Lie-admissible algebras and Operads

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Abstract

A Lie-admissible algebra gives a Lie algebra by anticommutativity. In this work we describe remarkable types of Lie-admissible algebras such as Vinberg algebras, pre-Lie algebras or Lie algebras. We compute the corresponding binary quadratic operads and study their duality. Considering Lie algebras as Lie-admissible algebras, we can define for each Lie algebra a cohomology with values in a Lie-admissible module. This permits to study some deformations of Lie algebras, in particular classes of Lie-admissible algebras such as Vinberg algebras or pre-Lie algebras.

1 Lie-admissible algebras

1.1 Definition

Let $\mathcal{A} = (A, \mu)$ be a finite dimensional algebra over a commutative field $\mathbb{K}$ of characteristic zero. In this notation, μ is the law of $\mathcal{A}$, that is a linear homomorphism

$$\mu : A \otimes A \rightarrow A$$

on the vector space A.

We denote by $a_\mu : A^{\otimes 3} \rightarrow A$ the associator of the law μ :

$$a_\mu (X_1, X_2, X_3) = \mu(\mu(X_1, X_2), X_3) - \mu(X_1, \mu(X_2, X_3)).$$

Let Σ_n be the symmetric group of degree n. For every $\sigma \in \Sigma_3$, we put

$$\sigma(X_1, X_2, X_3) = \left(X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, X_{\sigma^{-1}(3)}\right).$$

Definition 1.1 *The algebra $\mathcal{A} = (A, \mu)$ is called Lie-admissible if the law μ satisfies*

$$\sum_{\sigma \in \Sigma_3} (-1)^{\varepsilon(\sigma)} a_\mu \circ \sigma = 0. \qquad (*)$$

This definition ([1]) is equivalent to saying that the mapping $[,]_\mu : A \otimes A \rightarrow A$ defined by $[X, Y] = \mu(X, Y) - \mu(Y, X)$ is a Lie bracket. We will denote by $\mathcal{A}_L$ the corresponding Lie algebra.

1.2 Examples

1. Every associative algebra (not necessarily unitary) is a Lie-admissible algebra.
2. An algebra $\mathcal{A} = (A, \mu)$ is a Vinberg algebra ([13]) (also called left symmetric algebra ([9])) if its law satisfies

$$\mu(X, \mu(Y, Z)) - \mu(Y, \mu(X, Z)) = \mu(\mu(X, Y), Z) - \mu(\mu(Y, X), Z).$$

It is also a Lie-admissible algebra.
3. Of course a Lie algebra law is a law of Lie-admissible algebra, the Jacobi conditions implying $(*)$.
4. A pre-Lie algebra ([6]) (sometimes called right symmetric algebra ([5])) is defined by a law μ such that

$$\mu(\mu(X, Y), Z) - \mu(X, \mu(Y, Z)) = \mu(\mu(X, Z), Y) - \mu(X, \mu(Z, Y)).$$

It is a Lie-admissible algebra. Let us note that the opposite of a pre-Lie algebra is a Vinberg algebra.

Remarks. Every associative algebra is a Vinberg algebra and a pre-Lie algebra. A Vinberg algebra is a pre-Lie algebra if and only if all the associators are invariant under Σ_3

$$a_\mu \circ \sigma = a_\mu \circ \tau$$

for every τ and σ in Σ_3.

A Lie algebra is associative (respectively Vinberg, respectively pre-Lie algebra) if and only if it is 2-step nilpotent.

1.3 Geometrical interpretation

Let $\mathfrak{g}$ be a real Lie algebra equipped with an affine structure. The associated covariant derivative $\bigtriangledown$ satisfies

$$\begin{cases} \bigtriangledown_X Y - \bigtriangledown_Y X - [X, Y] = 0 \\ \bigtriangledown_X \bigtriangledown_Y - \bigtriangledown_Y \bigtriangledown_X = \bigtriangledown_{[X,Y]} \end{cases}$$

Then the product $\mu(X, Y) = \bigtriangledown_X Y$ endows the vector space $\mathfrak{g}$ with a Vinberg algebra structure. As we have $[X, Y] = \mu(X, Y) - \mu(Y, X)$, every Lie algebra provided with an affine structure is associated to a Lie-admissible algebra which is in this case a Vinberg algebra. More generally, every Lie algebra is associated to a Lie-admissible algebra. In fact it is sufficient to consider Levi Civita connections (which always exist). As the torsion vanishes, the covariant derivative satisfies the first of the previous equations and $\mu(X, Y) = \bigtriangledown_X Y$ is a Lie-admissible algebra such that $[X, Y] = [X, Y]_\mu$.

We may therefore consider the set of Lie-admissible algebras as the set of invariant linear torsion-free connections on Lie algebras. In fact, if μ is a law of admissible algebra, putting $\bigtriangledown_X Y = \mu(X, Y)$ then $\bigtriangledown$ defines a linear connection if the Bianchi identities are satisfied. Let us denote $R(X, Y)$ the curvature tensor corresponding to $\bigtriangledown$. As μ is an Lie-admissible law, we have $R(X, Y).Z +$

$R(Y,Z).X + R(Z,X).Y = 0$ and the first identity is satisfied. Considering the second one we can prove that

$$(\bigtriangledown_X R)(Y,Z) + (\bigtriangledown_Y R)(Z,X) + (\bigtriangledown_Z R)(X,Y) = 0.$$

Using the relations $(\bigtriangledown_X R)(Y,Z) = \bigtriangledown_X(R(Y,Z)) - R(\bigtriangledown_X Y, Z) - R(Y, \bigtriangledown_X Z)$, and $\bigtriangledown_{[X,Y]} = \bigtriangledown_{\bigtriangledown_X Y} - \bigtriangledown_{\bigtriangledown_Y X}$, we deduce the second identity.

1.4 Actions of the symmetric group $\sum_3$

Definition 1.2 *Let G be a subgroup of $\sum_3$. We say that the algebra law μ is G-associative if it is invariant by G, that is,*

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} a_\mu \circ \sigma = 0.$$

The subgroups of $\sum_3$ are well known. We have $G_1 = \{Id\}, G_2 = \{Id, \tau_{12}\}, G_3 = \{Id, \tau_{23}\}, G_4 = \{Id, \tau_{13}\}, G_5 = \{Id, (231), (312)\} = A_3$ (the alternating group), $G_6 = \sum_3$ where τ_{ij} is the transposition between i and j and (231) a 3-cycle. We may distinguish the following types of Lie-admissible algebras :

1. If μ is G_1-associative then μ is an associative law.
2. If μ is G_2-associative then μ is a law of Vinberg algebra. If A is finite-dimensional, the associated Lie algebra is provided with an affine structure.
3. If μ is G_3-associative then μ is a pre-Lie algebra.
4. If μ is G_4-associative then μ satisfies

$$(X.Y).Z - X.(Y.Z) = (Z.Y).X - Z.(Y.X)$$

5. If μ is G_5-associative then μ satisfies the generalized Jacobi condition :

$$(X.Y).Z + (Y.Z).X + (Z.X).Y = X.(Y.Z) + Y.(Z.X) + Z.(X.Y)$$

Moreover if the law is antisymmetric, then it is a law of Lie algebra.

6. If μ is G_6-associative then μ is a Lie-admissible law.

Example. Let us consider the vector space $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ of real infinitely derivable functions with values in $\mathbb{R}$. We can endow this space with the following algebra structures :

1. $\mu_1(f,g) = f.g$ and $(\mathcal{C}^\infty(\mathbb{R},\mathbb{R}),\mu_1)$ is associative (type 1)
2. $\mu_2(f,g) = f.g'$ and $(\mathcal{C}^\infty(\mathbb{R},\mathbb{R}),\mu_2)$ is a Vinberg algebra (type 2)
3. $\mu_3(f,g) = f'.g$ and $(\mathcal{C}^\infty(\mathbb{R},\mathbb{R}),\mu_3)$ is a pre-Lie algebra (type 3)
4. $\mu_4(f,g) = f'.g + f.g'$ and $(\mathcal{C}^\infty(\mathbb{R},\mathbb{R}),\mu_4)$ is invariant by G_4 (type 4).

