AVERAGING METHOD FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, the method of averaging which is well known for ordinary differential equations is extended, in a natural way, to functional differential equations. Our results are formulated in both classical mathematics and nonstandard analysis.

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1 INTRODUCTION

It is well known that the method of averaging is a powerful tool of investigation of many perturbation problems in nonlinear oscillations, and some in celestial mechanics. There is a rich literature for ordinary differential equations case ([2, 7, 11, 13, 15, 21, 22] and the references cited therein). The method is also extended to functional differential equations [8, 9, 10, 18, 24, 25] of the form

$$\dot{x}(t) = \varepsilon f(t, x_t) \tag{1}$$

where $\varepsilon > 0$ is a small parameter. Under suitable conditions, solutions of (1) can be approximated by those ones of the averaged ordinary differential equation

$$\dot{y}(t) = \varepsilon f^o(\tilde{y}), \qquad \tilde{y}(\theta) = y \quad \text{for } \theta \in [-r, 0] \tag{2}$$

where

$$f^o(u) = \lim_{T \longrightarrow \infty} \frac{1}{T} \int_0^T f(\tau, u) d\tau. \tag{3}$$

Notice that, if we let $t \longmapsto t/\varepsilon$ and $x(t/\varepsilon) = z(t)$, equation (1) becomes

$$\dot{z}(t) = f(\frac{t}{\varepsilon}, z_{t,\varepsilon}) \tag{4}$$

with $z_{t,\varepsilon}(\theta) = z(t + \varepsilon\theta)$, $\theta \in [-r, 0]$, which is an equation with a small delay.

The purpose of this paper is to consider a functional differential equation in the general case, that is

$$\dot{x}(t) = f(\frac{t}{\varepsilon}, x_t) \tag{5}$$

and to show that solutions of (5) may be approximed by those one of the averaged equation

$$\dot{y}(t) = f^o(y_t) \tag{6}$$

where f^o is given in (3). Notice that (6) is a functional differential equation and not an ordinary differential equation.

Among works devoted to study the general case (5), we will cite the paper by Hale and Verduyn Lunel [12]. Without going into details, we will emphasize that, in this work, the authors introduce an extension of the method of averaging to abstract evolutionary equations in Banach spaces. In particular, they rewrite a functional differential equation as an ordinary differential equation in an infinite dimensional Banach space and proceed formally from there. Our approach is different since all the analysis is kept in the associated natural phase space.

This paper is organized as follows. In Section 2, we state closeness of solutions of the original and averaged equations on finite time intervals (Theorem 1). This result generalizes the corresponding one of [14] and then its proof is directly related to [14] (and [22]). We also investigate the long time behaviour of solutions of the original equation (Theorem 2). This is done under the assumption that the averaged equation has an exponentially stable equilibrium. For this case, the idea of the proof is the same one used for ordinary differential equations in [21]. The proofs of Theorem 1 and 2 are established within an axiomatic description of A. Robinson's *Nonstandard Analysis* (NSA) [20], namely *Internal Set Theory* (IST), proposed by E. Nelson [19]. Section 3 starts with a short tutorial on IST. Then we present the nonstandard translates (Theorem 3 and 4) in the language of IST of Theorem 1 and 2. Finally, in Section 4 we first begin with some preliminary lemmas and then give the proofs of Theorem 3 and 4.

Let $r \geq 0$ be a given constant. Throughout this paper $\mathcal{C}_o = \mathcal{C}([-r, 0], \mathbb{R}^d)$ will denote the Banach space of all continuous functions of $[-r, 0]$ into $\mathbb{R}^d$ with the norm $|\phi| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$. Even though single bars are used for norms in different spaces, no confusion should arise. Let $t_0 \in \mathbb{R}$ and $L > t_0$. If x is a continuous function defined on $[t_0 - r, L]$ and $t \in [t_0, L]$, then $x_t \in \mathcal{C}_o$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. The functional $f : \mathbb{R}_+ \times \mathcal{C}_o \longrightarrow \mathbb{R}^d$ in (5) is always assumed continuous. Let $\phi \in \mathcal{C}_o$ and assume in (5) that $x_{t_0} = \phi$. Then (5) has a solution which is denoted $x(.) = x(.; t_0, \phi)$.

2 MAIN RESULTS

In this section we state hypotheses and present the main results on averaging for functional differential equations.

First, let us assume that the following conditions are satisfied:

H1: The functional f is continuous and bounded on $\mathbb{R}_+ \times \mathcal{C}_o$.
H2: The continuity of f in $u \in \mathcal{C}_o$ is uniform with respect to $\tau = t/\varepsilon \geq 0$.
H3: For all $u \in \mathcal{C}_o$ there exists a limit

$$f^o(u) := \lim_{T \longrightarrow \infty} \frac{1}{T} \int_0^T f(\tau, u) d\tau.$$

H4: The averaged equation (6) has the uniqueness of the solutions with the prescribed initial conditions.

