Valued deformations of algebras

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Abstract

We develop the notion of deformations using a valuation ring as ring of coefficients. This permits to consider in particular the classical Gerstenhaber deformations of associative or Lie algebras as infinitesimal deformations and to solve the equation of deformations in a polynomial frame. We consider also the deformations of the enveloping algebra of a rigid Lie algebra and we define valued deformations for some classes of non associative algebras.

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1.1 Rings of valuation

We recall briefly the classical notion of ring of valuation. Let $\mathbb{F}$ be a (commutative) field and A a subring of $\mathbb{F}$. We say that A is a ring of valuation of $\mathbb{F}$ if A is a local integral domain satisfying:

$$\text{If } x \in \mathbb{F} - A, \quad \text{then} \quad x^{-1} \in \mathfrak{m}.$$

where $\mathfrak{m}$ is the maximal ideal of A.

A ring A is called ring of valuation if it is a ring of valuation of its field of fractions.

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Examples : Let $\mathbb{K}$ be a commutative field of characteristic 0. The ring of formal series $\mathbb{K}[[t]]$ is a valuation ring. On other hand the ring $\mathbb{K}[[t_1, t_2]]$ of two (or more) indeterminates is not a valuation ring.

1.2 Versal deformations of Fialowski [F]

Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and A an unitary commutative local $\mathbb{K}$-algebra. The tensor product $\mathfrak{g} \otimes A$ is naturally endowed with a Lie algebra structure :

$$[X \otimes a, Y \otimes b] = [X, Y] \otimes ab.$$

If $\epsilon : A \longrightarrow \mathbb{K}$, is an unitary augmentation with kernel the maximal ideal $\mathfrak{m}$, a deformation λ of $\mathfrak{g}$ with base A is a Lie algebra structure on $\mathfrak{g} \otimes A$ with bracket $[,]_\lambda$ such that

$$id \otimes \epsilon : \mathfrak{g} \otimes A \longrightarrow \mathfrak{g} \otimes \mathbb{K}$$

is a Lie algebra homomorphism. In this case the bracket $[,]_\lambda$ satisfies

$$[X \otimes 1, Y \otimes 1]_\lambda = [X, Y] \otimes 1 + \sum Z_i \otimes a_i$$

where $a_i \in A$ and $X, Y, Z_i \in \mathfrak{g}$. Such a deformation is called infinitesimal if the maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^2 = 0$. An interesting example is described in [F]. If we consider the commutative algebra $A = \mathbb{K} \oplus (H^2(\mathfrak{g}, \mathfrak{g}))^*$ (where * denotes the dual as vector space) such that $dim(H^2) \leq \infty$, the deformation with base A is an infinitesimal deformation (which plays the role of an universal deformation).

1.3 Valued deformations of Lie algebra

Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and A a commutative $\mathbb{K}$-algebra of valuation. Then $\mathfrak{g} \otimes A$ is a $\mathbb{K}$-Lie algebra. We can consider this Lie algebra as an A-Lie algebra. We denote this last by $\mathfrak{g}_A$. If $dim_{\mathbb{K}}(\mathfrak{g})$ is finite then

$$dim_A(\mathfrak{g}_A) = dim_{\mathbb{K}}(\mathfrak{g}).$$

As the valued ring A is also a $\mathbb{K}$-algebra we have a natural embedding of the $\mathbb{K}$-vector space $\mathfrak{g}$ into the free A-module $\mathfrak{g}_A$. Without loss of generality we can consider this embedding to be the identity map.

Definition 1 *Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and A a commutative $\mathbb{K}$-algebra of valuation such that the residual field $\frac{A}{\mathfrak{m}}$ is isomorphic to $\mathbb{K}$ (or to a subfield of $\mathbb{K}$). A valued deformation of $\mathfrak{g}$ with base A is a A-Lie algebra $\mathfrak{g}'_A$ such that the underlying A-module of $\mathfrak{g}'_A$ is $\mathfrak{g}_A$ and that*

$$[X, Y]_{\mathfrak{g}'_A} - [X, Y]_{\mathfrak{g}_A}$$

is in the $\mathfrak{m}$-quasi-module $\mathfrak{g} \otimes \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of A.

The classical notion of deformation studied by Gerstenhaber ([G]) is a valued deformation. In this case $A = \mathbb{K}[[t]]$ and the residual field of A is isomorphic to $\mathbb{K}$. Likewise a versal deformation is a valued deformation. The algebra A is in this case the finite dimensional $\mathbb{K}$-vector space $\mathbb{K} \oplus (H^2(\mathfrak{g}, \mathfrak{g}))^*$ where H^2 denotes the second Chevalley cohomology group of $\mathfrak{g}$. The algebra law is given by

$$(\alpha_1, h_1).(\alpha_2, h_2) = (\alpha_1.\alpha_2, \alpha_1.h_2 + \alpha_2.h_1).$$

It is a local field with maximal ideal $\{0\} \oplus (H^2)^*$. It is also a valuation field because we can endowe this algebra with a field structure, the inverse of (α, h) being $((\alpha)^{-1}, -(\alpha)^{-2}h)$.

2 Decomposition of valued deformations

In this section we show that every valued deformation can be decomposed in a finite sum (and not as a serie) with pairwise comparable infinitesimal coefficients (that is in $\mathfrak{m}$). The interest of this decomposition is to avoid the classical problems of convergence.

