the one satisfying (1.4). We finish with some concluding remarks.

2 Boundary layer approximations and similarity solutions

Let us consider the problem of steady free convection around a vertical impermeable flat plate in a saturated porous medium, and consider a rectangular Cartesian coordinate system with the origin fixed at the leading edge of the vertical surface such that the x-axis is directed upwards along the wall and the y-axis is normal to it. If we assume that, in particular, the convective fluid and the porous medium are everywhere in local thermodynamic equilibrium, the governing equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x} + \rho g\right)$$

$$v = -\frac{k}{\mu}\frac{\partial p}{\partial y}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\rho = \rho_\infty[1 - \beta(T - T_\infty)]$$

where u and v are Darcy velocities in the x and y directions, ρ, μ and β are the density, viscosity and thermal expansion coefficient of the fluid, k is the permeability of the saturated porous medium, and λ is the equivalent thermal diffusivity; p is the pressure, T the temperature and g the acceleration due to gravity. The subscript ∞ denotes a value far from the plate.

For our coordinate system, the boundary conditions at the wall are

$$v(x,0)=0, \qquad T(x,0)=T_w(x)=T_\infty + Ax^\alpha \ , \qquad \alpha\in\mathbb{R},$$

where $A>0$. The boundary conditions far from the plate are

$$u=0 \quad \text{and} \quad T=T_\infty.$$

In terms of the stream function ψ satisfying

$$u=\frac{\partial\psi}{\partial y} \qquad \text{and} \qquad v=-\frac{\partial\psi}{\partial x},$$

the previous equations can be rewritten as

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{\rho_\infty\beta g k}{\mu}\frac{\partial T}{\partial y},$$

$$\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial T}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial T}{\partial y}\frac{\partial\psi}{\partial x}.$$

If we assume that convection takes place in a thin layer around the heating surface, we get the boundary layer approximation

$$\frac{\partial^2\psi}{\partial y^2}=\frac{\rho_\infty\beta g k}{\mu}\frac{\partial T}{\partial y}, \tag{2.1}$$

$$\frac{\partial^2 T}{\partial y^2}=\frac{1}{\lambda}\left(\frac{\partial T}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial T}{\partial y}\frac{\partial\psi}{\partial x}\right). \tag{2.2}$$

with

$$\frac{\partial\psi}{\partial x}(x,0)=0 \quad \text{and the matching condition} \quad \frac{\partial\psi}{\partial y}\to 0 \ \text{ as } \ y\to\infty.$$

We now define a local Rayleigh number as

$$Ra_x=\frac{\rho_\infty\beta g k(T_w(x)-T_\infty)x}{\mu\lambda}$$

and introduce the following dimensionless similarity variables:

$$\eta=(Ra_x)^{1/2}\frac{y}{x}, \qquad \psi(x,y)=\lambda(Ra_x)^{1/2}f(\eta), \qquad \theta(\eta)=\frac{T(x,y)-T_\infty}{T_w(x)-T_\infty}.$$

Since the temperature is assumed to decrease away from the wall, we have

$$0 \leq \theta \leq 1. \tag{2.3}$$

In terms of the new variables, it is easy to see that equations (2.1)-(2.2) become

$$f'' - \theta' = 0, \tag{2.4}$$

$$\theta'' + \frac{\alpha+1}{2} f\theta' - \alpha f'\theta = 0, \tag{2.5}$$

with

$$f(0) = 0, \qquad \theta(0) = 1,$$

and

$$f'(\infty) = 0, \qquad \theta(\infty) = 0, \tag{2.6}$$

where the primes indicate differentiation with respect to η. Finally, integrating (2.4), and taking into account the boundary conditions (2.6), we get

$$f' = \theta, \tag{2.6}$$

and (1.1)-(1.4) follow.

For more details on this model, see for example Cheng & Minkowicz 1977, Ene & Poliševski 1987, Kays & Crawford 1993.

3 Properties of the solutions

3.1. Definitions and preliminary remarks According to the previous sections, we will distinguish between solutions of $(\mathcal{P}_\alpha)$ for every C^3-function f satisfying (1.1)-(1.3) and physical solutions of $(\mathcal{P}_\alpha)$ for every C^3-function f satisfying (1.1)-(1.4).

By setting

$$v = f' + \frac{\alpha+1}{4} f^2 \tag{3.1}$$

we easily see that, if f is a solution of $(\mathcal{P}_\alpha)$, then v satisfies

$$v'' = \frac{3\alpha+1}{2} f'^2 \quad \text{on} \quad (0,\infty) \tag{3.2}$$

and

$$v(0) = 1 \quad \text{and} \quad v(\infty) = \frac{\alpha+1}{4} f(\infty)^2. \tag{3.3}$$

We see also that the value $\alpha = -\frac{1}{3}$ plays a particular role. Actually, if f is a solution of $(\mathcal{P}_{-\frac{1}{3}})$ then, from (3.2) and (1.2), we get (if we replace the variable η by t)

$$\forall t \geq 0, \qquad f'(t) + \frac{1}{6} f(t)^2 = \mu t + 1 \tag{3.4}$$

which is a Riccati equation (here μ is a constant which is in fact equal to $f''(0)$).