For any $\phi \in \mathcal{C}_0$ and any $t_0 \in \mathbb{R}$, the solution of (6) such that $y_{t_0} = \phi$ is denoted $y(.) = y(.; t_0, \phi)$ and $J = [t_0 - r, \omega)$ will denote its maximal interval of definition, $t_0 < \omega \leq +\infty$.

Remark 1 *In hypothesis H*4 *we anticipate the existence of solutions of (6). We will justify this* a posteriori*. Endeed, in Lemma* 5 *below we will show that f^o is continuous so that the existence is guaranteed.*

Under the above assumptions, we will prove a theorem on nearness of the solutions of (5) and (6) with the same initial conditions.

Theorem 1 (Averaging on Finite Time Intervals) *Let the hypotheses H1 to H4 hold. Let $\phi \in \mathcal{C}_0$ and $t_0 \in \mathbb{R}$. Let y be the solution of (6) with $y_{t_0} = \phi$. Then for any $L > t_0$, $L \in J$, and $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, any solution x of (5) with $x_{t_0} = \phi$ is defined at least on $[t_0, L]$ and the inequality $|x(t) - y(t)| < \delta$ holds for $t \in [t_0, L]$.*

One can also extend the validity of the averaging technique for all (future) time when the solution of (6) lies in the domain of exponential stability of an exponentially stable equilibrium. For this, let us first recall the concept of *exponential stability* of equilibriums.

Suppose that y_e is an equilibrium of (6), that is, $f^o(y_e) = 0$.

Definition 1 *y_e is said to be exponentially stable if there exist b, K and $\lambda > 0$ such that, for all $t_0 \in \mathbb{R}$ and all $\phi \in \mathcal{C}_o$, the solution $y(.) = y(.; t_0, \phi)$ of (6) for which $|\phi - y_e| < b$, is defined on $[t_0, \infty)$ and the inequality $|y(t) - y_e| \leq K e^{-\lambda(t - t_0)} |\phi - y_e|$ holds, for all $t \geq t_0$.*

Remark 2 *The ball $\mathcal{B}$ of center y_e and radius b where the stability is exponential will be called the domain of exponential stability of y_e.*

Now consider the following hypothesis which will be used in Theorem 2 above.

H5: The functional f is continuously differentiable with respect to the second variable.

As a next result of this section, we will prove validity of the approximation of the solutions of (5) and (6) with the same initial conditions, for all time.

Theorem 2 (Averaging for All Time) *Let the hypotheses H1, H2, H3, H4 and H5 be true. Let $\phi \in \mathcal{C}_0$ and $t_0 \in \mathbb{R}$. Let x be the solution of (5), and y be the solution of (6) with $x_{t_0} = y_{t_0} = \phi$. Assume that*

H6: *y_e is exponentially stable.*

H7: *ϕ lies in the domain of exponential stability of y_e.*

Then for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, x is defined on $[t_0, \infty)$ and the inequality $|x(t) - y(t)| < \delta$ holds for all $t \geq t_0$.

3 NONSTANDARD RESULTS

3.1 INTERNAL SET THEORY

In IST (*Internal Set Theory*) we adjoin to ordinary mathematics (say ZFC) a new undefined unary predicate symbol *st* (read standard). We call *internal*, the formulas of IST without any occurrence of the predicate *st* in them; otherwise, we call them *external*. Thus internal formulas are the formulas of ZFC. The axioms of IST are all axioms of ZFC, restricted to internal formulas (in other words, IST is an extension of ZFC), plus three others which govern the use of the new predicate. Thus *all theorems* of ZFC *remain valid* in IST. IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. Some of the theorems which are proved in IST are external and can be reformulated so that they become internal. Indeed, there is an algorithm (a well-known *reduction algorithm*) to reduce any external formula $F(x_1, \ldots, x_n)$ of IST without other free variables than $x_1, \ldots, x_n$, to an internal formula $F'(x_1, \ldots, x_n)$ with the same free variables, such that $F \equiv F'$, that is, $F \iff F'$ for all standard values of the free variables. In other words, any result which may be formalized within IST by a formula $F(x_1, \ldots, x_n)$ is equivalent to the classical property $F'(x_1, \ldots, x_n)$, provided the parameters $x_1, \ldots, x_n$ are restricted to standard values. We give the reduction of the frequently occurring formula $\forall x\ (\forall^{st} y\ A \Longrightarrow \forall^{st} z\ B)$ where A and B are internal formulas

$$\forall x\ (\forall^{st} y\ A \Longrightarrow \forall^{st} z\ B)\ \equiv\ \forall z\ \exists^{fin} y'\ \forall x\ (\forall y \in y'\ A \Longrightarrow B). \tag{7}$$

A real number x is called *infinitesimal*, denoted by $x \simeq 0$, if its absolute value $|x|$ is smaller than any standard strictly positive real number, *limited* if its absolute value $|x|$ is smaller than some standard real number, *unlimited*, denoted by $x \simeq \pm\infty$, if it is not limited, and *appreciable* if it is neither unlimited nor infinitesimal. Two real numbers x and y are *infinitely close*, denoted by $x \simeq y$, if their difference $x - y$ is infinitesimal.