2.1 Decomposition in $\mathfrak{m} \times \mathfrak{m}$

Let A be a valuation ring satisfying the conditions of definition 1. Let us denote by $\mathcal{F}_\mathcal{A}$ the field of fractions of A and $\mathfrak{m}^2$ the catesian product $\mathfrak{m} \times \mathfrak{m}$. Let $(a_1, a_2) \in \mathfrak{m}^2$ with $a_i \neq 0$ for $i = 1, 2$.

i) Suppose that $a_1.a_2^{-1} \in A$. Let be $\alpha = \pi(a_1.a_2^{-1})$ where π is the canonical projection on $\frac{A}{\mathfrak{m}}$. Clearly, there exists a global section $s : \mathbb{K} \to A$ which permits to identify α with $s(\alpha)$ in A. Then

$$a_1.a_2^{-1} = \alpha + a_3$$

with $a_3 \in \mathfrak{m}$. Then if $a_3 \neq 0$,

$$(a_1, a_2) = (a_2(\alpha + a_3), a_2) = a_2(\alpha, 1) + a_2a_3(0, 1).$$

If $\alpha \neq 0$ we can also write

$$(a_1, a_2) = aV_1 + abV_2$$

with $a, b \in \mathfrak{m}$ and V_1, V_2 linearly independent in $\mathbb{K}^2$. If $\alpha = 0$ then $a_1.a_2^{-1} \in \mathfrak{m}$ and $a_1 = a_2a_3$. We have

$$(a_1, a_2) = (a_2a_3, a_2) = ab(1, 0) + a(0, 1).$$

So in this case, $V_1 = (0, 1)$ and $V_2 = (1, 0)$. If $a_3 = 0$ then

$$a_1a_2^{-1} = \alpha$$

and

$$(a_1, a_2) = a_2(\alpha, 1) = aV_1.$$

This correspond to the previous decomposition but with $b = 0$.

ii) If $a_1.a_2^{-1} \in \mathcal{F}_A - A$, then $a_2.a_1^{-1} \in \mathfrak{m}$. We put in this case $a_2.a_1^{-1} = a_3$ and we have

$$(a_1, a_2) = (a_1, a_1.a_3) = a_1(1, a_3) = a_1(1, 0) + a_1a_3(0, 1)$$

with $a_3 \in \mathfrak{m}$. Then, in this case the point (a_1, a_2) admits the following decomposition :

$$(a_1, a_2) = aV_1 + abV_2$$

with $a, b \in \mathfrak{m}$ and V_1, V_2 linearly independent in $\mathbb{K}^2$. Note that this case corresponds to the previous but with $\alpha = 0$.

Then we have proved

Proposition 1 *For every point $(a_1, a_2) \in \mathfrak{m}^2$, there exist lineary independent vectors V_1 and V_2 in the $\mathbb{K}$-vector space $\mathbb{K}^2$ such that*

$$(a_1, a_2) = aV_1 + abV_2$$

for some $a, b \in \mathfrak{m}$.

Such decomposition est called of length 2 if $b \neq 0$. If not it is called of length 1.

2.2 Decomposition in $\mathfrak{m}^k$

Suppose that A is valuation ring satisfying the hypothesis of Definition 1. Arguing as before, we can conclude

Theorem 1 *For every $(a_1, a_2, ..., a_k) \in \mathfrak{m}^k$ there exist h ($h \leq k$) independent vectors $V_1, V_2, .., V_h$ whose components are in $\mathbb{K}$ and elements $b_1, b_2, .., b_h \in \mathfrak{m}$ such that*

$$(a_1, a_2, ..., a_k) = b_1V_1 + b_1b_2V_2 + ... + b_1b_2...b_hV_h.$$

The parameter h which appears in this theorem is called the length of the decomposition. This parameter can be different to k. It corresponds to the dimension of the smallest $\mathbb{K}$-vector space V such that $(a_1, a_2, ..., a_k) \in V \otimes \mathfrak{m}$.

If the coordinates a_i of the vector $(a_1, a_2, ..., a_k)$ are in A and not necessarily in its maximal ideal, then writing $a_i = \alpha_i + a'_i$ with $\alpha_i \in \mathbb{K}$ and $a'_i \in \mathfrak{m}$, we decompose

$$(a_1, a_2, ..., a_k) = (\alpha_1, \alpha_2, ..., \alpha_k) + (a'_1, a'_2, ..., a'_k)$$

and we can apply Theorem 1 to the vector $(a'_1, a'_2, ..., a'_k)$.

2.3 Uniqueness of the decomposition

Let us begin by a technical lemma.

Lemma 1 *Let V and W be two vectors with components in the valuation ring A. There exist V_0 and W_0 with components in $\mathbb{K}$ such that $V = V_0 + V_0'$ and $W = W_0 + W_0'$ and the components of V_0' and W_0' are in the maximal ideal $\mathfrak{m}$. Moreover if the vectors V_0 and W_0 are linearly independent then V and W are also independent.*

Proof. The decomposition of the two vectors V and W is evident. It remains to prove that the independence of the vectors V_0 and W_0 implies those of V and W. Let V, W be two vectors with components in A such that $\pi(V) = V_0$ and $\pi(W) = W_0$ are independent. Let us suppose that

$$xV + yW = 0$$

with $x, y \in A$. One of the coefficients xy^{-1} or yx^{-1} is not in $\mathfrak{m}$. Let us suppose that $xy^{-1} \notin \mathfrak{m}$. If $xy^{-1} \notin A$ then $x^{-1}y \in \mathfrak{m}$. Then $xV + yW = 0$ is equivalent to $V + x^{-1}yW = 0$. This implies that $\pi(V) = 0$ and this is impossible. Then $xy^{-1} \in A - \mathfrak{m}$. Thus if there exists a linear relation between V and W, there exists a linear relation with coefficients in $A - \mathfrak{m}$. We can suppose that $xV + yW = 0$ with $x, y \in A - \mathfrak{m}$. As $V = V_0 + V_0'$, $W = W_0 + W_0'$ we have

$$\pi(xV + yW) = \pi(x)V_0 + \pi(y)W_0 = 0.$$

Thus $\pi(x) = \pi(y) = 0$. This is impossible and the vectors V and W are independent as soon as V_0 and W_0 are independent vectors. □

Let $(a_1, a_2, ..., a_k) = b_1V_1 + b_1b_2V_2 + ... + b_1b_2...b_hV_h$ and $(a_1, a_2, ..., a_k) = c_1W_1 + c_1c_2W_2 + ... + c_1c_2...c_sW_s$ be two decompositions of the vector $(a_1, a_2, ..., a_k)$. Let us compare the coefficients b_1 and c_1. By hypothesis $b_1c_1^{-1}$ is in A or the inverse is in $\mathfrak{m}$. Then we can suppose that $b_1c_1^{-1} \in A$. As the residual field is a subfield of $\mathbb{K}$, there exists $\alpha \in \frac{A}{\mathfrak{m}}$ and $c_1 \in \mathfrak{m}$ such that

$$b_1c_1^{-1} = \alpha + b_{11}$$

thus $b_1 = \alpha c_1 + b_{11}c_1$. Replacing this term in the decompositions we obtain

$$(\alpha c_1 + b_{11}c_1)V_1 + (\alpha c_1 + b_{11}c_1)b_2V_2 + ... + (\alpha c_1 + b_{11}c_1)b_2...b_hV_h$$
$$= c_1W_1 + c_1c_2W_2 + ... + c_1c_2...c_sW_s.$$