If $\mu = 0$, the constant function $f = \sqrt{6}$ is a particular solution of (3.4). Then, solving (3.4) with the initial condition $f(0) = 0$ we obtain

$$f(t) = \sqrt{6}\ \tanh \frac{t}{\sqrt{6}}$$

which satisfies (1.3). This function is the unique physical solution of $(\mathcal{P}_{-\frac{1}{3}})$. In fact, according to (1.3), we must have $\mu \geq 0$ and if $\mu > 0$, we get $f'(t) > 1$ for small $t > 0$. As we will show in §6, there exists an infinite number of solutions to the problem $(\mathcal{P}_{-\frac{1}{3}})$, satisfying $f''(0) > 0$. From (3.4) we see that such solutions will be unbounded.

In one case, we can also find an explicit solution. By setting $f = g + a$, with $a \in \mathbb{R}$, then (1.1) can be rewritten as

$$g''' + \frac{\alpha+1}{2} a g'' = \alpha g'^2 - \frac{\alpha+1}{2} g g''.$$

One can show that there exists a solution to the linear equation

$$g''' + \frac{\alpha+1}{2} a g'' = 0$$

which satisfies $g(0) = -a$, $g'(0) = 1$ and $g'(\infty) = 0$ and now we see that the equality

$$\alpha g'^2 - \frac{\alpha+1}{2} g g'' = 0$$

holds if and only if $\alpha = 1$. By this way, we get as physical solution of $(\mathcal{P}_1)$, the function

$$f(t) = 1 - \mathrm{e}^{-t},$$

and we see also that this method works only for $\alpha = 1$.

Remark 3.1 The case $\alpha = -\frac{1}{3}$ is of special interest. In fact, the energy flux

$$E(x) = C_1 x^{\frac{3\alpha+1}{2}} \int_0^\infty f'(\eta)\theta(\eta)\mathrm{d}\eta$$

is constant for this special case, which corresponds to a horizontal line source embedded in a porous medium. Note also that the local heat transfer at the surface of the flat plate can be written as

$$q(x) = C_2 x^{\frac{3\alpha-1}{2}} [-\theta'(0)]$$

where the values of $[-\theta'(0)]$ increases from 0 at $\alpha = -\frac{1}{3}$ to 1 at $\alpha = 1$. Here we denote by C_1 and C_2 some constants depending on the physical quantities introduced in §2. See Cheng & Minkowicz 1977.

3.2. Qualitative properties of the solutions Let us introduce the initial value problem

$$\begin{cases} f''' + \frac{\alpha+1}{2} f f'' - \alpha f'^2 = 0 \\ f(0) = 0 \\ f'(0) = 1 \\ f''(0) = \mu \end{cases} \qquad (\mathcal{P}_{\alpha,\mu})$$

Let f be a solution of $(\mathcal{P}_{\alpha,\mu})$ defined on $[0,T)$, with $0 < T \leq \infty$. Clearly f' and f'' cannot vanish at the same point.

On the other hand, if we denote by F any anti-derivative of f, we get the equation

$$\left(f'' \,\mathrm{e}^{\frac{\alpha+1}{2}F}\right)' = \alpha \,\mathrm{e}^{\frac{\alpha+1}{2}F} f'^2. \tag{3.5}$$

Now, we can prove the following lemmas concerning the changes of concavity of solutions of $(\mathcal{P}_{\alpha,\mu})$.

Lemma 3.1 *Let us assume that $\alpha < 0$ (resp. $\alpha > 0$) and let f be a solution on $[0,T)$ of $(\mathcal{P}_{\alpha,\mu})$. For $t_0 \in [0,T)$, we have:*

$$f''(t_0) \leq 0 \text{ (resp. } f''(t_0) \geq 0) \quad \Longrightarrow \quad f'' < 0 \text{ (resp. } f'' > 0) \quad \text{on} \quad (t_0, T).$$

Proof If $\alpha < 0$, it follows from (3.5) and the fact that f' and f'' cannot vanish at the same point, that the function $t \longmapsto f''(t)\,\mathrm{e}^{\frac{\alpha+1}{2}F(t)}$ decreases and so

$$\forall t > t_0, \qquad f''(t) < f''(t_0)\,\mathrm{e}^{\frac{\alpha+1}{2}(F(t_0)-F(t))} \leq 0.$$

This completes the proof of this case. For $\alpha > 0$ the proof is similar. □

Now, we are able to give results about the behaviour of the solutions of the boundary value problem $(\mathcal{P}_\alpha)$. For the rest of this section, we will denote by f a solution, if it exists, of (1.1)-(1.3).