For x and y in a standard metric space E, the notation $x \simeq y$ means that the distance from x to y is infinitesimal. If there exists in that space a standard x_0 such that $x \simeq x_0$, the element x is called *nearstandard* in E and the standard point x_0 is called the *standard part* of x (it is unique) and is also denoted by ${}^o x$.

We may not use external formulas in the axiom schemes of ZFC, in particular we may not use external formulas to define subsets. The notations $\{x \in \mathbb{R} : x \text{ is limited}\}$ or $\{x \in \mathbb{R} : x \simeq 0\}$ are not allowed. Moreover we can prove that

Lemma 1 *There do not exist subsets $\mathcal{L}$ and $\mathcal{I}$ of $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, x is in $\mathcal{L}$ if and only if x is limited, or x is in $\mathcal{I}$ if and only if x is infinitesimal.*

It happens sometimes in classical mathematics that a property is assumed, or proved, on a certain domain, and that afterwards it is noticed that the character of the property and the nature of the domain are incompatible. So actually the property must be valid on a large domain. In the same manner, in Nonstandard Analysis, the result of Lemma 1 is frequently used to prove that the validity of a property exceeds the domain where it was established in direct way. Suppose that we have shown that A holds for every limited x, then we know that A holds for some unlimited x, for otherwise we could let $\mathcal{L} = \{x \in \mathbb{R} : A\}$. This statement is called the *Cauchy principle.* It has the following frequently used application.

Lemma 2 (Robinson's Lemma) *If g is a real function such that $g(t) \simeq 0$ for all limited $t \geq 0$, then there exists $\flat \simeq +\infty$ such that $g(t) \simeq 0$ for all $t \in [0, \flat]$.*

Proof. Indeed, the set of all $l \in \mathbb{R}$ such that $|g(t)| < 1/l$ for all $t \in [0, l]$, contains all limited l in $\mathbb{R}$, $l \geq 1$. By the Cauchy principle it must contain some unlimited $\flat$. $\oslash$

We conclude this section with two others applications of Cauchy's principle which will be used later.

Lemma 3 *If $\mathcal{P}(.)$ is an internal property such that $\mathcal{P}(\lambda)$ holds for all appreciable real numbers $\lambda > 0$, then there exists $0 < \lambda_0 \simeq 0$ such that $\mathcal{P}(\lambda_0)$ holds.*

Lemma 4 *Let $h : I \longrightarrow \mathbb{R}$ be a function such that $h(t) \simeq 0$ for all $t \in I$. Then* $\sup\{h(t) : t \in I\} \simeq 0$.

Remark 3 *For more informations on Nonstandard Analysis and its applications, the interested reader is referred to* [1, 3, 5, 4, 6, 16, 17, 19, 20, 23] *and the references therein.*

3.2 AVERAGING RESULTS

First we give nonstandard formulations of Theorem 1 and Theorem 2. Then, by use of the reduction algorithm, we show that the reduction of Theorem 3 and Theorem 4 bellow are Theorem 1 and Theorem 2 respectively.

Theorem 3 *Let* $f : \mathbb{R} \times \mathcal{C}_o \longrightarrow \mathbb{R}^d$ *be standard. Assume that all hypotheses in Theorem 1 hold. Let* $\phi \in \mathcal{C}_o$ *and* $t_0 \in \mathbb{R}$ *be standard. Let* y *be the solution of (6) with* $y_{t_0} = \phi$*. Let* $\varepsilon > 0$ *be infinitesimal. Then for any standard* $L > t_0$*,* $L \in J$*, any solution* x *of (5) with* $x_{t_0} = \phi$ *is defined at least on* $[t_0, L]$ *and satisfies* $x(t) \simeq y(t)$ *for all* $t \in [t_0, L]$*.*

Theorem 4 *Let* $f : \mathbb{R} \times \mathcal{C}_o \longrightarrow \mathbb{R}^d$ *be standard. Let* y_e *be a standard equilibrium of* (6)*. Assume that all hypotheses in Theorem* 2 *hold. Let* $\phi \in \mathcal{C}_o$ *and* $t_0 \in \mathbb{R}$ *be standard. Let* x *be the solution of (5), and* y *be the solution of (6) with* $x_{t_0} = y_{t_0} = \phi$*. Let* $\varepsilon > 0$ *be infinitesimal. Then* x *is defined on* $[t_0, \infty)$ *and satisfies* $x(t) \simeq y(t)$ *for all* $t \geq t_0$*.*

The proofs of Theorems 3 and 4 are postponed to Section 4. Theorem 3 and Theorem 4 are external statements. Let us show that the reduction of Theorem 3 (resp. Theorem 4) is Theorem 1 (resp. Theorem 2).