Simplifying by c_1, this expression is written

$$\alpha V_1 + m_1 = W_1 + m_2$$

where m_1, m_2 are vectors with coefficients $\in \mathfrak{m}$. From Lemma 1, if V_1 and W_1 are linearly independent, as its coefficients are in the residual field, the vectors $\alpha V_1 + m_1$ and $W_1 + m_2$ would be also linearly independent ($\alpha \neq 0$). Thus $W_1 = \alpha V_1$. One deduces

$$b_1V_1 + b_1b_2V_2 + ... + b_1b_2...b_hV_h = c_1(\alpha V_1) + c_1b_{11}V_1 + c_1b_{12}V_2 + ... + c_1b_{12}b_3...b_hV_h,$$

with $b_{12} = b_2(\alpha + b_{11})$. Then

$$b_{11}V_1 + b_{12}V_2 + ... + b_{12}b_3...b_hV_h = c_2W_2 + ... + c_2...c_sW_s.$$

Continuing this process by induction we deduce the following result

Theorem 2 *Let be $b_1V_1 + b_1b_2V_2 + ... + b_1b_2...b_hV_h$ and $c_1W_1 + c_1c_2W_2 + ... + c_1c_2...c_sW_s$ two decompositions of the vector $(a_1, a_2, ..., a_k)$. Then*
i. $h = s$,
ii. The flag generated by the ordered free family $(V_1, V_2, .., V_h)$ is equal to the flag generated by the ordered free family $(W_1, W_2, ..., W_h)$ that is $\forall i \in 1, .., h$

$$\{V_1, ..., V_i\} = \{W_1, ..., W_i\}$$

where $\{U_i\}$ designates the linear space genrated by the vectors U_i.

2.4 Geometrical interpretation of this decomposition

Let A be an $\mathbb{R}$ algebra of valuation. Consider a differential curve γ in $\mathbb{R}^3$. We can embed γ in a differential curve

$$\Gamma : \mathbb{R} \otimes A \to \mathbb{R}^3 \otimes A.$$

Let $t = t_0 \otimes 1 + 1 \otimes \epsilon$ an parameter infinitely close to t_0, that is $\epsilon \in \mathfrak{m}$. If M corresponds to the point of Γ of parameter t and M_0 those of t_0, then the coordinates of the point $M - M_0$ in the affine space $\mathbb{R}^3 \otimes A$ are in $\mathbb{R} \otimes \mathfrak{m}$. In the flag associated to the decomposition of $M - M_0$ we can considere a direct orthonormal frame (V_1, V_2, V_3). It is the Serret-Frenet frame to γ at the point M_0.

3 Decomposition of a valued deformation of a Lie algebra

3.1 Valued deformation of Lie algebras

Let $\mathfrak{g}'_A$ be a valued deformation with base A of the $\mathbb{K}$-Lie algebra $\mathfrak{g}$. By definition, for every X and Y in $\mathfrak{g}$ we have $[X, Y]_{\mathfrak{g}'_A} - [X, Y]_{\mathfrak{g}_A} \in \mathfrak{g} \otimes \mathfrak{m}$. Suppose that $\mathfrak{g}$ is finite dimensional and let $\{X_1, ..., X_n\}$ be a basis of $\mathfrak{g}$. In this case

$$[X_i, X_j]_{\mathfrak{g}'_A} - [X_i, X_j]_{\mathfrak{g}_A} = \sum_k C_{ij}^k X_k$$

with $C_{ij}^k \in \mathfrak{m}$. Using the decomposition of the vector of $\mathfrak{m}^{n^2(n-1)/2}$ with for components C_{ij}^k , we deduce that

$$\begin{aligned}[X_i, X_j]_{\mathfrak{g}'_A} - [X_i, X_j]_{\mathfrak{g}_A} &= a\epsilon_1\phi_1(X_i, X_j) + \epsilon_1\epsilon_2\phi_2(X_i, X_j) \\ &\quad + ... + \epsilon_1\epsilon_2...\epsilon_k\phi_k(X_i, X_j)\end{aligned}$$

where $\epsilon_s \in \mathfrak{m}$ and $\phi_1, ..., \phi_l$ are linearly independent. Then we have

$$\begin{array}{rcl} [X,Y]_{\mathfrak{g}'_A} - [X,Y]_{\mathfrak{g}_A} & = & \epsilon_1\phi_1(X,Y) + \epsilon_1\epsilon_2\phi_2(X,Y) \\ & & +... + \epsilon_1\epsilon_2...\epsilon_k\phi_k(X,Y) \end{array}$$

where the bilinear maps ϵ_i have values in $\mathfrak{m}$ and linear maps $\phi_i : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ are linearly independent.

If $\mathfrak{g}$ is infinite dimensional with a countable basis $\{X_n\}_{n\in\mathbb{N}}$ then the $\mathbb{K}$-vector space of linear map $T_2^1 = \{\phi : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\}$ also admits a countable basis.

Theorem 3 *If $\mu_{\mathfrak{g}'_A}$ (resp. $\mu_{\mathfrak{g}_A}$) is the law of the Lie algebra $\mathfrak{g}'_A$ (resp. $\mathfrak{g}_A$) then*

$$\mu_{\mathfrak{g}'_A} - \mu_{\mathfrak{g}_A} = \sum_{i\in I} \epsilon_1\epsilon_2...\epsilon_i\phi_i$$

where I is a finite set of indices, $\epsilon_i : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{m}$ are linear maps and ϕ_i's are linearly independent maps in T_2^1.