Proposition 3.1 *If $\alpha < 0$, then f is positive and strictly increasing on $(0,\infty)$. Moreover:*

- *if $f''(0) \leq 0$, then f is strictly concave on $[0,\infty)$,*
- *if $f''(0) > 0$, there exists $t_0 \in (0,\infty)$ such that f is strictly convex on $[0,t_0]$ and strictly concave on $[t_0,\infty)$.*

Proof Assume first that $f''(0) \leq 0$. It follows from Lemma 3.1, that $f'' < 0$ on $(0,\infty)$, that is to say f is strictly concave on $[0,\infty)$ and f' is strictly decreasing on $[0,\infty)$. But $f'(\infty) = 0$; thus $f' > 0$ on $[0,\infty)$ and f is strictly increasing. Finally, since $f(0) = 0$, we get $f > 0$ on $(0,\infty)$.

Now, assume that $f''(0) > 0$. There exists $t_0 \in (0, \infty)$ such that $f'' > 0$ on $[0, t_0)$ and $f''(t_0) = 0$. Indeed, if not, f' would be strictly increasing and thus, since $f'(0) = 1$, we could not have $f'(\infty) = 0$. Next, Lemma 3.1 implies that $f'' < 0$ on (t_0, ∞) and as in the previous case, we deduce that $f' > 0$ on $[t_0, \infty)$. Moreover, since $f' \geq 1$ on $[0, t_0]$, we have $f' > 0$ on $[0, \infty)$ and this completes the proof. □

Proposition 3.2 *If $\alpha > 0$, then $f''(0) < 0$, f is bounded and positive on $(0, \infty)$. Moreover:*

- *either f is strictly increasing and strictly concave on $[0, \infty)$,*
- *or there exists $t_0 \in (0, \infty)$ such that f is strictly concave on $[0, t_0]$ and f is strictly decreasing and strictly convex on $[t_0, \infty)$.*

Proof First, if $f''(0) \geq 0$, it follows from Lemma 3.1 that $f'' > 0$ on $(0, \infty)$. Hence f' is strictly increasing and we cannot have $f'(0) = 1$ and $f'(\infty) = 0$ together. So, $f''(0) < 0$.

Let us assume that f'' does not vanish on $(0, \infty)$. We then have $f'' < 0$ and f is strictly concave on $[0, \infty)$. Hence, f' is strictly decreasing and it follows from (1.3) that $f' > 0$. Consequently, f is strictly increasing and, since $f(0) = 0$, positive on $(0, \infty)$. Finally, suppose that f is unbounded, which means that $f(\infty) = \infty$. But, for $t > 0$, we have that

$$-\frac{f'''(t)}{f''(t)} = \frac{\alpha + 1}{2} f(t) - \alpha \frac{f'(t)^2}{f''(t)} \geq \frac{\alpha + 1}{2} f(t)$$

which implies

$$\lim_{t \to \infty} \frac{f'''(t)}{-f''(t)} = \infty.$$

Then, there exists t_1 such that

$$\forall t \geq t_1, \quad f'''(t) \geq -f''(t).$$

Now, by integrating between $s \geq t_1$ and ∞ we obtain

$$\forall s \geq t_1, \quad -f''(s) \geq f'(s).$$

(Let us remark that here $f''' > 0$ and thus we deduce from (1.3) that $f''(\infty) = 0$.) Integrating again we get

$$\forall t \geq t_1, \quad -f'(t) + f'(t_1) \geq f(t) - f(t_1)$$

and a contradiction with (1.3).

Let us now assume that f'' vanishes and denote by t_0 the point of $(0, \infty)$ such that

$$f'' < 0 \text{ on } [0, t_0) \quad \text{and} \quad f''(t_0) = 0.$$

Thanks to Lemma 3.1, we get $f'' > 0$ on (t_0, ∞). Therefore, f' is strictly increasing on $[t_0, \infty)$ and, due to (1.3), we obtain $f' < 0$ on $[t_0, \infty)$. So, f is strictly convex and strictly decreasing on $[t_0, \infty)$. It remains to prove that $f > 0$. For that, suppose there exists $t_1 > 0$

such that $f(t_1) \le 0$. Since f is strictly decreasing on $[t_0,\infty)$, we get $f<0$ on (t_1,∞). Therefore,

$$\forall t > t_1, \qquad f'''(t) = \alpha f'(t)^2 - \frac{\alpha+1}{2} f(t) f''(t) > 0$$

and f' is convex on $[t_1,\infty)$. But, this is a contradiction with (1.3) and $f'(t_1)<0$. So, $f>0$ on $[t_0,\infty)$. On the other hand, $f(0)=0$, $f(t_0)>0$ and f is concave on $[0,t_0]$, thus $f>0$ on $(0,t_0]$. Finally, f is positive and clearly bounded. The proof is now complete. □