Reduction of Theorem 3. Without any loss of generality, let $t_0 = 0$. Let B be the formula "If $\delta > 0$ then any solution x of (5) with $x_{t_0} = \phi$ is defined at least on $[0, L]$ and the inequality $|x(t) - y(t)| < \delta$ holds for all $t \in [0, L]$". To say that "any solution x of (5) with $x_{t_0} = \phi$ is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$" is the same as saying $\forall^{st}\delta\ B$. Then Theorem 3 asserts that

$$\forall\varepsilon\ (\forall^{st}\eta\ \varepsilon < \eta \implies \forall^{st}\delta\ B). \tag{8}$$

In this formula L is standard and ε, η and δ range over the strictly positive real numbers. By (7), formula (8) is equivalent to

$$\forall\delta\ \exists^{fin}\eta'\ \forall\varepsilon\ (\forall\eta \in \eta'\ \varepsilon < \eta \implies B). \tag{9}$$

For η' a finite set, $\forall\eta \in \eta'\ \varepsilon < \eta$ is the same as $\varepsilon < \varepsilon_0$ for $\varepsilon_0 = \min\eta'$, and so formula (9) is equivalent to

$$\forall\delta\ \exists\varepsilon_0\ \forall\varepsilon\ (\varepsilon < \varepsilon_0 \implies B).$$

This shows that for any standard $L > 0$, $L \in J$, the statement of Theorem 1 holds, thus by transfer, it holds for any $L > 0$, $L \in J$. ⊘

The reduction of Theorem 4 to Theorem 2 follows almost verbatim the reduction of Theorem 3 to Theorem 1 and is left to the reader.

4 PROOFS OF THEOREMS 3 AND 4

4.1 PRELIMINARY LEMMAS

First Part: Hereafter we are giving some results we need for the proof of Theorem 3. We assume that all assumptions in Theorem 3 hold. Let us give external formulations of conditions H1, H2 and H3 respectively:

H1': $\forall^{st}\tau \geq 0 \quad \forall^{st}u \in \mathcal{C}_o \quad \forall\tau' \geq 0 \quad \forall u' \in \mathcal{C}_o$:

$$\tau' \simeq \tau \text{ and } u' \simeq u \Longrightarrow f(\tau, u') \simeq f(\tau, u).$$

And, there exists some standard constant M such that

$$\forall^{st}\tau \geq 0 \quad \forall^{st}u \in \mathcal{C}_o : |f(\tau, u)| \leq M$$

(and by transfer the inequality holds for all $\tau \geq 0$ and $u \in \mathcal{C}_o$).

H2': $\forall^{st}u \in \mathcal{C}_o \quad \forall u' \in \mathcal{C}_o \quad \forall\tau \geq 0 : \quad u' \simeq u \Longrightarrow f(\tau, u') \simeq f(\tau, u)$.

H3': There is a standard functional $f^o : \mathcal{C}_o \longrightarrow \mathbb{R}^d$ such that

$$\forall^{st}u \in \mathcal{C}_o \quad \forall T \simeq +\infty : \quad f^o(u) \simeq \frac{1}{T}\int_0^T f(\tau, u)d\tau.$$

Lemma 5 *The functional f^o is continuous and satisfies*

$$f^o(u) \simeq \frac{1}{T}\int_0^T f(\tau, u)d\tau$$

for all $u \in \mathcal{C}_o$, u nearstandard, and all $T \simeq +\infty$.

Proof. Let $u, {}^ou \in \mathcal{C}_o$ such that ${}^ou$ is standard and $u \simeq {}^ou$. Let $\nu > 0$ be infinitesimal. By condition H3 there exists $T_0 > 0$ such that, for $T > T_0$

$$\left| f^o(u) - \frac{1}{T}\int_0^T f(\tau, u)d\tau \right| < \nu.$$

Hence for some $T \simeq +\infty$ we have

$$f^o(u) \simeq \frac{1}{T}\int_0^T f(\tau, u)d\tau.$$

By condition H2' we have $f(\tau, u) \simeq f(\tau, {}^ou)$ for $\tau \geq 0$. Therefore

$$f^o(u) \simeq \frac{1}{T}\int_0^T f(\tau, {}^ou)d\tau.$$

By condition H3' we deduce that $f^o(u) \simeq f^o({}^o u)$. Thus f^o is continuous. Moreover for $T \simeq +\infty$ we have

$$f^o(u) \simeq f^o({}^o u) \simeq \frac{1}{T}\int_0^T f(\tau, {}^o u)d\tau \simeq \frac{1}{T}\int_0^T f(\tau, u)d\tau.$$

$\oslash$

Lemma 6 *There exists* $\mu > 0$ *such that whenever* $t \geq 0$ *is limited and* $u \in \mathcal{C}_o$ *is nearstandard there exists* $\alpha > 0$ *such that* $\mu < \alpha \simeq 0$ *and*

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, u)d\tau \simeq f^o(u).$$

Proof. Let $t \geq 0$ be limited and let $u \in \mathcal{C}_o$ be nearstandard.