3.2 Equations of valued deformations

We will prove that the classical equations of deformation given by Gerstenhaber are still valid in the general frame of valued deformations. Neverless we can prove that the infinite system described by Gerstenhaber and which gives the conditions to obtain a deformation, can be reduced to a system of finite rank. Let

$$\mu_{\mathfrak{g}'_A} - \mu_{\mathfrak{g}_A} = \sum_{i\in I} \epsilon_1\epsilon_2...\epsilon_i\phi_i$$

be a valued deformation of μ (the bracket of $\mathfrak{g}$). Then $\mu_{\mathfrak{g}'_A}$ satisfies the Jacobi equations. Following Gerstenhaber we consider the Chevalley-Eilenberg graded differential complex $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$ and the product $\circ$ defined by

$$(g_q \circ f_p)(X_1, ..., X_{p+q}) = \sum (-1)^{\epsilon(\sigma)} g_q(f_p(X_{\sigma(1)}, ..., X_{\sigma(p)}), X_{\sigma(p+1)}, ..., X_{\sigma(q)})$$

where σ is a permutation of $1, ..., p+q$ such that $\sigma(1) < ... < \sigma(p)$ and $\sigma(p+1) < ... < \sigma(p+q)$ (it is a (p,q)-schuffle); $g_q \in \mathcal{C}^q(\mathfrak{g}, \mathfrak{g})$ and $f_p \in \mathcal{C}^p(\mathfrak{g}, \mathfrak{g})$. As $\mu_{\mathfrak{g}'_A}$ satisfies the Jacobi identities, $\mu_{\mathfrak{g}'_A} \circ \mu_{\mathfrak{g}'_A} = 0$. This gives

$$(\mu_{\mathfrak{g}_A} + \sum_{i\in I} \epsilon_1\epsilon_2...\epsilon_i\phi_i) \circ (\mu_{\mathfrak{g}_A} + \sum_{i\in I} \epsilon_1\epsilon_2...\epsilon_i\phi_i) = 0. \qquad (1)$$

As $\mu_{\mathfrak{g}_A} \circ \mu_{\mathfrak{g}_A} = 0$, this equation becomes :

$$\epsilon_1(\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A}) + \epsilon_1 U = 0$$

where U is in $\mathcal{C}^3(\mathfrak{g}, \mathfrak{g}) \otimes \mathfrak{m}$. If we symplify by ϵ_1 which is supposed non zero if not the deformation is trivial, we obtain

$$(\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A})(X,Y,Z) + U(X,Y,Z) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. As $U(X,Y,Z)$ is in the module $\mathfrak{g} \otimes \mathfrak{m}$ and the first part in $\mathfrak{g} \otimes A$, each one of these vectors is null. Then

$$(\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A})(X,Y,Z) = 0.$$

Proposition 2 *For every valued deformation with base A of the $\mathbb{K}$-Lie algebra $\mathfrak{g}$, the first term ϕ appearing in the associated decomposition is a 2-cochain of the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ belonging to $Z^2(\mathfrak{g},\mathfrak{g})$.*

We thus rediscover the classical result of Gerstenhaber but in the broader context of valued deformations and not only for the valued deformation of basis the ring of formal series.

In order to describe the properties of other terms of equations (1) we use the super-bracket of Gerstenhaber which endows the space of Chevalley-Eilenberg cochains $\mathcal{C}(\mathfrak{g},\mathfrak{g})$ with a Lie superalgebra structure. When $\phi_i \in \mathcal{C}^2(\mathfrak{g},\mathfrak{g})$, it is defines by

$$[\phi_i, \phi_j] = \phi_i \circ \phi_j + \phi_j \circ \phi_i$$

and $[\phi_i, \phi_j] \in \mathcal{C}^3(\mathfrak{g},\mathfrak{g})$.

Lemma 2 *Let us suppose that $I = \{1, ..., k\}$. If*

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i \in I} \epsilon_1 \epsilon_2 ... \epsilon_i \phi_i$$

is a valued deformation of μ, then the 3-cochains $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$, $1 \leq i, j \leq k-1$, generate a linear subspace V of $\mathcal{C}^3(\mathfrak{g},\mathfrak{g})$ of dimension less or equal to $k(k-1)/2$. Moreover, the 3-cochains $[\phi_i, \phi_j]$, $1 \leq i, j \leq k-1$, form a system of generators of this space.

Proof. Let V be the subpace of $\mathcal{C}^3(\mathfrak{g},\mathfrak{g})$ generated by $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$. If ω is a linear form on V of which kernel contains the vectors $[\phi_i, \phi_j]$ for $1 \leq i, j \leq (k-1)$, then the equation (1) gives

$$\epsilon_1 \epsilon_2 ... \epsilon_k \omega([\phi_1, \phi_k]) + \epsilon_1 \epsilon_2^2 ... \epsilon_k \omega([\phi_2, \phi_k]) + ... + \epsilon_1 \epsilon_2^2 ... \epsilon_k^2 \omega([\phi_k, \phi_k]) + \epsilon_2 \omega([\mu, \phi_2])$$
$$+ \epsilon_2 \epsilon_3 \omega([\mu, \phi_3]) ... + \epsilon_2 \epsilon_3 ... \epsilon_k \omega([\mu, \phi_k]) = 0.$$

As the coefficients which appear in this equation are each one in one $\mathfrak{m}^p$, we have necessarily

$$\omega([\phi_1, \phi_k]) = ... = \omega([\phi_k, \phi_k]) = \omega([\mu, \phi_2]) = ... = \omega([\mu, \phi_k]) = 0$$

and this for every linear form ω of which kernel contains V. This proves the lemma.

From this lemma and using the descending sequence

$$\mathfrak{m} \supset \mathfrak{m}^{(2)} \supset ... \supset \mathfrak{m}^{(p)} ...$$

where $\mathfrak{m}^{(p)}$ is the ideal generated by the products $a_1 a_2 ... a_p$, $a_i \in \mathfrak{m}$ of length p, we obtain :

Proposition 3 *If*
$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i \in I} \epsilon_1 \epsilon_2 ... \epsilon_i \phi_i$$
is a valued deformation of μ, then we have the following linear system :

$$\begin{cases} \delta\phi_1 = 0 \\ \delta\phi_2 = a_{11}^2[\phi_1, \phi_1] \\ \delta\phi_3 = a_{12}^3[\phi_1, \phi_2] + a_{22}^3[\phi_1, \phi_1] \\ ... \\ \delta\phi_k = \sum_{1 \leq i \leq j \leq k-1} a_{ij}^k[\phi_i, \phi_j] \\ [\phi_1, \phi_k] = \sum_{1 \leq i \leq j \leq k-1} b_{ij}^1[\phi_i, \phi_j] \\ .... \\ [\phi_{k-1}, \phi_k] = \sum_{1 \leq i \leq j \leq k-1} b_{ij}^{k-1}[\phi_i, \phi_j] \end{cases}$$

where $\delta\phi_i = [\mu, \phi_i]$ is the coboundary operator of the Chevalley cohomology of the Lie algebra $\mathfrak{g}$.