Suppose that μ is invariant by G_5. If this law is also antisymmetric then it is a Lie algebra law. This shows that the definition of G_5-invariance gives another generalization of "non-commutative" Lie algebras than the notion of Leibniz algebras.

1.5 G-cogebras

In this section we introduce the notion of cogebra dualizing the G_i-associative algebras. This leads to present the definition in the category $Vect_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces.

Definition 1.3 *A G_i-associative algebra is a pair (A, μ) where A is a vector space and $\mu : A \otimes A :\longrightarrow A$ a linear mapping satisfying the following axiom (G_i-ass):*
The square

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{(\mu \otimes Id)_{G_i}} & A \otimes A \\ {\scriptstyle (Id \otimes \mu)_{G_i}} \downarrow & & \downarrow {\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

commutes, where $(Id \otimes \mu)_{G_i}$ is the linear mapping defined by :

$$(Id \otimes \mu)_{G_i} = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} (Id \otimes \mu) \circ \sigma.$$

Let Δ be a comultiplication on a vector space C:

$$\Delta : C \longrightarrow C \otimes C.$$

We define the bilinear mapping $G_i \circ (\Delta \otimes Id)$ by

$$G_i \circ (\Delta \otimes Id) = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma \circ (\Delta \otimes Id).$$

Definition 1.4 *A G_i-cogebra is a vector space C equipped with a comultiplication $\Delta : C \longrightarrow C \otimes C$ such that the following diagram is commutative :*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta} \downarrow & & \downarrow {\scriptstyle G_i \circ (Id \otimes \Delta)} \\ C \otimes C & \xrightarrow{G_i \circ (\Delta \otimes Id)} & C \otimes C \otimes C. \end{array}$$

The next results relate G_i-algebras and G_i-cogebras.

Proposition 1.1 *The dual space of a G_i-cogebra is a G_j-associative algebra .*

Proof. Let (C, Δ) a G_i-cogebra. Let us consider the maps $\lambda : C^* \otimes C^* \rightarrow (C \otimes C)^*$ given by $\lambda(f \otimes g)(v \otimes u) = f(u) \otimes g(v)$ and $\tau : C^* \otimes C^* \rightarrow C^* \otimes C^*$ defined by $\tau(u \otimes v) = v \otimes u$. Let us consider the law $\mu = \Delta^* \circ \lambda \circ \tau$. It provides the dual space C^* with a G_j-associative algebra. In fact we have

$$\mu(f_1 \otimes f_2)(x) = \Delta^* \circ \lambda(f_2 \otimes f_1)(x) = \lambda(f_2 \otimes f_1)(\Delta(x))$$

for all $f_1, f_2 \in C^*, x \in C$. We denote $X \otimes Y = \Delta(x), X_1 \otimes X_2 = \Delta(X)$ and $Y_1 \otimes Y_2 = \Delta(Y)$ (we use the sigma notation and we forget the sigma). This gives

$$\mu(f_1 \otimes f_2)(x) = f_1(X)f_2(Y).$$

The associator of μ is written :

$$\mu(\mu(f_1 \otimes f_2) \otimes f_3)(x) - \mu(f_1 \otimes \mu(f_2 \otimes f_3))(x) = (f_1 \otimes f_2 \otimes f_3)(X_1 \otimes X_2 \otimes Y - X \otimes Y_1 \otimes Y_2)$$

and the property of the comultiplication Δ is equivalent to

$$\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma \circ (\Delta \otimes Id)(\Delta(x)) = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma \circ (Id \otimes \Delta)(\Delta(x))$$

that is

$$\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma(\Delta(X) \otimes Y) = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma(X \otimes \Delta(Y))$$

or

$$\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma(X_1 \otimes X_2 \otimes Y - X \otimes Y_1 \otimes Y_2) = 0.$$

This relation proves that the associator of μ satisfies the same relation.

Proposition 1.2 *The dual vector space of a finite dimensional G_i-associative algebra has a G_i-cogebra structure.*

Proof. Let A a finite dimensional G_i-associative algebra and let $\{e_i, i = 1, ..., n\}$ be a basis of A. If $\{f_i\}$ is the dual basis then $\{f_i \otimes f_j\}$ is a basis of $A^* \otimes A^*$. Let us put

$$\Delta(f) = \sum_{i,j} f(\mu(e_i \otimes e_j)) f_i \otimes f_j.$$

In particular

$$\Delta(f_k) = \sum_{i,j} C_{ij}^k f_i \otimes f_j$$

where C_{ij} are the structure constants of μ related to the basis $\{e_i\}$. Then Δ is the comultiplication of a G_i-cogebra.

2 Lie-Admissible operads

2.1 Binary quadratic operads

Let $\mathbb{K}[\Sigma_n]$ be the $\mathbb{K}$-group algebra of the symmetric group Σ_n. An operad $\mathcal{P}$ is defined by a sequence of $\mathbb{K}$-vector spaces $\mathcal{P}(n)$, $n \geq 1$ such that $\mathcal{P}(n)$ is a right module over $\mathbb{K}[\sum_n]$ with composition maps

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1) \qquad i = 1, ..., n$$

satisfying some "associative" properties, the May Axioms ([11], [12]).

Any $\mathbb{K}[\Sigma_2]$-module E generates a free operad noted $\mathcal{F}(E)$ satisfying $\mathcal{F}(E)(1) = \mathbb{K}$, $\mathcal{F}(E)(2) = E$. In particular if $E = \mathbb{K}[\Sigma_2]$, the free module $\mathcal{F}(E)(n)$ admits as a basis the "parenthized products" of n variables indexed by $\{1, 2, ..., n\}$. For instance a basis of $\mathcal{F}(E)(2)$ is given by $(x_1.x_2)$ and $(x_2.x_1)$, and a basis of $\mathcal{F}(E)(3)$ is given by

$$\{((x_i.x_j).x_k)\,,(x_i.(x_j.x_k).)\,, i \neq j \neq k \neq i, i, j, k \in \{1, 2, 3\}\}\,.$$

Let E be a $\mathbb{K}[\Sigma_2]$-module and R a $\mathbb{K}[\Sigma_3]$-submodule of $\mathcal{F}(E)(3)$. We denote $\mathcal{R}$ the ideal generated by R, that is the intersection of all the ideals $\mathcal{I}$ of $\mathcal{F}(E)$ such that $\mathcal{I}(1) = 0, \mathcal{I}(2) = 0$ and $\mathcal{I}(3) = R$.

We call binary quadratic operad generated by E and R the operad $\mathcal{P}(\mathbb{K}, E, R)$, also denoted $\mathcal{F}(E)/\mathcal{R}$ and defined by

$$\mathcal{P}(\mathbb{K}, E, R)(n) = (\mathcal{F}(E)/\mathcal{R})(n) = \frac{\mathcal{F}(E)(n)}{\mathcal{R}(n)}$$

Thus an operad $\mathcal{P}$ is binary quadratic operad if and only if there exists a $\mathbb{K}[\Sigma_2]$-module E and a $\mathbb{K}[\Sigma_3]$-submodule R of $\mathcal{F}(E)(3)$ such that $\mathcal{P} \simeq \mathcal{F}(E)/\mathcal{R}$.

Example. The associative operad $\mathcal{A}ss$, the Lie operad $\mathcal{L}ie$, the Leibniz operad $\mathcal{L}eib$ are binary quadratic operads ([7],[10]).