i) Suppose t/ε is limited. Let $S > 0$ be unlimited such that $\varepsilon S \simeq 0$. Then

$$\frac{1}{S}\int_{t/\varepsilon}^{t/\varepsilon+S} f(\tau, u)d\tau = \left(1 + \frac{t}{\varepsilon S}\right)\frac{1}{t/\varepsilon + S}\int_0^{t/\varepsilon+S} f(\tau, u)d\tau - \frac{1}{S}\int_0^{t/\varepsilon} f(\tau, u)d\tau.$$

By Lemma 5 we have

$$\frac{1}{t/\varepsilon + S}\int_0^{t/\varepsilon+S} f(\tau, u)d\tau \simeq f^o(u).$$

Since

$$\frac{1}{S}\int_0^{t/\varepsilon} f(\tau, u)d\tau \simeq 0 \qquad \text{and} \qquad t/\varepsilon S \simeq 0$$

we have

$$\frac{1}{S}\int_{t/\varepsilon}^{t/\varepsilon+S} f(\tau, u)d\tau \simeq f^o(u).$$

Then, it suffices to choose $\mu = \varepsilon$ and take $\alpha = \varepsilon S$.

ii) Suppose t/ε is unlimited. Let $S > 0$. We write

$$\begin{aligned}\frac{1}{S}\int_{t/\varepsilon}^{t/\varepsilon+S} f(\tau, u)d\tau \quad &= \quad \frac{1}{t/\varepsilon + S}\int_0^{t/\varepsilon+S} f(\tau, u)d\tau \\ &\quad + \frac{t}{\varepsilon S}\left(\frac{1}{t/\varepsilon + S}\int_0^{t/\varepsilon+S} f(\tau, u)d\tau - \frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, u)d\tau\right).\end{aligned}$$

By Lemma 5 we have

$$\frac{1}{t/\varepsilon + S}\int_0^{t/\varepsilon+S} f(\tau, u)d\tau \simeq f^o(u) \simeq \frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, u)d\tau.$$

Let us denote by

$$\eta(S) = \frac{t}{\varepsilon S}\left(\frac{1}{t/\varepsilon + S}\int_0^{t/\varepsilon+S} f(\tau, u)d\tau - \frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau, u)d\tau\right).$$

$\eta(S)$ is infinitesimal for all S such that $t/\varepsilon S$ is limited. By Lemma 2 this property holds for some S for which $t/\varepsilon S$ is unlimited. S can be choosen so that $S > 1$ and $t/\varepsilon S \simeq +\infty$. Since t is limited we have $\varepsilon S \simeq 0$. Then, it suffices to choose $\mu = \varepsilon$ and take $\alpha = \varepsilon S$. $\oslash$

Lemma 7 *Let $\phi \in \mathcal{C}_o$ be standard. Let x be a solution of* (5) *on $I = [-r, b)$ with $x_0 = \phi$, and let $L_1 > 0$ be standard such that $[0, L_1] \subset I$. Then x is S-continuous and nearstandard on $[0, L_1]$, and there exist some positive integer N_o and some infinitesimal partition $\{t_n : n = 0, ..., N_o + 1\}$ of $[0, L_1]$ such that $t_0 = 0$, $t_{N_o} < L_1 \leq t_{N_o+1}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$ and*

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f(\tau, x_{t_n}) d\tau \simeq f^o(x_{t_n}).$$

Proof. First, as f is bounded on $\mathbb{R}_+ \times \mathcal{C}_o$, it follows that x is S-continuous on $[0, L_1]$. Endeed, if $t \simeq t'$, with $t, t' \in [0, L_1]$ then $|x(t) - x(t')| \leq M|t - t'| \simeq 0$. Furthermore, for $t \in [0, L_1]$, we have $x(t) = \phi(0) + \int_0^t f(\frac{\tau}{\varepsilon}, x_\tau) d\tau$ where $\phi(0)$ is limited. This implies that x is nearstandard on $[0, L_1]$.

Note that, from what precedes, it is not difficult to deduce that x_t is nearstandard (in $\mathcal{C}_o$) for all $t \in [0, L_1]$.

Next, let $\mu > 0$ be given as in Lemma 6 and let $A_\mu = \{\lambda \in \mathbb{R} \;/\; \forall t \in [0, L_1] \; \exists \alpha \in \mathbb{R} : \mathcal{P}_\mu(t, \alpha, \lambda)\}$ where

$$\mathcal{P}_\mu(t, \alpha, \lambda) \equiv \mu < \alpha < \lambda \quad \text{and} \quad \left| \frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x_t) d\tau - f^o(x_t) \right| < \lambda.$$