Let us suppose that the dimension of V is the maximum $k(k-1)/2$. In this case we have no other relations between the generators of V and the previous linear system is complete, that is the equation of deformations does not give other relations than the relations of this system. The following result shows that, in this case, such deformation is isomorphic, as Lie algebra laws, to a "polynomial" valued deformation.

Proposition 4 *Let be $\mu_{\mathfrak{g}'_A}$ a valued deformation of μ such that*

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i=1,...,k} \epsilon_1 \epsilon_2 ... \epsilon_i \phi_i$$

and dimV=k(k-1)/2. Then there exists an automorphism of $\mathbb{K}^n \otimes \mathfrak{m}$ of the form $f = Id \otimes P_k(\epsilon)$ with $P_k(X) \in \mathbb{K}^k[X]$ satisfying $P_k(0) = 1$ and $\epsilon \in \mathfrak{m}$ such that the valued deformation $\mu_{\mathfrak{g}''_A}$ defined by

$$\mu_{\mathfrak{g}''_A}(X, Y) = f^{-1}(\mu_{\mathfrak{g}'_A}(f(X), f(Y)))$$

is of the form
$$\mu_{\mathfrak{g}_A"} = \mu_{\mathfrak{g}_A} + \sum_{i=1,...,k} \epsilon^i \varphi_i$$

where $\varphi_i = \sum_{j \leq i} \phi_j$.

Proof. Considering the Jacobi equation

$$[\mu_{\mathfrak{g}'_A}, \mu_{\mathfrak{g}'_A}] = 0$$

and writting that dim$V$=$k(k-1)/2$, we deduce that there exist polynomials $P_i(X) \in \mathbb{K}[X]$ of degree i such that

$$\epsilon_i = a_i \epsilon_k \frac{P_{k-i}(\epsilon_k)}{P_{k-i+1}(\epsilon_k)}$$

with $a_i \in \mathbb{K}$. Then we have

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i=1,\dots,k} a_1 a_2 ... a_i (\epsilon_k)^i \frac{P_{k-i}(\epsilon_k)}{P_k(\epsilon_k)} \phi_i.$$

Thus

$$P_k(\epsilon_k)\mu_{\mathfrak{g}'_A} = P_k(\epsilon_k)\mu_{\mathfrak{g}_A} + \sum_{i=1,\dots,k} a_1 a_2 ... a_i (\epsilon_k)^i P_{k-i}(\epsilon_k)\phi_i.$$

If we write this expression according the increasing powers we obtain the announced expression. $\square$

Let us note that, for such deformation we have

$$\begin{cases} \delta\varphi_2 + [\varphi_1, \varphi_1] = 0 \\ \delta\varphi_3 + [\varphi_1, \varphi_2] = 0 \\ \dots \\ \delta\varphi_k + \sum_{i+j=k} [\varphi_i, \varphi_j] = 0 \\ \sum_{i+j=k+s} [\varphi_i, \varphi_j] = 0. \end{cases}$$

3.3 Particular case : one-parameter deformations of Lie algebras

In this section the valuation ring A is $\mathbb{K}[[t]]$. Its maximal ideal is $t\mathbb{K}[[t]]$ and the residual field is $\mathbb{K}$. Let $\mathfrak{g}$ be a $\mathbb{K}$- Lie algebra. Consider $\mathfrak{g} \otimes A$ as an A-algebra and let be $\mathfrak{g}'_A$ a valued deformation of $\mathfrak{g}$. The bracket $[,]_t$ of this Lie algebra satisfies

$$[X,Y]_t = [X,Y] + \sum t^i \phi_i(X,Y).$$

Considered as a valued deformation with base $\mathbb{K}[[t]]$, this bracket can be written

$$[X,Y]_t = [X,Y] + \sum_{i=1}^{i=k} c_1(t)...c_i(t)\psi_i(X,Y)$$

where $(\psi_1, ..., \psi_k)$ are linearly independent and $c_i(t) \in t\mathbb{C}[[t]]$. As $\phi_1 = \psi_1$, this bilinear map belongs to $Z^2(\mathfrak{g}, \mathfrak{g})$ and we find again the classical result of Gerstenhaber. Let V be the $\mathbb{K}$-vector space generated by $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$, $i, j = 1, ..., k-1$, μ being the law of $\mathfrak{g}$. If $\dim V = k(k-1)/2$ we will say that one-parameter deformation $[,]_t$ is of maximal rank.

Proposition 5 *Let*

$$[X,Y]_t = [X,Y] + \sum t^i \phi_i(X,Y)$$

be a one-parameter deformation of $\mathfrak{g}$. *If its rank is maximal then this deformation is equivalent to a polynomial deformation*

$$[X,Y]'_t = [X,Y] + \sum_{i=1,\dots,k} t^i \varphi_i$$

with $\varphi_i = \sum_{j=1,\dots,i} a_{ij}\psi_j$.

Corollary 1 *Every one-parameter deformation of maximal rank is equivalent to a local non valued deformation with base the local algebra* $\mathbb{K}[t]$.

Recall that the algebra $\mathbb{K}[t]$ is not an algebra of valuation. But every local ring is dominated by a valuation ring. Then this corollary can be interpreted as saying that every deformation in the local algebra $\mathbb{C}[t]$ of polynomials with coefficients in $\mathbb{C}$ is equivalent to a "classical"-Gerstenhaber deformation with maximal rank.

4 Deformations of the enveloping algebra of a rigid Lie algebra

4.1 Valued deformation of associative algebras

Let us recall that the category of $\mathbb{K}$-associative algebras is a monoidal category.