2.2 Lie-Admissible operads

Let $\mathcal{F}(E)$ be the free operad generated by $E = \mathbb{K}[\Sigma_2]$. Consider the $\mathbb{K}[\Sigma_3]$-submodule R generated by the vector

$$\begin{aligned} u \;=\; & x_1.(x_2.x_3) + x_2.(x_3.x_1) + x_3.(x_1.x_2) - x_2.(x_1.x_3) - x_3.(x_2.x_1) \\ & - x_1.(x_3.x_2) - (x_1.x_2).x_3 - (x_2.x_3).x_1 - (x_3.x_1).x_2 + (x_2.x_1).x_3 \\ & + (x_3.x_2).x_1 + (x_1.x_3).x_2 \end{aligned}$$

The Lie-Admissible operad, denoted $\mathcal{L}ieAdm$ is the binary quadratic operad defined by

$$\mathcal{L}ieAdm = \mathcal{F}(E)/\mathcal{R}$$

For each of the above six types of Lie-admissible algebras there exists the corresponding operad. We obtain the following binary quadratic operads :

1. $\mathcal{A}ss$ corresponding to associative algebras.

2. $\mathcal{V}inb$ corresponding to Vinberg algebras. Here the $\mathbb{K}[\sum_3]$-submodule is generated by the vectors

$$x_1.(x_2.x_3) - (x_1.x_2).x_3 - x_2.(x_1.x_3) + (x_2.x_1).x_3.$$

3. $\mathcal{P}re\mathcal{L}ie$ corresponding to pre-Lie algebras (G_3-associative). The vectors which generate the ideal R are :

$$x_1.(x_2.x_3) - (x_1.x_2).x_3 - x_1.(x_3.x_2) + (x_1.x_3).x_2.$$

This operad had appeared recently (see for example ([4])). It is clear that the operads $\mathcal{P}reLie$ and $\mathcal{V}inb$ are isomorphic.

4. $G_4 - \mathcal{A}ss$ corresponding to G_4-associative algebras. The ideal R is generated by

$$x_1.(x_2.x_3) - (x_1.x_2).x_3 - x_3.(x_2.x_1) + (x_3.x_2).x_1.$$

5. $G_5 - \mathcal{A}ss$ corresponding to G_5-associative algebras. R is generated by

$$x_1.(x_2.x_3) - (x_1.x_2).x_3 + x_2.(x_3.x_1) - (x_2.x_3).x_1 + x_3.(x_1.x_2) - (x_3.x_1).x_2.$$

6. $\mathcal{L}ieAdm$. This operad has already been described above

2.3 The composition product of $\mathcal{L}ieAdm$

Let $\circ_i$ the fundamental Gerstenhaber product defined in [6]. It permits to describe the composition product of the operad $\mathcal{A}dm$ as well as for other operads defined above. For every $v^n \in \mathcal{L}ieAdm(n)$ and $w^m \in \mathcal{L}ieAdm(m)$ we put

$$v^n \odot w^m = \sum_{i=1,\dots,n} \sum_{\sigma \in \sum_{n+m-1}} (-1)^{\epsilon(\sigma)} \sigma(v^n \circ_i w^m).$$

As we can define a composition product for each operad we will index each product with the index of the subgroup of $\sum_3$ which characterizes the operad. Thus $\odot_1$ correspond to the composition product of $\mathcal{A}ss$, $\odot_2$ for $\mathcal{V}inb$ etc and $\odot_6$ for $\mathcal{L}ieAdm$. Let us note that $\odot_1$ corresponds to $\circ$ in [6]. We will study with more details these product later in the section 5.

2.4 The dual operad $\mathcal{L}ieAdm^!$

Let us consider the binary quadratic operad $\mathcal{P}(\mathbb{K}, E, R)$. Then the dual binary quadratic operad is definied by

$$\mathcal{P}^! = \mathcal{P}(\mathbb{K}, E^\vee, R^\perp)$$

where $E^\vee$ is the dual of E tensored by the signum representation of Σ_n and $R^\perp$ the orthogonal complement to R in $\mathcal{F}(E^\vee)(3) = \mathcal{F}(E)(3)^\vee$. This identification is nontrivial and uses a pairing discussed later on.
Before studying the dual operads of each Lie-Admissible operad defined above, let us introduce some classes of associative algebras.

Definition 2.1 *An associative algebra A is called 3-order abelian if we have*

$$X_1 X_2 X_3 = X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)}$$

for all $\sigma \in \Sigma_3$ and for all $X_i \in A$.

Unitary 3-order abelian algebras are the commutative algebras. But there exists non commutative 3-order abelian algebras. For example, the five dimensional associative algebra defined by

$$e_1^2 = e_2, \quad e_1^3 = e_3, \quad e_1 e_4 = e_5, \quad e_4 e_1 = e_3 + e_5, \quad e_4^2 = e_3.$$

If A is a 3-order abelian algebra, then the subalgebra $\mathcal{D}(A)$ generated by the product xy is abelian. Then A is an extension

$$0 \rightarrow V \rightarrow A \rightarrow A_1 \rightarrow 0$$

where A_1 is abelian and V satisfying $vx = xv$ for all $x \in A_1$ and $v \in V$. In this case, the corresponding Lie algebra is 2-step nilpotent. In fact, as we have $abc = bac$, therefore $[a, b]c = 0$. We have also $c[a, b] = 0$ thus $[[a, b], c] = 0$. Return to the geometrical interpretation. Such a Lie algebra is provided with an affine connection with trivial torsion and curvature. Moreover the connection satisfies

$$\nabla_X = 0$$

for all X in the center and

$$\nabla_X \nabla_Y = \nabla_Y \nabla_X$$

for any generators X and Y.

Let us consider now the scalar product on $\mathcal{F}(E)(3)$ defined by

$$< i(jk), i(jk) >= sgn\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

$$< (ij)k, (ij)k >= -sgn\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

Let R be the $\mathbb{K}[\Sigma_3]$-submodule which determines the $\mathcal{L}ieAdm$ operad. The annihilator $R^{\perp}$ respect to this scalar product is of dimension 11. Let R' be the $\mathbb{K}[\Sigma_3]$-submodule of $\mathcal{F}(E)(3)$ generated by the vectors

$$(x_1x_2)x_3 - x_1(x_2x_3),$$
$$(x_1x_2)x_3 - (x_1x_3)x_2,$$
$$(x_1x_2)x_3 - (x_2x_1)x_3.$$

Then $\dim R' = 11$ and $< u, v >= 0$ for all $v \in R'$ where u is the vector which generates R. This implies $R' \simeq R^{\perp}$ and $(\mathcal{F}(E)/\mathcal{R})^!$ is by definition the quadratic operad $\mathcal{F}(E)/\mathcal{R}^{\perp}$.

Proposition 2.1 *The dual operad of $\mathcal{L}ieAdm$ is the binary quadratic operad whose corresponding algebras are associative and satisfy*

$$abc = acb = bac$$

that is they are 3-order abelian.

Return to the remaining classes of Lie-admissible algebras. One knows that $\mathcal{A}ss^! = \mathcal{A}ss$ ([7]) and $\mathcal{P}reLie^! = \mathcal{P}erm$ ([4]). Recall that the last operad describes associative algebras satisfying also :

$$abc = acb$$

Proposition 2.2 *The dual operads of $\mathcal{V}inb$, $G_4 - \mathcal{A}ss$, $G_5 - \mathcal{A}ss$ are quadratic operads whose corresponding algebras are associative algebras satisfying respectively :*
- for $\mathcal{V}inb^!$: $abc = bac$
- for $G_4 - \mathcal{A}ss^!$: $abc = cba$
- for $G_5 - \mathcal{A}ss^!$: $abc = bca = cab$.

Sketch of proof. $R_2^\perp$ is the $\mathbb{K}[\sum_3]$-sub-module of $\mathcal{F}(E)(3)$ generated by the vectors

$$(x_1.(x_2.x_3) - (x_1.x_2).x_3),\ (x_1.(x_2.x_3) - x_2.(x_1.x_3))$$

for all $x_1, x_2, x_3 \in E$.
Similary

$$R_4^\perp = \langle (x_1.(x_2.x_3) - (x_1.x_2).x_3), (x_1.(x_2.x_3) - x_3.(x_2.x_1)) \rangle$$
$$R_5^\perp = \langle (x_1.(x_2.x_3) - (x_1.x_2).x_3), (x_1.(x_2.x_3) - x_2.(x_3.x_1)),$$
$$(x_1.(x_2.x_3) - x_3.(x_1.x_2)) \rangle$$

and $\dim R_4^\perp = 9$, $\dim R_5^\perp = 10$. The rest follows from the definition of the quadratic dual.