By Lemma 6 the set A_μ contains all the standard real numbers $\lambda > 0$. By Lemma 3 there exists $\lambda_0 \simeq 0$ in A_μ, that is, there exists $0 < \lambda_0 \simeq 0$ such that for all $t \in [0, L_1]$ there exists $\alpha \in \mathbb{R}$ such that $\mathcal{P}_\mu(t, \alpha, \lambda_0)$ holds. By the axiom of choise there exists a function $c : [0, L_1] \longrightarrow \mathbb{R}$ such that $c(t) = \alpha$, that is, $\mathcal{P}_\mu(t, c(t), \lambda_0)$ holds for all $t \in [0, L_1]$. Since $c(t) > \mu$ for all $t \in [0, L_1]$, the conclusion of the lemma is immediate. $\oslash$

Lemma 8 *Let $\phi \in \mathcal{C}_o$ be standard. Let x be a solution of* (5) *on $I = [-r, b)$ with $x_0 = \phi$, and let $L_1 > 0$ be standard such that $[0, L_1] \subset I$. Then for all $t \in [0, L_1]$*

$$\int_0^t f(\frac{\tau}{\varepsilon}, x_\tau) d\tau \simeq \int_0^t f^o(x_\tau) d\tau.$$

Proof. By Lemma 7 there exists $\{t_n : n = 0, ..., N_o + 1\}$ such that $t_0 = 0$, $t_{N_o} < L_1 \leq t_{N_o+1}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$ and

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f(\tau, x_{t_n}) d\tau \simeq f^o(x_{t_n}). \tag{10}$$

Let $t \in [0, L_1]$ and let N be a positive integer such that $t_N < t \leq t_{N+1}$. We have

$$\begin{aligned}
\int_0^t f(\frac{\tau}{\varepsilon}, x_\tau) d\tau - \int_0^t f^o(x_\tau) d\tau &= \int_0^t \Big(f(\frac{\tau}{\varepsilon}, x_\tau) - f^o(x_\tau) \Big) d\tau \\
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \Big(f(\frac{\tau}{\varepsilon}, x_\tau) - f^o(x_\tau) \Big) d\tau \\
&\qquad + \int_{t_N}^{t} \Big(f(\frac{\tau}{\varepsilon}, x_\tau) - f^o(x_\tau) \Big) d\tau \\
&\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \Big(f(\frac{\tau}{\varepsilon}, x_\tau) - f^o(x_\tau) \Big) d\tau
\end{aligned} \tag{11}$$

since

$$\begin{aligned}
|\int_{t_N}^t \Big(f(\frac{\tau}{\varepsilon}, x_\tau) - f^o(x_\tau) \Big) d\tau| &\leq \int_{t_N}^t \Big(|f(\frac{\tau}{\varepsilon}, x_\tau)| + |f^o(x_\tau)| \Big) d\tau \\
&\leq 2M(t - t_N) \leq 2M(t_{N+1} - t_N) \leq 2M\alpha \simeq 0
\end{aligned}$$

where $\alpha = \max\{\alpha_n\} \simeq 0$ (see Lemma 4) and M is a standard bound for f (condition H1') and then for f^o too.

By Lemma 7 we have $x_\tau \simeq x_{t_n}$ for $\tau \in [t_n, t_{n+1}]$ and x_{t_n} is nearstandard, so that by condition H2' and Lemma 5 (the continuity of f^o) it follows respectively that $f(\frac{\tau}{\varepsilon}, x_\tau) = f(\frac{\tau}{\varepsilon}, x_{t_n}) + \gamma_n(\tau)$ and $f^o(x_\tau) = f^o(x_{t_n}) + \delta_n(\tau)$ with $\gamma_n(\tau) \simeq 0 \simeq \delta_n(\tau)$. Hence, from (11) it follows that

$$\begin{aligned}
\int_0^t f(\frac{\tau}{\varepsilon}, x_\tau) d\tau - \int_0^t f^o(x_\tau) d\tau &\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \Big(f(\frac{\tau}{\varepsilon}, x_{t_n}) - f^o(x_{t_n}) + \eta_n(\tau) \Big) d\tau \\
&= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \Big(f(\frac{\tau}{\varepsilon}, x_{t_n}) - f^o(x_{t_n}) \Big) d\tau \\
&\qquad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \eta_n(\tau) d\tau
\end{aligned}$$

where $\eta_n(\tau) = \gamma_n(\tau) + \delta_n(\tau)$, and therefore

$$\int_0^t f(\frac{\tau}{\varepsilon}, x_\tau) d\tau - \int_0^t f^o(x_\tau) d\tau \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \Big(f(\frac{\tau}{\varepsilon}, x_{t_n}) - f^o(x_{t_n}) \Big) d\tau$$

since $\left| \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \eta_n(\tau) d\tau \right| \leq \overline{\eta} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} d\tau = \overline{\eta}.t_N$, where $\overline{\eta} = \sup\{\eta_n : 0 \leq n \leq N-1\}$ and $\eta_n = \sup\{|\eta_n(\tau)| : t_n \leq \tau \leq t_{n+1}\}$. By Lemma 4, $\overline{\eta}$ is infinitesimal and so is $\overline{\eta}.t_N$.