Definition 2 *Let* $\mathfrak{a}$ *be a* $\mathbb{K}$*-associative algebra and* A *an* $\mathbb{K}$*-algebra of valuation of such that the residual field* $\frac{A}{\mathfrak{m}}$ *is isomorphic to* $\mathbb{K}$ *(or to a subfield* $\mathbb{K}'$ *of* $\mathbb{K}$*). A valued deformation of* $\mathfrak{a}$ *with base* A *is an* A*-associative algebra* $\mathfrak{a}'_A$ *such that the underlying* A*-module of* $\mathfrak{a}'_A$ *is* $\mathfrak{a}_A$ *and that*

$$(X.Y)_{\mathfrak{a}'_A} - (X.Y)_{\mathfrak{a}_A}$$

belongs to the $\mathfrak{m}$*-quasi-module* $\mathfrak{a} \otimes \mathfrak{m}$ *where* $\mathfrak{m}$ *is the maximal ideal of* A.

The classical one-parameter deformation is a valued deformation. As in the Lie algebra case we can develop the decomposition of a valued deformation. It is sufficient to change the Lie bracket by the associative product and the Chevalley cohomology by the Hochschild cohomology.

The most important example concerning valued deformations of associative algebras is those of the associative algebra of smooth fonctions of a manifold. But we will be interested here by associative algebras that are the enveloping algebras of Lie algebras. More precisely, what can we say about the valued deformations of the enveloping algebra of a rigid Lie algebra?

4.2 Complex rigid Lie algebras

In this section we suppose that $\mathbb{K} = \mathbb{C}$. Let $\mathcal{L}_n$ be the algebraic variety of structure constants of n-dimensional complex Lie algebra laws. The basis of $\mathbb{C}^n$ being fixed, we can identify a law with its structure constants. Let us consider the action of the linear group $Gl(n, \mathbb{C})$ on $\mathcal{L}_n$:

$$\mu'(X, Y) = f^{-1}\mu(f(X), f(Y)).$$

We denote by $\mathcal{O}(\mu)$ the orbit of μ.

Definition 3 *The law* $\mu \in \mathcal{L}_n$ *is called rigid if* $\mathcal{O}(\mu)$ *is Zariski-open in* $\mathcal{L}_n$.

Let $\mathfrak{g}$ be a n-dimensional complex Lie algebra with product μ and $\mathfrak{g}_A$ a valued deformation with base A. As before $\mathcal{F}_A$ is the field of fractions of A.

Definition 4 *Let A be a valued $\mathbb{C}$-algebra. We say that $\mathfrak{g}$ is A-rigid if for every valued deformation $\mathfrak{g}'_A$ of $\mathfrak{g}_A$ there exists a $\mathcal{F}_A$-linear isomorphism between $\mathfrak{g}'_A$ and $\mathfrak{g}_A$.*

Let $\mu_{\mathfrak{g}'_A}$ be a valued deformation of $\mu_{\mathfrak{g}_A}$. If we write $\mu_{\mathfrak{g}'_A} - \mu_{\mathfrak{g}_A} = \phi$, then $\phi(X,Y) \in \mathfrak{g}\otimes\mathfrak{m}$ for all $X,Y \in \mathfrak{g}\otimes A$. If $\mu_{\mathfrak{g}_A}$ is rigid, there exits $f \in Gl_n(\mathfrak{g}\otimes\mathcal{F}_A)$ such that

$$f^{-1}(\mu_{\mathfrak{g}'_A}(f(X),f(Y))) = \mu_{\mathfrak{g}_A}(X,Y).$$

Thus

$$\mu_{\mathfrak{g}_A}(f(X),f(Y)) - f(\mu_{\mathfrak{g}_A}(X,Y)) = \phi(f(X),f(Y)).$$

As $\mathfrak{g}_A$ is invariant by f, $\phi(f(X),f(Y)) \in \mathfrak{g}\otimes\mathfrak{m}$. So we can decompose f as $f = f_1 + f_2$ with $f_1 \in Aut(\mathfrak{g}_A)$ and $f_2 : \mathfrak{g}_A \to \mathfrak{g}\otimes\mathfrak{m}$. Let f' be $f' = f \circ f_1^{-1}$. Then

$$f'^{-1}(\mu_{\mathfrak{g}'_A}(f'(X),f'(Y))) = \mu_{\mathfrak{g}_A}(X,Y)$$

and $f' = Id + h$ with $h : \mathfrak{g}_A \to \mathfrak{g}\otimes\mathfrak{m}$. Thus we have proved

Lemma 3 *If $\mu_{\mathfrak{g}_A}$ is A-rigid for every valued deformation $\mu_{\mathfrak{g}'_A}$ there exits $f \in Gl_n(\mathfrak{g}\otimes\mathcal{F}_A)$ of the type $f = Id + h$ with $h : \mathfrak{g}_A \to \mathfrak{g}\otimes\mathfrak{m}$ such that*

$$f^{-1}(\mu_{\mathfrak{g}'_A}(f(X),f(Y))) = \mu_{\mathfrak{g}_A}(X,Y)$$

for every $X,Y \in \mathfrak{g}_A$.

Remark. If $f = Id + h$ then $f^{-1} = Id + k$. As $\mathfrak{g}_A$ is invariant by f, the linear map k satisfies $k : \mathfrak{g}_A \to \mathfrak{g}\otimes\mathfrak{m}$.

Theorem 4 *If the residual field of the valued ring is isomorphic to $\mathbb{C}$ then the notions of A-rigidity and of rigidity are equivalent.*

Proof. Let us suppose that for every valued algebra of residual field $\mathbb{C}$, the Lie algebra $\mathfrak{g}$ is A-rigid. We will consider the following special valued algebra: let $\mathbb{C}^*$ be non standard extension of $\mathbb{C}$ in the Robinson sense ([Ro]). If $\mathbb{C}_l$ is the subring of non-infinitely large elements of $\mathbb{C}^*$ then the subring $\mathfrak{m}$ of infinitesimals is the maximal ideal of $\mathbb{C}_l$ and $\mathbb{C}_l$ is a valued ring. Let us consider $A = \mathbb{C}_l$. In this case we have a natural embedding of the variety of A-Lie algebras in the variety of $\mathbb{C}$-Lie algebras. Up this embedding (called the transfert principle in the Robinson theory), the set of A-deformations of $\mathfrak{g}_A$ is an infinitesimal neighbourhood of $\mathfrak{g}$ contained in the orbit of $\mathfrak{g}$. Then $\mathfrak{g}$ is rigid. □

Examples. If $A = \mathbb{C}[[t]]$ then $\mathbb{K}' = \mathbb{C}$ and we find again the classical approach to the rigidity. We have another example, yet used in the proof of Theorem 3, considering a non standard extension $\mathbb{C}^*$ of $\mathbb{C}$. In this context the notion of rigidity has been developed in [A] (such a deformation is called perturbation). This work has allowed to classifiy complex finite dimensional rigid Lie algebras up the dimension eight.