2.5 Koszul duality

Recall that a quadratic operad $\mathcal{P}$ is called Koszul if for every $\mathcal{P}$-free algebra $F_{\mathcal{P}}(V)$ one has $H_i^{\mathcal{P}}(F_{\mathcal{P}}(V)) = 0, \quad i > 0$.

Proposition 2.3 *The operads $\mathcal{A}ss$, $\mathcal{V}inb$, $\mathcal{P}reLie$ are Koszul operads. The operads $G_4 - \mathcal{A}ss$ and $G_5 - \mathcal{A}ss$ are not Koszul operads.*

Proof. It follows from [G.K] and [G] thqt the operads $\mathcal{A}ss$, $\mathcal{P}reLie$ are Koszul operads. Considering the relations between $\mathcal{P}reLie$ and $\mathcal{V}inb$, these two operads are basically equivalent. For the two others, we will show that there are not Koszul inspecting their Poincare series and using the Ginzburg-Kapranov criterium. These series are defined for a general operad $\mathcal{P}$ by

$$g_{\mathcal{P}}(x) := \sum_{i=1}^{\infty} (-1)^n dim \mathcal{P}(n) \frac{x^n}{n!},$$

The Poincare series for a Koszul operad and for its dual operad are connected by the functional equation

$$g_{\mathcal{P}}(g_{\mathcal{P}^!}(x)) = x.$$

But we have

$$g_{\mathcal{G}_4-\mathcal{A}ss}(x) = -x+x^2-\frac{3}{2}x^3+\frac{59}{4!}x^4+... \quad , \quad g_{\mathcal{G}_4-\mathcal{A}ss^!}(x) = -x+x^2-\frac{1}{2}x^3-\frac{1}{4}x^4+...$$

$$g_{\mathcal{G}_5-\mathcal{A}ss}(x) = -x+x^2-\frac{10}{3!}x^3+\frac{39}{4!}x^4+... \quad , \quad g_{\mathcal{G}_5-\mathcal{A}ss^!}(x) = -x+x^2-\frac{1}{3}x^3+\frac{1}{12}x^4+...$$

These series do not satisfy the above functional equation. This finishes the proof of the proposition.

3 The categories G_i-ASS

3.1 Functorial correspondence

Let LIE and $LIE-AD$ be the categories of Lie algebras and Lie-admissible algebras over $\mathbb{K}$. The correspondence

$$T : LIE-AD \longrightarrow LIE$$

is a functor whose dual functor corresponds to the application

$$T^* : LIE \longrightarrow LIE-AD$$

with $T^*(\mu) = \frac{1}{2}\mu$ (we identify a Lie algebra with its law).

3.2 The functor $A \otimes -$

It is well known that the category ASS of associative algebras on $\mathbb{K}$ is a tensorial category. The categories $LIE-AD$ and G_i-ASS for $i \neq 1$ do not have this property. In this section we will prove that the functor $A \otimes -$ determines a duality between the algebras over the operad $G_i-\mathcal{A}ss$ and algebras on the dual operad $G_i-\mathcal{A}ss^!$.

Theorem 3.1 *Let A be a G_i-associative algebra. Then $A \otimes -$ is a covariant functor*

$$A \otimes - : (G_i-ASS)^! \longrightarrow G_i-ASS$$

where $(G_i-ASS)^!$ is the category of algebras corresponding to the algebras over the dual operad $G_i-\mathcal{A}ss^!$.

Proof. Let B be an associative algebra. Then $A \otimes B$ is an algebra for the classical product $(a_1 \otimes b_1).(a_2 \otimes b_2) = a_1.a_2 \otimes b_1.b_2$. This product is in $G_i-\mathcal{A}ss$ if and only if B is an algebra on $G_i-\mathcal{A}ss^!$-algebra. In fact

$$((a_1 \otimes b_1).(a_2 \otimes b_2)).(a_3 \otimes b_3) = (a_1.a_2).a_3 \otimes (b_1.b_2).b_3 = (a_1.a_2).a_3 \otimes b_1.b_2.b_3$$

because B is associative. Let μ is the law of the algebra $A \otimes B$ and put $X_i = a_i \otimes b_i$.

First, let us prove the theorem for the category $G_2 - ASS$. In this case A is a Vinberg algebra. Then
$\mu(X_1, \mu(X_2, X_3)) - \mu(X_2, \mu(X_1, X_3)) - \mu(\mu(X_1, X_2), X_3)) + \mu(\mu(X_2, X_1), X_3))$
$= a_1.(a_2.a_3) \otimes b_1.b_2.b_3 - a_2.(a_1.a_3) \otimes b_2.b_1.b_3 - (a_1.a_2).a_3 \otimes b_1.b_2.b_3$
$+(a_2.a_1).a_3 \otimes b_2.b_1.b_3$
$= (a_1.(a_2.a_3) - a_2.(a_1.a_3) - (a_1.a_2).a_3 + (a_2.a_1).a_3) \otimes b_1.b_2.b_3$ because B is in the category $G_2 - ASS^!$,
$= 0 \otimes b_1.b_2.b_3$ because A is a Vinberg algebra
$= 0$.
Then $A \otimes B$ also is a Vinberg algebra.
The demonstration is the same for the other $(G_i - \mathcal{A}ss)$-algebras, taking the adapted relation (i.e for $i = 3$, we consider $\mu(\mu(X,Y),Z) - \mu(X, \mu(Y,Z)) = \mu(\mu(X,Z),Y) - \mu(X, \mu(Z,Y))$, etc...).

Applications. This theorem enables one to construct interesting classes of Vinberg algebras and to give new examples of Lie algebras equipped with affine structure. For example suppose $dimA = dimB = 2$. Then :
1. A is commutative and isomorphic to

$$A_1 : \begin{cases} X_1.X_1 = X_1 \\ X_1.X_2 = X_2.X_1 = X_2 \\ X_2.X_2 = X_2 \end{cases} \qquad A_2 : \begin{cases} X_1.X_1 = X_1 \\ X_1.X_2 = X_2.X_1 = X_2 \\ X_2.X_2 = 0 \end{cases}$$

$$A_3 : \begin{cases} X_1.X_1 = X_1 \\ X_1.X_2 = X_2.X_1 = X_2 \\ X_2.X_2 = -X_1 \end{cases} \qquad A_4 : \{X_1.X_1 = X_2$$

$$A_5 : \{ \; X_1.X_1 = X_1 \qquad A_6 : \{ \; X_i.X_j = 0$$

2. A is non commutative and isomorphic to

$$A_7 : \begin{cases} X_1.X_1 = \frac{b^2+2e}{e} X_1 - b\frac{b^2+e}{e^2} X_2 \\ X_1.X_2 = bX_1 - \frac{b^2-e}{e} X_2 \\ X_2.X_1 = bX_1 - \frac{b^2}{e} X_2 \\ X_2.X_2 = eX_1 - bX_2 \end{cases} \qquad A_8 : \begin{cases} X_1.X_1 = aX_1 + cX_2 \\ X_1.X_2 = X_2 \\ X_2.X_1 = X_2.X_2 = 0 \end{cases}$$

$$A_9 : \begin{cases} X_1.X_1 = aX_1 \\ X_1.X_2 = (a+1)X_2 \\ X_2.X_1 = aX_2 \\ X_2.X_2 = 0 \end{cases}$$

In this case the Lie algebra associated to A is the two-dimensional solvable abelian Lie algebra.

Let us classify $Vinb^!$ algebras of dimension 2.