Let $0 \leq n \leq N-1$. By means of (10) we have

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} \Big(f(\frac{\tau}{\varepsilon}, x_{t_n}) - f^o(x_{t_n}) \Big) d\tau &= \int_{t_n}^{t_n+\alpha_n} \Big(f(\frac{\tau}{\varepsilon}, x_{t_n}) - f^o(x_{t_n}) \Big) d\tau \\
&= \int_{t_n}^{t_n+\alpha_n} f(\frac{\tau}{\varepsilon}, x_{t_n}) d\tau - \alpha_n . f^o(x_{t_n}) \\
&= \varepsilon \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(\tau, x_{t_n}) d\tau - \alpha_n . f^o(x_{t_n}) \\
&= \alpha_n \left(\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(\tau, x_{t_n}) d\tau - f^o(x_{t_n}) \right) \\
&= \alpha_n . \beta_n \quad \text{with } \beta_n \simeq 0.
\end{aligned}$$

Therefore

$$\int_0^t f(\frac{\tau}{\varepsilon}, x_\tau) d\tau - \int_0^t f^o(x_\tau) d\tau \simeq \sum_{n=0}^{N-1} \alpha_n . \beta_n \simeq 0$$

since $| \sum_{n=0}^{N-1} \alpha_n . \beta_n | \leq \overline{\beta} \sum_{n=0}^{N-1} \alpha_n = \overline{\beta} \sum_{n=0}^{N-1} (t_{n+1} - t_n) = \overline{\beta} . t_N$, where $\overline{\beta} = \max\{ |\beta_n| : 0 \leq n \leq N-1 \}$. By Lemma 4, $\overline{\beta}$ is infinitesimal and so is $\overline{\beta} . t_N$. This completes the proof of Lemma 8. $\oslash$

Lemma 9 *Let* $\phi \in \mathcal{C}_o$ *be standard. Let* x *be a solution of* (5) *on* $I = [-r, b)$ *with* $x_0 = \phi$, *and let* $L_1 > 0$ *be standard such that* $[0, L_1] \subset I$. *Then the shadow of* x *on* $[0, L_1]$ *coincides with the solution* y *of* (6) *on this interval so that* $x(t) \simeq y(t)$ *for all* $t \in [0, L_1]$.

Proof. First, by Lemma 7, x is S-continuous and nearstandard on $[0, L_1]$.

Next, by means of Lemma 8, for $t \in [0, L_1]$, we have

$$x(t) = \phi(0) + \int_0^t f(\frac{\tau}{\varepsilon}, x_\tau) d\tau \simeq \phi(0) + \int_0^t f^o(x_\tau) d\tau.$$

If ${}^o x$ is the shadow of x on $[0, L_1]$, it is not difficult to verify that the function z defined as

$$z(t) = \begin{cases} {}^o x(t), & \text{for } t \in [0, L_1] \\ \phi(t), & \text{for } t \in [-r, 0] \end{cases}$$

is a solution of (6). By hypothesis H4 we have $z \equiv y$ on $[-r, L_1]$ so that $x(t) \simeq {}^o x(t) = z(t) = y(t)$ for $t \in [0, L_1]$. $\oslash$

Second Part: In this part, we are giving in Lemma 10 below, the external formulation of an equilibrium exponential stability definition. This result is needed for the proof of Theorem 4.

Lemma 10 *The equilibrium* y_e *of (6) is exponentially stable if and only if it admits a standard domain of exponential stability, that is, there exist standard* b, K *and* $\lambda > 0$ *such that, for all standard* $t_0 \in \mathbb{R}$ *and all standard* $\phi \in \mathcal{C}_o$*, the solution* $y(.) = y(.;t_0,\phi)$ *of (6) for which* $|\phi - y_e| < b$*, is defined on* $[t_0, \infty)$ *and the inequality* $|y(t) - y_e| \leq Ke^{-\lambda(t-t_0)}|\phi - y_e|$ *holds for all* $t \geq t_0$*.*

Proof. The conclusion of the lemma is obtained by successive use of transfer principle. ⊘

4.2 PROOF OF THEOREM 3

For notation simplicity, let $t_0 = 0$. Let $L > 0$ be standard in J. Let K be a standard tubular neighborhood of diameter ρ around $\Gamma = y([0, L])$. Let I be the maximal interval of definition of x. Define the set $A = \{L_1 \in I \cap [0, L] \ / \ x([0, L_1]) \subset K\}$. A is non empty ($0 \in A$) and bounded above by L. Let L_0 be a lower upper bound of A. There is $L_1 \in A$ such that $L_0 - \varepsilon^2 < L_1 \leq L_0$. By continuation, there is L_2, L_2 appreciable, such that x remains defined on $[0, L_1 + \varepsilon L_2]$. Likewise, by continuation, y remains defined in particular on the same interval. By Lemma 9, we have $x(t) \simeq y(t)$ for $t \in [0, L_1 + \varepsilon L_2]$. Suppose $L_1 + \varepsilon L_2 \leq L$. Then, $[0, L_1 + \varepsilon L_2] \subset I$ and $x([0, L_1 + \varepsilon L_2]) \subset K$, implie that $L_1 + \varepsilon L_2 \in A$, which is a contradiction with $L_1 + \varepsilon L_2 > L_0$. Thus $L_1 + \varepsilon L_2 > L$, that is, we have $x(t) \simeq y(t)$ for all $t \in [0, L] \subset [0, L_1 + \varepsilon L_2]$. ⊘