Proposition 3.3 *If $\alpha=0$, then f is strictly increasing, strictly concave and positive on $(0,\infty)$. Moreover $f''(0)<0$ and f is bounded.*

Proof By using equation (3.5), we get

$$\forall t \ge 0, \qquad f''(t) = f''(0)\, \mathrm{e}^{\frac{1}{2}(F(0)-F(t))},$$

and we see that either $f'' \equiv 0$, or $f''>0$, or $f''<0$. But, the equalities $f'(0)=1$ and $f'(\infty)=0$ show that, necessarily, we must have $f''<0$ and also $f'>0$ and $f>0$ on $(0,\infty)$. To see that f is bounded we argue as in the proof of the previous proposition. □

Remark 3.2 If $\alpha \in [-\frac{1}{3},0)$ and $f''(0)>0$, then it follows from (3.1)-(3.2) that

$$\forall t \ge 0, \qquad f'(t) + \frac{\alpha+1}{4} f(t)^2 \ge f''(0)t + 1,$$

and thus f is unbounded.

3.3. Some identities Let us now give some miscellaneous results, which will end this section.

Let f be a solution of (1.1)-(1.2). Multiplying the equation (1.1) by 1, t, f, f'' and integrating by parts, we obtain, for all t such that $f(t)$ exists, the following useful identities:

$$f''(t) - f''(0) + \frac{\alpha+1}{2} f'(t) f(t) = \frac{3\alpha+1}{2} \int_0^t f'(s)^2 \mathrm{d}s, \tag{3.6}$$

$$t f''(t) + \frac{\alpha+1}{2} t f'(t) f(t) - f'(t) - \frac{\alpha+1}{4} f(t)^2 + 1 = \frac{3\alpha+1}{2} \int_0^t s f'(s)^2 \mathrm{d}s, \tag{3.7}$$

$$f(t) f''(t) - \frac{1}{2} f'(t)^2 + \frac{1}{2} + \frac{\alpha+1}{2} f'(t) f(t)^2 = (2\alpha+1) \int_0^t f(s) f'(s)^2 \mathrm{d}s, \tag{3.8}$$

$$\frac{1}{2} f''(t)^2 - \frac{1}{2} f''(0)^2 - \frac{\alpha}{3} f'(t)^3 + \frac{\alpha}{3} = -\frac{\alpha+1}{2} \int_0^t f(s) f''(s)^2 \mathrm{d}s. \tag{3.9}$$

Proposition 3.4 *Let $\alpha \in \mathbb{R}$ and f be a solution of $(\mathcal{P}_\alpha)$. We have*

$$\lim_{t\to\infty} f''(t) = 0 \tag{3.10}$$

and there exists an increasing sequence (t_n) tending to ∞ and such that

$$\lim_{n\to\infty} f'''(t_n) = \lim_{n\to\infty} f(t_n)f''(t_n) = 0 \tag{3.11}$$

Proof Since $f'(\infty) = 0$, there exists an increasing sequence (x_n) going to ∞ and satisfying

$$\lim_{n\to\infty} f''(x_n) = 0$$

(one can take x_n such that $f''(x_n) = f'(n+1) - f'(n)$). On the other hand, since $f \geq 0$, the function

$$t \longmapsto \int_0^t f(s)f''(s)^2 \mathrm{d}s$$

is increasing, and we deduce from (3.9) that $\lim_{t\to\infty} f''(t)^2$ exists. Then (3.10) holds. Furthermore, choosing (t_n) such that $f'''(t_n) = f''(n+1) - f''(n)$, and using (1.1) and (1.3) we get (3.11). □

4 Nonexistence results

In what follows, we are going to prove that the problem $(\mathcal{P}_\alpha)$ has no solution when $\alpha \leq -\frac{1}{2}$. The arguments used being essentially different, we will distinguish in the proof the cases $\alpha \in (-\infty, -1]$ and $\alpha \in (-1, -\frac{1}{2}]$. Also, we prove that for $\alpha \in (-\frac{1}{2}, -\frac{1}{3})$, we have no *physical* solution.