1. B is commutative and isomorphic to A_i, $i = 1, .., 7$.
2. B is non commutative and isomorphic to

$$B_7 : \begin{cases} e_1.e_1 = e_1 \\ e_1.e_2 = e_2 \\ e_2.e_1 = e_2.e_2 = 0 \end{cases}$$

If A and B are commutative, the corresponding Lie algebra is the 4-dimensional abelian Lie algebra.
Suppose A is commutative and B is not commutative. In this case the bracket of the Lie algebra associated to $A \otimes B$ is given by

$$[X_i \otimes e_j, X_k \otimes e_l] = X_i X_k \otimes e_j e_l - X_k X_i \otimes e_l e_j = X_i X_k \otimes [e_j, e_l]$$

with $[e_1, e_2] = e_2$.
When we put $f_{ij} = X_i \otimes e_j$, we obtain the following list of Lie algebras, $\mathfrak{g}_{i7}$, that are underlying the Vinberg algebras $A_i \otimes B_7$:

g_{17}	$[f_{11}, f_{12}] = f_{12}$,	$[f_{11}, f_{22}] = f_{22}$,	$[f_{12}, f_{21}] = -f_{22}$,	$[f_{21}, f_{22}] = f_{22}$
g_{27}	$[f_{11}, f_{12}] = f_{12}$,	$[f_{11}, f_{22}] = f_{22}$,	$[f_{12}, f_{21}] = -f_{22}$	
g_{37}	$[f_{11}, f_{12}] = f_{12}$,	$[f_{11}, f_{22}] = f_{22}$,	$[f_{12}, f_{21}] = -f_{22}$,	$[f_{21}, f_{22}] = -f_{12}$
g_{47}	$[f_{11}, f_{12}] = f_{22}$			
g_{57}	$[f_{11}, f_{12}] = f_{12}$			
g_{67}	*abelian*			

Likewise, if A is a non commutative Vinberg algebra and B a commutative $Vinb^!$ algebra then the bracket of the corresponding Lie algebra satisfies

$$[X_i \otimes e_j, X_k \otimes e_l] = X_i X_k \otimes e_j e_l - X_k X_i \otimes e_l e_j = [X_i, X_k] \otimes e_j e_l$$

For the algebras $A_i \otimes B_j$, $i = 7, 8, 9$, let us note that the corresponding Lie algebras $\mathfrak{g}_{ij}$ satisfy $\mathfrak{g}_{7j} = \mathfrak{g}_{8j} = \mathfrak{g}_{9j}$ for $j = 1, ..., 7$. Using the same previous notations we obtain the following Lie algebras :

g_{i1}	$[f_{11}, f_{21}] = f_{21}$,	$[f_{11}, f_{22}] = f_{22}$,	$[f_{12}, f_{21}] = f_{22}$,	$[f_{12}, f_{22}] = f_{22}$
g_{i2}	$[f_{11}, f_{21}] = f_{21}$,	$[f_{11}, f_{22}] = f_{22}$,	$[f_{12}, f_{21}] = f_{22}$	
g_{i3}	$[f_{11}, f_{21}] = f_{22}$,	$[f_{11}, f_{22}] = f_{22}$,	$[f_{12}, f_{21}] = f_{22}$,	$[f_{12}, f_{22}] = -f_{21}$
g_{i4}	$[f_{11}, f_{21}] = f_{22}$			
g_{i5}	$[f_{11}, f_{21}] = f_{21}$			
g_{i6}	*abelian*			

Finally, let us look at the case $A = A_i$, $i = 7, 8, 9$ and $B = B_7$. We obtain :

g_{77} : $[f_{11}, f_{12}] = \frac{b^2+2e}{e} f_{12} - b\frac{b^2+e}{e^2} f_{22}$, $[f_{11}, f_{21}] = f_{21}$,
$[f_{11}, f_{22}] = bf_{12} - \frac{b^2-e}{e} f_{22}$
$[f_{12}, f_{21}] = -bf_{12} + \frac{b^2}{e} f_{22}$, $[f_{21}, f_{22}] = ef_{12} - bf_{22}$

g_{87} : $[f_{11}, f_{12}] = af_{12}$, $[f_{11}, f_{21}] = f_{21}$, $[f_{11}, f_{22}] = (a+1) f_{22}$,
$[f_{12}, f_{21}] = -af_{22}$.

g_{97} : $[f_{11}, f_{12}] = af_{12} + cf_{22}$, $[f_{11}, f_{21}] = f_{21}$, $[f_{11}, f_{22}] = f_{22}$

Comparing it with the classification of 4-dimensional real Lie algebras presented in [15], we obtain :

Theorem 3.2 *The following solvable Lie algebras have an affine structure of tensorial type :*
$\mathfrak{g}_{3,2}(1)\oplus\mathbb{R}$, $\mathfrak{g}_{4,5}(1,a)$, $\mathfrak{g}_{4,6}(1)$, $\mathfrak{g}_{4,9}(\alpha)$, $\mathfrak{g}_{4,10}$, $\mathfrak{g}_2\oplus\mathfrak{g}_2$, $\mathfrak{g}_{4,1}$
$\mathfrak{g}_{4,2}$, $\mathfrak{g}_2\oplus\mathbb{R}^2$, $\mathfrak{g}_{3,1}(1)\oplus\mathbb{R}$, $\mathbb{R}^4$.

4 Cohomology of Lie-admissible algebras

4.1 Cohomology of $\mathcal{L}ie\mathcal{A}dm$-algebras

Let $\mathcal{P}$ be a binary quadratic operad. A $\mathcal{P}$-algebra is given by a $\mathbb{K}$-vector space V and an operad morphism

$$\varphi:\mathcal{P}\longrightarrow\mathcal{E}nd(V)$$

where $\mathcal{E}nd(V)$ is the operad of endomorphismes of V defined by

$$\mathcal{E}nd(V)(n)=Hom_{\mathbb{K}}(V^{\otimes^n},V).$$

Let A be a $\mathcal{L}ie\mathcal{A}dm$-algebra. In the following definition we denote, for each Σ_n-space V, $V^\vee:=V^*\otimes_{\Sigma_n} sgn$, the dual V^* tensored with the signum representation. The cochain complex for define the cohomology $H^*_{\mathcal{L}ie\mathcal{A}dm}(A,A)$ is given by

$$C^n_{\mathcal{L}ie\mathcal{A}dm}(A)=Hom_{\mathbb{K}}((\mathcal{L}ie\mathcal{A}dm^!)^\vee(n)\otimes_{\sum_n}A^{\otimes^n},A)$$

As every $\mathcal{L}ie\mathcal{A}dm^!$-algebra is an associative 3-order commutative algebra, the complex $C^*_{\mathcal{L}ie\mathcal{A}dm}(A)$ is

$$A\longrightarrow Hom(A\otimes A,A)\longrightarrow Hom(\wedge^3(A),A)\longrightarrow Hom(\wedge^4(A),A)\longrightarrow ...$$

The differential is induced by the composition maps of operad $\mathcal{L}ie\mathcal{A}dm$. Before we give a precise formula, we need to recall the definition of te Nijenhuis-Gerstenhaber products.

Let f and g be n-, respectively m-, linear mappings on a vector space V. The fundamental product of Gerstenhaber $\circ_i$ is in this case written

$$f\circ_i g(X_1,...,X_{n+m-1})=f(X_1,..,X_{i-1},g(X_i,..,X_{i+m-1}),\\ X_{i+m},..,X_{n+m-1})\ .$$

for $i=1,..,n$ and the product $\circ$ is given by

$$f\circ g(X_1,...,X_{n+m-1})=\textstyle\sum_{i=1,..,n}(-1)^{(i-1)(m-1)}f(X_1,..,X_{i-1},g(X_i,..,X_{i+m-1}),\\ X_{i+m},..,X_{n+m-1})\ .$$

In section 2.3 we have noted by $\odot_6$ the composition product of the operad $\mathcal{L}ie\mathcal{A}dm$. We can exprime this product using the Gerstenhaber product $\circ$. Let P be the antisymmetrization. It is defined by

$$P(f)(X_1,...,X_n)=\sum_{\sigma\in\sum_n}(-1)^{\varepsilon(\sigma)}f(X_{\sigma(1)},X_{\sigma(2)},...,X_{\sigma(n)}).$$

Definition 4.1 $f \odot_6 g = P(f \circ g)$

If we developp this expression we obtain :