4.3 PROOF OF THEOREM 4

Let $t_0 = 0$. On $[-r, 0]$ we have $x(t) = y(t) = \phi(t)$ and therefore the conclusion of the theorem holds. By Theorem 3, the approximation $x(t) \simeq y(t)$ is satisfied for all $t \in [0, L]$, $L > t_0$, L standard. Let $t_1 > t_0$, t_1 standard. t_1 will be chosen convenably later.

Now, for $n = 0, 1, 2, ...$, let $I_n = [nt_1, (n+1)t_1]$. The collection $\{I_n\}_{n \geq 0}$ is a partition of the positive time axis so that $\mathbb{R}_+ = [0, \infty) = \bigcup_{n \geq 0} I_n$. On each interval I_n, $n \geq 1$, we define y_n as the solution of (6) with initial function $y_n(t) = x(t)$ for $t \in [nt_1 - r, nt_1]$. By Theorem 3, the approximation $x(t) \simeq y_n(t)$ holds for all $t \in I_n$. From the definition of exponential stability and its properties we have, for $t \geq nt_1$

$$|y(t) - y_n(t)| \leq Ke^{-\lambda(t - nt_1)} \sup_{s \in [nt_1 - r, nt_1]} |y(s) - y_n(s)| \tag{12}$$

where K and λ are positive and standard.

Using the triangle inequality, we have, for $s \in [nt_1 - r, nt_1]$

$$|y(s) - y_n(s)| \leq |y(s) - y_{n-1}(s)| + |y_n(s) - y_{n-1}(s)|. \tag{13}$$

However, by Theorem 3, we have $y_n(s) = x(s) \simeq y_{n-1}(s)$ for all $s \in [nt_1 - r, nt_1]$ and then, by Lemma 4, $\max\limits_{n\geq 0} \sup\limits_{s\in[nt_1-r,nt_1]} |y_n(s) - y_{n-1}(s)| \leq \alpha \simeq 0$.

Take $t_1 \geq r$. From (12) and (13) it follows that, for $t \geq nt_1$

$$|y(t) - y_n(t)| \leq Ke^{-\lambda(t-nt_1)} \left(\sup_{s\in[nt_1-r,nt_1]} |y(s) - y_{n-1}(s)| + \alpha \right) \tag{14}$$

so that

$$\begin{aligned} \sup_{s\in[(n+1)t_1-r,(n+1)t_1]} |y(s) - y_n(s)| &\leq K \sup_{s\in[(n+1)t_1-r,(n+1)t_1]} e^{-\lambda(s-nt_1)} \times \\ &\quad \times \left(\sup_{s\in[nt_1-r,nt_1]} |y(s) - y_{n-1}(s)| + \alpha \right) \\ &= Ke^{-\lambda(t_1-r)} \left(\sup_{s\in[nt_1-r,nt_1]} |y(s) - y_{n-1}(s)| + \alpha \right), \end{aligned}$$

or equivalently

$$|y - y_n|_n \leq Ke^{-\lambda(t_1-r)} \left(|y - y_{n-1}|_{n-1} + \alpha \right), \qquad n = 1, 2, ...$$

where $|y - y_n|_n := \sup\limits_{s\in[(n+1)t_1-r,(n+1)t_1]} |y(s) - y_n(s)|$.

Suppose $K > 1$ and choose t_1 such that $Ke^{-\lambda(t_1-r)} < 1$. Since $|y - y_0|_0 = 0$ we deduce that

$$|y - y_n|_n \leq \frac{Ke^{-\lambda(t_1-r)}}{1 - Ke^{-\lambda(t_1-r)}} \alpha.$$

Return now to inequality (14). For $t \in I_n$, $n \geq 0$, we have

$$|y(t) - y_n(t)| \leq Ke^{-\lambda(t-nt_1)} \left(\frac{Ke^{-\lambda(t_1-r)}}{1 - Ke^{-\lambda(t_1-r)}} + 1 \right) \alpha \leq \frac{K\alpha}{1 - Ke^{-\lambda(t_1-r)}}.$$

That is, $y(t) \simeq y_n(t)$ on I_n.

Thus, for $t \in I_n$

$$x(t) \simeq y_n(t) \quad \text{and} \quad y(t) \simeq y_n(t) \quad \Longrightarrow \quad x(t) \simeq y(t).$$

As n is chosen arbitrarily, this completes the proof. ⊘

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