4.3 Deformation of the enveloping algebra of a Lie algebra

Let $\mathfrak{g}$ be a finite dimensional $\mathbb{K}$-Lie algebra and $\mathcal{U}(\mathfrak{g})$ its enveloping algebra. In this section we consider a particular valued deformation of $\mathcal{U}(\mathfrak{g})$ corresponding to the valued algebra $\mathbb{K}[[t]]$. In [P], the following result is proved:

Proposition 6 *If $\mathfrak{g}$ is not rigid then $\mathcal{U}(\mathfrak{g})$ is not $\mathbb{K}[[t]]$-rigid.*

Recall that if the Hochschild cohomology $H^*(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g}))$ of $\mathcal{U}(\mathfrak{g})$ satisfies

$$H^2(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) = 0$$

then $\mathcal{U}(\mathfrak{g})$ is $\mathbb{K}[[t]]$-rigid. By the Cartan-Eilenberg theorem, we have that

$$H^n(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) = H^n(\mathfrak{g},\mathcal{U}(\mathfrak{g})).$$

Theorem 5 (P) *Let $\mathfrak{g}$ be a rigid Lie algebra. If $H^2(\mathfrak{g},\mathbb{C}) \neq 0$, then $\mathcal{U}(\mathfrak{g})$ is not $\mathbb{K}[[t]]$-rigid.*

¿From [C] and [A] every solvable complex Lie algebra decomposes as $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is the niladical of $\mathfrak{g}$ and $\mathfrak{t}$ a maximal exterior torus of derivations in the Malcev sense. Recall that the rank of $\mathfrak{g}$ is the dimension of $\mathfrak{t}$. A direct consequence of Petit's theorem is that for every complex rigid Lie algebra of rank equal or greater than 2 its envelopping algebra is not rigid.

Theorem 6 *Let $\mathfrak{g}$ be a complex finite dimensional rigid Lie algebra of rank* 1*. Then*

$$dim\ H^2(\mathfrak{g},\mathbb{C}) = 0$$

if and only if 0 *is not a root of the nilradical* $\mathfrak{n}$.

Proof. Suppose first that 0 is not a root of $\mathfrak{n}$ that is for every $X \neq 0 \in \mathfrak{t}$, 0 is not an eigenvalue of the semisimple operator adX. Let θ be in $Z^2(\mathfrak{g},\mathbb{C})$. Let $(X, Y_i)_{i=1,\dots,n-1}$ a basis of $\mathfrak{n}$ adapted to the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$. In particular we have

$$[X, Y_i] = \lambda_i Y_i$$

with $\lambda_i \in \mathbb{N}^*$ for all $i = 1, .., n-1$ ([A.G]). As $d\theta = 0$ we have for all $i, j = 1, ..., n-1$

$$d\theta(X, Y_i, Y_j) = \theta(X, [Y_i, Y_j]) + \theta(Y_i, [Y_j, X]) + \theta(Y_j, [X, Y_i]) = 0,$$

for all $1 \leq i, j \leq k-1$, and this gives

$$(\lambda_i + \lambda_j)\theta(Y_i, Y_j) = \theta(X, [Y_i, Y_j]). \qquad (1)$$

If $(\lambda_i + \lambda_j)$ is not a root, then $[Y_i, Y_j] = 0$ and this implies that $\theta(Y_i, Y_j) = 0$. If not, $(\lambda_i + \lambda_j) = \lambda_k$ is a root. Let us note $Y_k^1, ..., Y_k^{n_k}$ the eigenvectors of the chosen basis corresponding to the root λ_k. We have

$$[Y_i, Y_j] = \sum_{s=1}^{n_k} a_{ij}^s(k) Y_k^s.$$

Let us consider the dual basis $\{\omega_0, \omega_1, ..., \omega_{n-1}\}$ of $\{X, Y_1, ..., Y_{n-1}\}$. We have

$$d\omega_k^s = \lambda_k \omega_0 \wedge \omega_k^s + \sum_{l,m} a_{lm}^s(k) \omega_l \wedge \omega_m$$

where the pairs (l, m) are such that $\lambda_l + \lambda_m = \lambda_k$. Then we deduce from (1)

$$\sum a_{ij}^s(k)\theta(X, Y_k^s) - \lambda_k \theta(Y_i, Y_j) = 0.$$

Let us fix λ_k. If we write

$$\begin{aligned} \theta = \quad & \textstyle\sum_{l,m,\lambda_l+\lambda_m=\lambda_k} b_{lm}(k)\omega_l \wedge \omega_m + \sum_{r,s,\lambda_r+\lambda_s \neq \lambda_k} c_{rs}(k)\omega_r \wedge \omega_s \\ & + \textstyle\sum_k \sum_{s=1}^{n_k} \beta_k^s \omega_0 \wedge \omega_k^s \end{aligned}$$

then, for every pair (i, j) such that $\lambda_i + \lambda_j = \lambda_k$, (1) gives

$$-\lambda_k b_{ij}(k) + \sum_{s=1}^{n_k} a_{ij}^s(k)\beta_k^s = 0$$

and

$$b_{ij}(k) = \sum_{s=1}^{n_k} \frac{a_{ij}^s(k)}{\lambda_k}\beta_k^s.$$

The expression of θ becomes

$$\begin{aligned} \theta = \quad & \textstyle\sum_k \sum_{s=1}^{n_k} \beta_k^s \omega_0 \wedge \omega_k^s + \sum_{i,j,\lambda_i+\lambda_j=\lambda_k} \frac{a_{ij}^s(k)}{\lambda_k}\beta_k^s \omega_i \wedge \omega_j \\ & + \textstyle\sum_{r,s,\lambda_r+\lambda_s \neq \lambda_k} c_{rs}(k)\omega_r \wedge \omega_s. \end{aligned}$$