$f \odot_6 g(X_1, ..., X_{n+m-1}) = \sum_{i=1,..,n} \sum_{\sigma \in \sum_{n+m-1}} (-1)^{\varepsilon(\sigma)} (-1)^{(i-1)(m-1)} f(X_{\sigma(1)},$
$..., X_{\sigma(i-1)}, g(X_{\sigma(i)}, .., X_{\sigma(i+m-1)}), X_{\sigma(i+m)}, .., X_{\sigma(n+m-1)})$

where Σ_p refers to the $p-$symmetric group.
For example if $f = g = \mu$ and $n = m = 2$ then

$$\mu \odot_6 \mu(X_1, X_2, X_3) = \sum_{\sigma \in \sum_3} (-1)^{\varepsilon(\sigma)} \{\mu(\mu(X_{\sigma(1)}, X_{\sigma(2)}), X_{\sigma(3)}) - \mu(X_{\sigma(1)}, \mu(X_{\sigma(2)}, X_{\sigma(3)})\}$$

and μ is a Lie-admissible law if and only if $\mu \odot_6 \mu = 0$. Likewise if μ is a bilinear mapping and f and endomorphism of V, then

$$\mu \odot_6 f(X_1, X_2) = \mu(f(X_1), X_2) - \mu(f(X_2), X_1) + \mu(X_1, f(X_2)) - \mu(X_2, f(X_1))$$

and

$$f \odot_6 \mu(X_1, X_2) = f(\mu(X_1, X_2)) - f(\mu(X_2, X_1)) = f([X_1, X_2]_\mu)$$

as soon as $\mu \odot_6 \mu = 0$.

Remark In [13] Nijenhuis denotes the product $\circ$ by $f\overline{o}g$

Lemma 4.1 *We have the following identities :*

$$P(P(f) \odot_1 g) = (n + m - 1)! P(f \odot_1 g) = P(f \odot_1 P(g)).$$

This can be prove directly.
We deduce that the following bracket

$$[f, g]^{\odot_6} = f \odot_6 g - (-1)^{(n-1)(m-1)} g \odot_6 f.$$

satisfies :

1. $[g, f]^{\odot_6} = (-1)^{(n-1)(m-1)+1} [f, g]^{\odot_6}$
2. $(-1)^{(n-1)(p-1)} [[f, g]^{\odot_6}, h]^{\odot_6} + (-1)^{(m-1)(n-1)} [[g, h]^{\odot_6}, f]^{\odot_6} + (-1)^{(p-1)(m-1)} [[h, f]^{\odot_6}, g]^{\odot_6} = 0$

where h is a p-linear mapping on V.

Consequence. $[,]^{\odot_6}$ is a bracket of a graded Lie algebra (on the space of multilinear mappings).

Definition 4.2 *Let ϕ be in $C^n_{\mathcal{L}ieAdm}(A)$, where A is a $\mathcal{L}ieAdm$-algebra with multiplication μ. The differential operator*

$$\delta_\mu : C^n_{\mathcal{L}ieAdm}(A) \longrightarrow C^{n+1}_{\mathcal{L}ieAdm}(A)$$

is defined by

$$\delta_\mu \phi = -[\mu, \phi]^{\odot_6}$$

4.2 Particular cases

i) $n=2$

Let φ be a bilinear mapping. Then

$$\begin{array}{l}
-\delta_\mu\varphi\left(X_1,X_2,X_3\right)= \\
\mu\left(\varphi\left(X_1,X_2\right),X_3\right)-\mu\left(X_1,\varphi\left(X_2,X_3\right)\right)+\mu\left(\varphi\left(X_2,X_3\right),X_1\right) \\
-\mu\left(X_2,\varphi\left(X_3,X_1\right)\right)-\mu\left(X_3,\varphi\left(X_1,X_2\right)\right)+\mu\left(\varphi\left(X_3,X_1\right),X_2\right) \\
-\mu\left(\varphi\left(X_2,X_1\right),X_3\right)+\mu\left(X_2,\varphi\left(X_1,X_3\right)\right)-\mu\left(\varphi\left(X_3,X_2\right),X_1\right) \\
+\mu\left(X_3,\varphi\left(X_2,X_1\right)\right)+\mu\left(X_1,\varphi\left(X_3,X_2\right)\right)-\mu\left(\varphi\left(X_1,X_3\right),X_2\right) \\
+\varphi\left(\mu\left(X_1,X_2\right),X_3\right)-\varphi\left(X_1,\mu\left(X_2,X_3\right)\right)+\varphi\left(\mu\left(X_2,X_3\right),X_1\right) \\
-\varphi\left(X_2,\mu\left(X_3,X_1\right)\right)-\varphi\left(X_3,\mu\left(X_1,X_2\right)\right)+\varphi\left(\mu\left(X_3,X_1\right),X_2\right) \\
-\varphi\left(\mu\left(X_2,X_1\right),X_3\right)+\varphi\left(X_2,\mu\left(X_1,X_3\right)\right)-\varphi\left(\mu\left(X_3,X_2\right),X_1\right) \\
+\varphi\left(X_3,\mu\left(X_2,X_1\right)\right)+\varphi\left(X_1,\mu\left(X_3,X_2\right)\right)-\varphi\left(\mu\left(X_1,X_3\right),X_2\right)
\end{array}$$

ii) $n=1$

Let f be in $\mathcal{C}^1$. Then

$$\begin{array}{rcl}
\delta_\mu f\left(X_1,X_2\right) & = & -\mu\left(f(X_1),X_2\right)-\mu\left(X_1,f(X_2)\right)+f\left(\mu\left(X_1,X_2\right)\right) \\
 & & +\mu\left(f(X_2),X_1\right)+\mu\left(X_2,f(X_1)\right)-f\left(\mu\left(X_2,X_1\right)\right)
\end{array}$$

iii) $n=0$

We are going to give an explicit definition of $\delta_\mu(X)$. Let X be in A . Consider the map

$$h_X : Y \mapsto \mu\left(X,Y\right)-\mu\left(Y,X\right)$$

$$\begin{array}{rcl}
-\delta_\mu h_X\left(X_1,X_2\right) & = & \mu\left(h_X(X_1),X_2\right)+\mu\left(X_1,h_X(X_2)\right)-h_X\left(\mu\left(X_1,X_2\right)\right) \\
 & & -\mu\left(h_X(X_2),X_1\right)-\mu\left(X_2,h_X(X_1)\right)+h_X\left(\mu\left(X_2,X_1\right)\right) \\
 & = & \mu\left(\mu\left(X,X_1\right),X_2\right)-\mu\left(\mu\left(X_1,X\right),X_2\right)+\mu\left(X_1,\mu\left(X,X_2\right)\right) \\
 & & -\mu\left(X_1,\mu\left(X_2,X\right)\right)-h_X\left(\mu\left(X_1,X_2\right)\right)+h_X\left(\mu\left(X_2,X_1\right)\right) \\
 & & -\mu\left(\mu\left(X,X_2\right),X_1\right)+\mu\left(\mu\left(X_2,X\right),X_1\right)-\mu\left(X_2,\mu\left(X,X_1\right)\right) \\
 & & +\mu\left(X_2,\mu\left(X_1,X\right)\right)
\end{array}$$

$$\begin{array}{rcl}
-\delta_\mu h_X\left(X_2,X_1\right) & = & \mu\left(\mu\left(X,X_2\right),X_1\right)-\mu\left(\mu\left(X_2,X\right),X_1\right)+\mu\left(X_2,\mu\left(X,X_1\right)\right) \\
 & & -\mu\left(X_2,\mu\left(X_1,X\right)\right)-h_X\left(\mu\left(X_2,X_1\right)\right)+h_X\left(\mu\left(X_1,X_2\right)\right) \\
 & & -\mu\left(\mu\left(X,X_1\right),X_2\right)+\mu\left(\mu\left(X_1,X\right),X_2\right)-\mu\left(X_1,\mu\left(X,X_2\right)\right) \\
 & & +\mu\left(X_1,\mu\left(X_2,X\right)\right)
\end{array}$$

Then

$$\delta_\mu h_X\left(X_2,X_1\right)-\delta_\mu h_X\left(X_1,X_2\right)=\mu\odot_6\mu\left(X,X_1,X_2\right)=0$$

$$\delta_\mu h_X\left(X_1,X_2\right)=\delta_\mu h_X\left(X_2,X_1\right)$$

Let us consider

$$\mathcal{C}^0=\{X\in\mathcal{A}\quad/\quad P\left(\delta_\mu h_X\right)=\delta_\mu h_X\}$$

For $X \in \mathcal{C}^0, \quad \delta_\mu h_X = 0$. Then we can define $B^1(\mathcal{A},\mathcal{A})$ putting

$$\delta(X) = h_X$$

and $Z^1(\mathcal{A},\mathcal{A}) = \{f \in \mathcal{C}^1 / \delta f = 0\}$.
Then $H^0(\mathcal{A},\mathcal{A})$ is well defined.