Theorem 4.1 *For $\alpha \leq -\frac{1}{2}$, the problem $(\mathcal{P}_\alpha)$ has no solution.*

Proof for $\alpha \leq -1$ Suppose there exists a solution f to the problem $(\mathcal{P}_\alpha)$. For $t \geq 0$, we have

$$f'''(t) = \alpha f'(t)^2 - \frac{\alpha+1}{2} f(t)f''(t).$$

By using Proposition 3.1, we see that there exists $t_0 \in [0, \infty)$ such that

$$\forall t > t_0, \qquad f'''(t) < 0$$

implying that f' is concave on $[t_0, \infty)$. But the concavity of f' does not allow $f' > 0$ and $f'(\infty) = 0$ together. □

Proof for $\alpha \in (-1, -\frac{1}{2}]$ Suppose that f is a solution of $(\mathcal{P}_\alpha)$. By Proposition 3.1, we know that $f \geq 0$ and $f' \geq 0$. Hence, using (3.8), (3.11) and (1.3), we can write

$$0 \geq \lim_{n\to\infty} (2\alpha+1) \int_0^{t_n} f(s)f'(s)^2 \mathrm{d}s = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{\alpha+1}{2} f'(t_n)f(t_n)^2 \right) \geq \frac{1}{2}$$

which is absurd. □

Theorem 4.2 *For $\alpha \in (-\frac{1}{2}, -\frac{1}{3})$, the problem $(\mathcal{P}_\alpha)$ has no physical solution. Moreover, if f is a solution of $(\mathcal{P}_\alpha)$, then,*

$$f''(0) = -\frac{3\alpha+1}{2}\int_0^\infty f'(s)^2\,ds \quad \text{and} \quad \lim_{t\to\infty} f(t)f'(t) = 0. \tag{4.1}$$

Proof Let $\alpha \in (-\frac{1}{2}, -\frac{1}{3})$ and f be a solution of $(\mathcal{P}_\alpha)$. From relation (3.6) and Proposition 3.1, it follows that

$$0 \leq -\frac{3\alpha+1}{2}\int_0^t f'(s)^2 \mathrm{d}s \leq f''(0) - f''(t).$$

Hence, using (3.10), we have that $f' \in L^2(0,\infty)$ and

$$0 < -\frac{3\alpha+1}{2}\int_0^\infty f'(s)^2 \mathrm{d}s \leq f''(0).$$

The nonexistence of physical solution to $(\mathcal{P}_\alpha)$ then follows from the positivity of $f''(0)$.

Now, coming back to (3.6) we see that $l = (ff')(\infty)$ exists and is nonnegative. Suppose that $l > 0$. Then, there is a t_0 such that

$$t \geq t_0 \Longrightarrow f(t)f'(t) < \frac{3l}{2}.$$

Therefore, for $t \geq t_0$, we have

$$\int_{t_0}^t 2f'(s)f(s)\mathrm{d}s \leq 3l(t-t_0)$$

which gives $f(t)^2 \leq 3l(t-t_0) + f(t_0)^2$. Then,

$$f'(t)^2 \sim \frac{l^2}{f(t)^2} \geq \frac{l^2}{3l(t-t_0)+f(t_0)^2} \sim \frac{l}{3t} \quad \text{when} \quad t \to \infty$$

contradicting the fact that $f' \in L^2(0,\infty)$. So, $l = 0$ and the first relation of (4.1) follows by passing to the limit in (3.6). □

5 Existence for $\alpha \geq -\frac{1}{3}$

In this part, we will prove that, for $\alpha \geq -\frac{1}{3}$, the problem $(\mathcal{P}_\alpha)$ has at least one *physical solution.*

To get existence of such a solution for $\alpha > 0$, it is possible to rewrite the differential equation (1.1) as a first-order system, and adapt an idea given in Coppel 1960 and Hartman 1964 for a similar problem involving the Falkner-Skan equation (see also Utz 1978 and Belhachmi, Brighi & Taous 2000a). But, for easy of presentation, we prefer to give a proof in term of the original variable f.

Theorem 5.1 *If $\alpha \geq -\frac{1}{3}$, the problem $(\mathcal{P}_\alpha)$ has a physical solution f_*, which is strictly concave and also satisfies*

$$\forall t \geq 0, \qquad 0 \leq f_*(t) \leq \frac{2}{\sqrt{\alpha+1}}. \tag{5.1}$$

Proof of existence for $\alpha \in [-\frac{1}{3}, 0]$ We already have existence for $\alpha = -\frac{1}{3}$, so we can assume $-\frac{1}{3} < \alpha \leq 0$. Let us consider the initial value problem $(\mathcal{P}_{\alpha,\mu})$ introduced in §3.2 with $\mu \leq 0$. Denote by f_μ the solution of $(\mathcal{P}_{\alpha,\mu})$ and by $[0, T^\mu)$, its right maximal interval of existence. Note that we can have $T^\mu < \infty$; for example, this is the case for $\alpha = 0$ and large values of $-\mu$ (see Coppel 1960 or Belhachmi, Brighi & Taous 2000b).