Remark. In the same way, we can define the cohomology $H^*_{G_i-\mathcal{A}ss}(A,A)$ for a $G_i\mathcal{A}ss$-algebra. We denote by $f \odot_i g$ the corresponding composition product. We have yet given the expression of this product for $i=1$ and $i=6$. In other cases we put :

$f \odot_2 g(X_1,...,X_{n+m-1}) = \sum_{\sigma\in\sum_{n+m-2}} (-1)^{\varepsilon(\sigma)} \{\sum_{i=1,..,n-1} (-1)^{(i-1)(m-1)}$
$f(X_{\sigma(1)}.., X_{\sigma(i-1)}, g(X_{\sigma(i)},..,X_{\sigma(m+i-1)}), X_{\sigma(m+i)},.,X_{\sigma(n+m-2)}, X_{n+m-1}) +$
$(-1)^{(n-1)} f(X_{\sigma(1)},..,X_{\sigma(n-1)}, g(X_{\sigma(n)},..,X_{\sigma(m+n-2)}, X_{n+m-1})\}$.

$f \odot_3 g(X_1,...,X_{n+m-1}) = \sum_{\sigma\in\sum_{n+m-2}} (-1)^{\varepsilon(\sigma)} \{f(g(X_1,X_{\sigma(2)},..,X_{\sigma(m)}),$
$X_{\sigma(m+1)},..,X_{\sigma(m+n-1)}) + \sum_{i=2,..,n-1} (-1)^{(i-1)(m-1)} f(X_1,\, X_{\sigma(2)},..,\, X_{\sigma(i)},$
$g(X_{\sigma(i+1)},..,X_{\sigma(m+i)}),.,X_{\sigma(n+m-1)})\}$

$f \odot_4 g(X_1,...,X_{n+m-1}) = \sum_{\sigma\in\sum_{n+m-2}} \sum_{i=0,..,n-1} (-1)^{\varepsilon(\sigma)+i(m-1)} f(X_{\sigma(1)},$
$X_{\sigma(2)}, X_{\sigma(3)},...,X_{\sigma(i)}, g(X_{\sigma(i+1)},..,X_{\sigma(m+i)}),...,X_{\sigma(n+m-1)})$.

$f \odot_5 g(X_1,...,X_{n+m-1}) = \sum_{\sigma\in A_{n+m-1}} \sum_{i=0,..,n-1} (-1)^{i(m-1)} f(X_{\sigma(1)}, X_{\sigma(2)},$
$X_{\sigma(3)},...,X_{\sigma(i)}, g(X_{\sigma(i+1)},..,X_{\sigma(m+i)}),...,X_{\sigma(n+m-1)})$.

where A_p refers to the alternating group.

4.3 Lie-admissible cohomology of Lie algebras

If $\mathfrak{g}$ is a Lie algebra of law μ, it is also a Lie-admissible algebra. Then it is possible to consider the following cohomologies

1. $H^*_{\mathcal{L}ieAdm}(\mathfrak{g},\mathfrak{g})$
2. $H^*_{\mathcal{L}ie}(\mathfrak{g},\mathfrak{g})$

Definition 4.3 *Let $\mathfrak{g}$ be a Lie algebra. We call Lie-admissible cohomology of $\mathfrak{g}$ with value in $\mathfrak{g}$ the cohomology $H^*_{\mathcal{L}ieAdm}(\mathfrak{g},\mathfrak{g})$.*

Remark A 2-cochain corresponding to the Chevalley's cohomology is alternated and satisfies

$$\delta\varphi = 4\delta^c\varphi.$$

5 Lie-admissible modules over a Lie algebra

5.1 Module over a Lie-admissible algebra

Let $\mathcal{A}=(A,\mu)$ be a Lie-admissible algebra and M a vector space over $\mathbb{K}$.

Definition 5.1 *M is an $\mathcal{A}$-module if there are bilinear maps*

$$\begin{array}{rcl} \lambda & : & A\otimes M\rightarrow M \\ \rho & : & M\otimes A\rightarrow M \end{array}$$

satisfying :

$\lambda(X,\lambda(Y,v)-\lambda(Y,\lambda(X,v)-\lambda([X,Y]_\mu,v)-\lambda(X,\rho(v,Y)+\rho(\lambda(X,v),Y)+\rho(v,[X,Y]_\mu)-\rho(\rho(v,X),Y)-\rho(\lambda(Y,v),X)+\lambda(Y,\rho(v,X))+\rho(\rho(v,Y),X)=0$

for all $X,Y\in A$ and $v\in M$.

For example, the vector space A is an $\mathcal{A}$-module.

Proposition 5.1 *Let $\mathcal{A}=(A,\mu)$ be a Lie-admissible algebra and M an $\mathcal{A}$-module defined by the mapping λ and ρ. Then the bilinear mapping*

$$\widehat{\lambda}:A\otimes M\rightarrow M$$

defined by

$$\widehat{\lambda}(X,v)=\lambda(X,v)-\rho(v,X)$$

equips the vector space M with an $\mathcal{A}_L$-module structure where $\mathcal{A}_L$ is the Lie algebra $(A,[,]_\mu)$.

We find again the same result established by Nijenhuis in [N] for the Vinberg algebra.

5.2 Modules over G_i-associative algebras

If $\mathcal{A}_V=(A,\mu_V)$ is a Vinberg algebra an $\mathcal{A}_V$-module M is given by the mappings λ and ρ satisfying the two conditions

$$\left\{\begin{array}{l} \lambda(X,\lambda(Y,v)-\lambda(Y,\lambda(X,v)-\lambda([X,Y]_\mu,v)=0 \\ \lambda(X,\rho(v,Y)-\rho(\lambda(X,v),Y)-\rho(v,\mu(X,Y))+\rho(\rho(v,X),Y)=0. \end{array}\right.$$

Considering $\mathcal{A}_V$ as a Lie-admissible algebra, then M is also a module on this Lie-admissible algebra.

The notion of $\mathcal{A}$-module is well known in the case of Lie-admissible algebras of type 1 (associative). It is easy to write the definitions of modules over algebras of type 4 and 5. We find :

- type 4 :

$$\left\{\begin{array}{l} \lambda(\mu(X,Y),v)-\lambda(X,\lambda(Y,v)-\rho(\rho(v,Y),X)+\rho(v,\mu(Y,X))=0 \\ \rho(\lambda(X,v),Y)-\lambda(X,\rho(v,Y)-\rho(\lambda(Y,v),X)+\lambda(Y,\rho(v,X))=0 \end{array}\right.$$

-type 5 :
$\lambda(\mu(X,Y),v) - \lambda(X,\lambda(Y,v) + \rho(\lambda(X,v),Y) - \lambda(X,\rho(v,Y) - \rho(v,\mu(X,Y)) + \rho(\rho(v,X),Y) = 0$.
We can see that if μ is antisymmetric (i.e. a law of Lie algebra) then we find again the definition of module on Lie algebra considering $\rho(v,X) = -\lambda(X,v)$.