Thanks to Lemma 3.1, we see that f_μ is affine or strictly concave. Moreover, if $-\mu$ is large enough, then f'_μ becomes negative from some $t_1 > 0$. Indeed, if f'_μ does not vanish, then $T^\mu = \infty$, $0 < f'_\mu \leq 1$ and $f_\mu \geq 0$. Therefore, we deduce from the identity (3.6) that

$$f''_\mu(t) - \mu \leq \frac{3\alpha+1}{2} \int_0^t f'_\mu(s)^2 \mathrm{d}s \leq \frac{3\alpha+1}{2} t.$$

Upon integration one obtains

$$\forall t > 0, \qquad \frac{3\alpha+1}{4} t^2 + \mu t + 1 > 0,$$

from which we get $\mu > -\sqrt{3\alpha+1}$.

Now, if we set $A = \{\mu \leq 0 \ ; \ f'_\mu > 0\}$ and $\mu_* = \inf A$, we have $\mu_* > -\infty$. Let us remark that A is not empty, since $0 \in A$; in fact, if $f'_0(t_0) = 0$, then (3.6) gives $f''_0(t_0) > 0$ and a contradiction.

Let f_* be the solution of $(\mathcal{P}_{\alpha,\mu_*})$. If $\mu_* \notin A$, then f'_* becomes negative from some point, and so $f''_* < 0$ leads to a contradiction. Therefore $\mu_* \in A$. It follows that $T_* = \infty$ and there exists $\lambda_* \geq 0$ such that

$$\lim_{t \to \infty} f'_*(t) = \lambda_*.$$

Let T be any positive number. Let (μ_n) be an increasing sequence converging to μ_*. Because of the lower semicontinuity of the map $\mu \longmapsto T^\mu$, there is an integer N such that for all $n \geq N$, the solution f_n of $(\mathcal{P}_{\alpha,\mu_n})$ is defined on $[0, T]$. By continuity, we have

$$\lim_{n \to \infty} f_n(T) = f_*(T).$$

But, if we write the identity (3.7) at t_n, with t_n the point where f_n achieves its maximum, we get

$$\forall t \geq 0, \qquad f_n(t) \leq \frac{2}{\sqrt{\alpha+1}}.$$

Thus the estimate (5.1) holds and necessarily $\lambda_* = 0$. Finally, f_* is a physical solution of $(\mathcal{P}_\alpha)$. □

Proof of existence for $\alpha > 0$ Let us consider again the initial value problem $(\mathcal{P}_{\alpha,\mu})$ with $\mu \le 0$, and denote by f_μ its solution and by $[0, T^\mu)$ its right maximal interval of existence. First of all, let us remark that $f_\mu(t)$ exists as long as $f'_\mu(t) \ge 0$ and $f''_\mu(t) \le 0$. Thanks to Lemma 3.1 and since for every $\mu \le 0$ we cannot have $f'_\mu(t) = f''_\mu(t) = 0$ at some point t, it follows that there are just three possibilities:

(a) f''_μ becomes positive from a point t_0 for which $f'_\mu > 0$,

(b) f'_μ becomes negative from a point t_1 for which $f''_\mu < 0$,

(c) f''_μ and f'_μ do not vanish.

Since $f''_0(0) = 0$ and $f'''_0(0) = \alpha > 0$, we see that f_0 is of type (a) with $t_0 = 0$, hence, by continuity, this must be so for f_μ if $-\mu > 0$ is small.

On the other hand, as long as $f'_\mu(t) \ge 0$ and $f''_\mu(t) \le 0$, we have $f_\mu(t) \ge 0$ and $f'_\mu(t) \le 1$, in such a way that, by using (3.6) we get

$$f''_\mu(t) \le \mu + \frac{3\alpha+1}{2}t \qquad \text{and} \qquad f'_\mu(t) \le 1 + \mu t + \frac{3\alpha+1}{4}t^2.$$

Consequently, if we choose $-\mu$ large enough, then f_μ is of type (b).

Now, if $A = \{\mu < 0\ ;\ f_\mu \text{ is of type (a)}\}$ and $B = \{\mu < 0\ ;\ f_\mu \text{ is of type (b)}\}$, then $A \ne \emptyset$, $B \ne \emptyset$ and $A \cap B = \emptyset$. Moreover, both of A and B are open sets and therefore there is a $\mu_* < 0$ such that the solution f_* of $(\mathcal{P}_{\alpha,\mu_*})$ is of type (c) and exists on the whole interval $[0,\infty)$. Since, for this solution we have $f''_* < 0$ and $f'_* > 0$, there exists $\lambda_* \ge 0$ such that

$$\lim_{t\to\infty} f'_*(t) = \lambda_*.$$

Suppose that $\lambda_* > 0$. We have

$$f'''_* = -\frac{\alpha+1}{2}f_* f''_* + \alpha(f'_*)^2 \ge \alpha(f'_*)^2 > \alpha\lambda_*^2$$

from which, by integrating, we get

$$\forall t \ge 0, \qquad f''_*(t) > \mu_* + \alpha\lambda_*^2 t,$$

and a contradiction with the negativity of f''_*. Finally, $\lambda_* = 0$. It remains to prove (5.1). We know from Proposition 3.2, that f_* is bounded. Hence, using the concavity of f_*, one can show that

$$\lim_{t\to\infty} t f'_*(t) f_*(t) = \lim_{t\to\infty} t f''_*(t) = 0$$

and passing to the limit in (3.7), we get

$$\frac{\alpha+1}{4}f_*(\infty)^2 = 1 - \frac{3\alpha+1}{2}\int_0^\infty s f'_*(s)^2 \mathrm{d}s. \tag{5.2}$$

and (5.1) follows. □

Remark 5.1 By using the concavity of f_*, we have $sf_*'(s) \le f_*(s)$ and

$$\int_0^\infty sf_*'(s)^2 \mathrm{d}s \le \frac{1}{2} f_*(\infty)^2.$$

Therefore (5.2) gives the following estimate:

$$\frac{2}{\sqrt{4\alpha+2}} \le f_*(\infty) \le \frac{2}{\sqrt{\alpha+1}}.$$

6 The question of uniqueness

This section deals with uniqueness questions of solutions or physical solutions of the problem $(\mathcal{P}_\alpha)$. The case $\alpha = -\frac{1}{3}$ is very interesting ; we saw in the §3 that the problem $(\mathcal{P}_{-\frac{1}{3}})$ has one and only one physical solution, given by

$$f(t) = \sqrt{6}\ \tanh \frac{t}{\sqrt{6}},$$

whereas we will prove there are an infinite number of solutions of (1.1)-(1.3).

For $\alpha \in [0, \frac{1}{3}]$, we prove that $(\mathcal{P}_\alpha)$ has a unique solution, which is the physical one, but for $\alpha > \frac{1}{3}$ we just are able to get uniqueness for the physical solution.

If $-\frac{1}{3} < \alpha < 0$, we do not know if uniqueness holds, even for physical solutions.

Theorem 6.1 *For $\alpha \ge 0$, the problem $(\mathcal{P}_\alpha)$ has one and only one physical solution.*

Proof Let f_1 and f_2 be two solutions of (1.1)-(1.4). Denote $\mu_i = f_i''(0)$ for $i = 1, 2$ and assume that $\mu_1 > \mu_2$. Let us now introduce the function $h = f_1 - f_2$. One has $h(0) = 0$, $h'(0) = 0$ and $h''(0) = \mu_1 - \mu_2 > 0$. Since $h'(\infty) = 0$, there is a point $t_0 > 0$, such that $h' > 0$ on $(0, t_0]$, $h''(t_0) = 0$ and

$$h'''(t_0) \le 0. \tag{6.1}$$

Moreover $h(t_0) > 0$. Now, using the equality $f_1''(t_0) = f_2''(t_0)$ and since $f_i'' < 0$ and $f_i' > 0$, we can write

$$h'''(t_0) = f_1'''(t_0) - f_2'''(t_0) = \alpha h'(t_0)(f_1'(t_0) + f_2'(t_0)) - \frac{\alpha+1}{2} f_1''(t_0) h(t_0)$$

which gives $h'''(t_0) > 0$ and a contradiction with (6.1). □

Theorem 6.2 *For $\alpha \in [0, \frac{1}{3}]$, the problem $(\mathcal{P}_\alpha)$ has no other solution than the physical one.*

Proof For $\alpha = 0$, this follows from Proposition 3.3. Assume now that $\alpha \in (0, \frac{1}{3})$. Let f be a solution of $(\mathcal{P}_\alpha)$ and suppose that f does not satisfies (1.4). Therefore, it follows from Proposition 3.2, that there exists $t_0 > 0$ such that $f''(t_0) = 0$, $f' < 0$ and $f'' > 0$ on (t_0, ∞). Since $f''(\infty) = 0$ there is a point t_1 such that $f'''(t_1) = 0$ and $f''''(t_1) \leq 0$. But

$$f'''' = -\frac{\alpha+1}{2} f f''' + \frac{3\alpha - 1}{2} f' f''$$

and thus we get a contradiction, since

$$f''''(t_1) = \frac{3\alpha - 1}{2} f'(t_1) f''(t_1) > 0.$$

Finally, let f be a solution of $(\mathcal{P}_{\frac{1}{3}})$. As in the Homann case, we have