5.3 Lie-admissible modules over Lie algebras

Let $\mathcal{A}$ =(A,μ) be a Lie algebra. Consider this algebra as a Lie-admissible algebra that we will note $\mathcal{A}_{ad}$=(A,μ) to distinguish the two structures. It is clear that every module M on the Lie algebra $\mathcal{A}$ is also a module over the Lie-admissible algebra $\mathcal{A}_{ad}$. But the converse is false.

Definition 5.2 *We call Lie-admissible module over the Lie algebra* $\mathcal{A}$ =(A,μ) *every module on the Lie-admissible algebra* $\mathcal{A}_{ad}$=(A,μ).

Let $\mathfrak{g}$ be the solvable non abelian 2-dimensional Lie algebra. There exists a basis $\{X_1, X_2\}$ such that $[X_1, X_2] = X_2$. Every one dimensional module on the Lie algebra $\mathfrak{g}$ is given by the mapping λ (here $\rho = -\lambda$) defined by

$$\begin{array}{rcl}\lambda(X_1,v) & = & \alpha v \\ \lambda(X_2,v) & = & 0\end{array}$$

On the other hand, a Lie-admissible module over $\mathfrak{g}$ is determined by the maps λ and ρ given by

$$\left\{\begin{array}{l}\lambda(X_1,v) = \alpha v \\ \lambda(X_2,v) = \beta v\end{array}\right. ; \quad \left\{\begin{array}{l}\rho(v,X_1) = \gamma v \\ \rho(v,X_2) = \beta v\end{array}\right.$$

Suppose now that M is a n-dimensional Lie-admissible module on $\mathfrak{g}$. Then if A, B, C, D are the matrices of linear operators $\lambda(X_1,.), \lambda(X_2,.), \rho(.,X_1), \rho(.,X_2)$ in a given basis of M, then these matrices satisfy

$$[(B-D),(C-A)] = B-D$$

We can describe in this way all structures of Lie-admissible modules over $\mathfrak{g}$.

Now consider the Lie algebra $\mathfrak{g} = sl(2,\mathbb{C})$. By a similar computation we can see that every n-dimensional Lie-admissible module over $sl(2,\mathbb{C})$ is described by the following matrix representations :

$$\left\{\begin{array}{l}[A_1 - B_1, A_2 - B_2] = 4(A_2 - B_2) \\ [A_1 - B_1, A_3 - B_3] = -4(A_3 - B_3) \\ [A_2 - B_2, A_3 - B_3] = 2(A_1 - B_1)\end{array}\right. .$$

Such representation is also completely reducible.

6 Deformations

Let us denote $\mathcal{L}A_n$ the algebraic variety of structure constants n-dimensional Lie-admissible algebras over an algebraically closed field $\mathbb{K}$ of characteristic 0. We have a natural fibration

$$\pi : \mathcal{L}A_n \to \mathcal{L}_n,$$

where $\mathcal{L}_n$ denotes the algebraic variety of n-dimensional Lie algebras :

$$\pi(\mu) = [,]_\mu$$

This fibration admits a global section

$$s : \mathcal{L}_n \to \mathcal{L}A_n$$

defined by

$$s(\mu) = \frac{1}{2}\mu.$$

Let us denote $T_{s(\mu)}\mathcal{L}A_n$ and $T_{s(\mu)}\pi^{-1}((\mu))$ the tangent spaces to the variety and to the fiber at the point $s(\mu)$. We know that $T_{s(\mu)}\mathcal{L}A_n$ is identified to the space of 2-cocycles $Z^2_{\mathcal{L}ieAdm}(A, A)$.

Lemma 6.1 $T_{s(\mu)}\pi^{-1}(\mu) \simeq \{\varphi : A^{\otimes 2} \to A \,/\, \varphi(x, y) = \varphi(y, x)\} \simeq S^2(A)$.

Proof. In fact $\frac{1}{2}\mu + t\varphi$ is a linear deformation of $\frac{1}{2}\mu$ in the fiber $\pi^{-1}(\mu)$ then $\varphi(x, y) - \varphi(y, x) = 0$. Moreover every symmetric bilinear mapping is a cocycle that is in $Z^2_{\mathcal{L}ieAdm}(A, A)$.

Proposition 6.2 *Every λ belonging to $\pi^{-1}(\mu)$ can be written $\lambda = \frac{1}{2}\mu + \varphi$ with φ a symmetric bilinear application. (Likewise every $\lambda = \frac{1}{2}\mu + \varphi$ with $\varphi \in S^2(A)$ is in $\pi^{-1}(\mu)$.*

Proof. The law λ is in $\pi^{-1}(\mu)$. This implies that $\lambda(x, y) - \lambda(y, x) = \mu(x, y)$. Let φ be $\lambda - \frac{1}{2}\mu$. A simple computation gives that $\varphi(x, y) - \varphi(y, x) = 0$. The other part is also obvious. □

Theorem 6.3 *The fiber $\pi^{-1}(\mu)$ is an abelian group with the following product*

$$\mu_1 \star \mu_2 = \mu_1 + \mu_2 - \frac{1}{2}\mu,$$

for all $\mu_i \in \pi^{-1}(\mu)$, $\quad i \in \{1, 2\}$.

Proof. As $\mu_i \in \pi^{-1}(\mu)$ we have $\mu_i = \frac{1}{2}\mu + \varphi_i$ with $\varphi_i \in S^2(A)$. Then $(\mu_1 \star \mu_2)(x, y) = \frac{1}{2}\mu + (\varphi_1 + \varphi_2)$ and $\varphi_1 + \varphi_2 \in S^2(A)$ So $\star$ is an internal product. It's obviously associative. The unit element for the law $\star$ is the element $\frac{1}{2}\mu$ and each element $\mu_i \in \pi^{-1}(\mu)$ has an inverse $(\mu_i)^{-1} = -\mu_i + \mu$. Observe that $(\mu_i)^{-1}$

is well in $\pi^{-1}(\mu)$ because $(\mu_i)^{-1}(x,y) - (\mu_i)^{-1}(y,x) = -(\mu_i(x,y) - \mu_i(y,x)) + (\mu(x,y) - \mu(y,x)) = \mu(x,y)$.
We have that $(\pi^{-1}(\mu), \star)$ is an abelian group. $\square$.

Recall the geometric problem concerning the existence of affine structures on solvable Lie algebras. A Lie algebra $\mathfrak{g}$ carries an affine structure if and only if there exists a Vinberg algebra whose associated Lie algebra is $\mathfrak{g}$. We know that there exists nilpotent Lie algebras without affine structure ([2]). In this case the fiber $\pi^{-1}(\mu)$ does not cut the subvariety of Vinberg laws. For $\varphi \in S^2(A) \subset Z^2_{\mathcal{L}ieAdm}(A,A)$ fixed, the staightline $\frac{1}{2}\mu + t\varphi$ is in the fiber $\pi^{-1}(\mu)$. This line cut the subvariety of Vinberg if and only if there is t_0 such that $\frac{1}{2}\mu + t_0\varphi$ is a Vinberg law. Considering φ for $t_0\varphi$ (we can always suppose that $t_0 = 1$) we obtain :

Proposition 6.4 *The deformation $\frac{1}{2}\mu + \varphi$ is in $Vinb_n$ if and only if the symmetric mapping satisfies*

$$\begin{aligned}
&4\varphi\left(\mu\left(X_2, X_1\right), X_3\right) + 2\mu\left(X_1, \varphi\left(X_2, X_3\right)\right) + 2\varphi\left(X_1, \mu\left(X_2, X_3\right)\right) \\
&+4\varphi\left(X_1, \varphi\left(X_2, X_3\right)\right) - 2\mu\left(X_2, \varphi\left(X_1, X_3\right)\right) - 2\varphi\left(X_2, \mu\left(X_1, X_3\right)\right) \\
&-4\varphi\left(X_2, \varphi\left(X_1, X_3\right)\right) + \mu\left(\mu\left(X_2, X_1\right), X_3\right) = 0.
\end{aligned}$$

for all $X_1, X_2, X_3 \in \mathfrak{g}$.

This proposition can be considered as a criterium of existence of an affine structure on a given Lie algebra.

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