$$f'''' = -\frac{2}{3} f f'''$$

in such a way that, if there is a t_0 satisfying $f'''(t_0) = 0$, then f is the solution of the initial value problem

$$\begin{cases} f'''' + \frac{2}{3} f f''' = 0 \quad \text{on} \quad (0, \infty), \\ f(t_0) = \lambda_1, \\ f'(t_0) = \lambda_2, \\ f''(t_0) = \lambda_3, \\ f'''(t_0) = 0, \end{cases}$$

and necessarily, we have

$$f(t) = \frac{\lambda_3}{2}(t - t_0)^2 + \lambda_2 (t - t_0) + \lambda_1$$

and a contradiction with $f'(\infty) = 0$. Since $f'''(0) = \frac{1}{3}$, then $f''' > 0$ and f'' is increasing. But $f''(\infty) = 0$, thus $f'' < 0$ and f' satisfies (1.4). □

We are going to prove now that we have an infinite number of solutions to $(\mathcal{P}_{-\frac{1}{3}})$. To this end, let us consider the following initial value problem:

$$\begin{cases} f'(t) + \frac{1}{6} f(t)^2 = at + 1 \\ f(0) = 0 \end{cases} \qquad (\mathcal{R}_a)$$

where $a \in \mathbb{R}$. If for some a, the problem $(\mathcal{R}_a)$ has a solution f defined on $[0, \infty)$ and satisfying $f'(\infty) = 0$, then it is a solution of $(\mathcal{P}_{-\frac{1}{3}})$.

Theorem 6.3 *For any $a \geq 0$, the problem $(\mathcal{R}_a)$ has one and only one solution f_a, defined on $[0, \infty)$ and satisfying*

$$\lim_{t \to \infty} f_a'(t) = 0.$$

It is a solution of $(\mathcal{P}_{-\frac{1}{3}})$.

Proof As we have already seen, the case $a = 0$ has an exact solution. Now, for $a > 0$, let f be the solution on $[0, T)$ of the problem $(\mathcal{R}_a)$. We have $f'(0) = 1$ and $f''(0) = a$, and since f and f' are positive and increasing as long as $f'' > 0$, we deduce from the equality

$$f'' = -\frac{ff'}{3} + a, \tag{6.2}$$

that f'' has a zero, at some $t_0 > 0$. Further, according to Lemma 3.1, we have $f'' < 0$ for $t \in (t_0, T)$. Since $f' \geq 1$ on $[0, t_0)$, and cannot vanish on (t_0, T), because if $f'(t_1) = 0$, we get from (6.2) that $f''(t_1) = a > 0$, a contradiction. Therefore $T = \infty$ and $f' > 0$ on $[0, \infty)$. Since $f'' < 0$ for $t > t_0$, the limit $f'(\infty) = c$ exists and is nonnegative. If $c > 0$, then $ff' \to \infty$ and (6.2) shows that $f'' \to -\infty$ and f' must become negative, a contradiction. Hence $c = 0$. The proof is now complete. □

7 Conclusion

For the full problem (1.1)-(1.4), we have proved that there is no solution for $\alpha < -\frac{1}{3}$, at least one solution for $-\frac{1}{3} < \alpha < 0$ and one and only one solution for $\alpha = -\frac{1}{3}$ and $\alpha \geq 0$. Our feeling is that uniqueness still holds for $-\frac{1}{3} < \alpha < 0$, but up to now, we are unable to prove this fact. Another open question for these values of α, is to know if a solution of (1.1)-(1.4) is bounded. It is easy to see that it is the case if and only if $f' \in L^2(0, \infty)$. Actually, we know that the solution of (1.1)-(1.4) constructed in the proof of Theorem 5.1 is bounded, and the bound is the one given in Remark 5.1, but if there are other solutions, we cannot be positive about boundedness.

For the problem (1.1)-(1.3), many questions are still without answer. For example, if $\alpha \in (-\frac{1}{2}, -\frac{1}{3})$ is there a solution? Numerical attempts seem to indicate that there should be (see Banks 1983 or Ingham & Brown 1986). For $\alpha > \frac{1}{3}$, we do not know if there exist solutions f for which (1.4) does not hold. Taking into account that we know for $\alpha = -\frac{1}{3}$ there are infinite number of solutions, and for $\alpha \in [0, \frac{1}{3}]$ there si no other solution than the one satisfying (1.4), it is not very easy to guess what happens in the other cases.

Acknowledgements

The authors wish to thank an anonymous referee for valuable hints and remarks, that led to simplify some proofs, and in particular rewrite the second part of the proof of Theorem 5.1, and the proof of Theorem 6.3. They also thank J.R. Ockendon for bringing some engineering literature to their attention.

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