Thus

$$\begin{aligned} \theta \quad = \quad & \textstyle\sum_k (\frac{1}{\lambda_k} \sum_{s=1}^{n_k} \beta_k^s (\lambda_k \omega_0 \wedge \omega_k^s + \sum_{i,j,\lambda_i+\lambda_j=\lambda_k} a_{ij}^s(k)\omega_i \wedge \omega_j) \\ & + \textstyle\sum_{r,s,\lambda_r+\lambda_s \neq \lambda_k} c_{rs}(k)\omega_r \wedge \omega_s \\ = \quad & \textstyle\sum_s \beta_k^s d\omega_k^s + \sum_{k' \neq k} \sum_{s=1}^{n_{k'}} \beta_{k'}^s \omega_0 \wedge \omega_{k'}^s \\ & + \textstyle\sum_{r,s,\lambda_r+\lambda_s \neq \lambda_k} c_{rs}(k)\omega_r \wedge \omega_s. \end{aligned}$$

If we continue this method for all the non simple roots (that is which admit a decomposition as sum of two roots, we obtain the heralded result.
For the converse, if 0 is a root, then the cocycle

$$\theta = \omega_0 \wedge \omega_0'$$

where ω_0' is related with the eigenvector associated to the root 0 is not integrable.
□

Remark. It is easy to verify that every solvable rigid Lie algebra of rank greater or equal to 2 cannot have 0 as root. Likewise every solvable rigid Lie algebra of rank 1 and of dimension less than 8 has not 0 as root. This confirm in small dimension the following conjecture [Ca]:
If $\mathfrak{g}$ is a complex solvable finite dimensional rigid Lie algebra of rank 1, then 0 is not a root.

Consequences. If $H^2(\mathfrak{g},\mathbb{C}) \neq 0$, there exits $\theta \in \wedge^2\mathfrak{g}^*$ such that $[\theta]_{H^2} \neq 0$. If $rg(\mathfrak{g}) \geq 2$, then we can suppose that $\theta \in \wedge^2\mathfrak{t}^*$ and ω defines a non trivial deformation of $\mathcal{U}(\mathfrak{g})$. If $rg(\mathfrak{g}) = 1$, then 0 is not a root of $\mathfrak{t}$. The Hochschild Serre sequence gives:

$$\begin{aligned} H^2_{CE}(\mathfrak{g},\mathcal{U}(\mathfrak{g})) &= (\wedge^2\mathfrak{t}^* \otimes Z(\mathcal{U}(\mathfrak{g}))) \oplus (\mathfrak{t}^* \otimes H^1_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t) \oplus H^2_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t \\ &= \mathfrak{t}^* \otimes H^1_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t \oplus H^2_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t \end{aligned}$$

But from the previous proof, if θ is a non trivial 2-cocycle of $Z^2_{CE}(\mathfrak{g},\mathbb{C})$ then $i(X)\theta \neq 0$ for every $X \in \mathfrak{t}$, $X \neq 0$. The 1-form $\omega = i(X)\theta$ is closed. Then θ corresponds to a cocycle belonging to $\mathfrak{t}^* \otimes Z^1_{CE}(\mathfrak{n},\mathcal{U}(\mathfrak{g}))^t$ and defines a deformation of $\mathcal{U}(\mathfrak{g})$.

Theorem 7 *Let $\mathfrak{g}$ be a solvable complex rigid Lie algebra. If its rank is greater or equal to 2 or if the rank is 1 and 0 is a root, then the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is not rigid.*

Remark. In [P], T.Petit discribes some examples of deformations of the enveloping algebra of a rigid Lie algebra $\mathfrak{g}$ in small dimension and satisfying $H^2_{CE}(\mathfrak{g},\mathbb{C}) = 0$. For this, he shows that every deformation of the linear Poisson structure on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ induces a non trivial deformation of $\mathcal{U}(\mathfrak{g})$. This reduces the problem to find non trivial deformation of the linear Poisson structure.

4.4 Poisson algebras

Recall that a Poisson algebra $\mathcal{P}$ is a (commutatitive) associative algebra endowed with a second algebra law satisfying the Jacobi's identity and the Liebniz rule

$$[a, bc] = b[a, c] + [a, b]c$$

for all $a, b, c \in \mathcal{P}$. The tensor product $\mathcal{P}_1 \otimes \mathcal{P}_2$ of two Poisson algebras is again a Poisson algebra with the following associative and Lie products on $\mathcal{P}_1 \otimes \mathcal{P}_2$:

$$(a_1 \otimes a_2).(b_1 \otimes b_2) = (a_1.b_1) \otimes (a_2.b_2)$$

$$[(a_1 \otimes a_2), (b_1 \otimes b_2)] = ([a_1, b_1] \otimes a_2.b_2) + (a_1.b_1 \otimes [a_2, b_2])$$

for all $a_1, b_1 \in \mathcal{P}_1, a_2, b_2 \in \mathcal{P}_2$. We can verify easily that these laws satisfy the Leibniz rule.

Every commutative associative algebra has a natural Poisson structure, putting $[a, b] = ab - ba = 0$. Then the tensor product of a Poisson algebra by a valued algebra is as well a Poisson algebra. In this context we have the notion of valued deformation. For example, if we take as valued algebra the algebra $\mathbb{C}[[t]]$, then the Poisson structure of $\mathcal{P} \otimes \mathbb{C}[[t]]$ is given by

$$(a_1 \otimes a_2(t)).(b_1 \otimes b_2(t)) = (a_1.b_1) \otimes (a_2(t).b_2(t))$$

$$[(a_1 \otimes a_2(t)), (b_1 \otimes b_2(t))] = [a_1, b_1] \otimes a_2(t).b_2(t)$$

because $\mathbb{C}[[t]]$ is a commutative associative algebra.

Remark. As we have a tensorial category it is natural to look if we can define a Brauer Group for Poisson algebras. As the associative product corresponds to the classical tensorial product of associative algebras, we can consider only Poisson algebras which are finite dimensonal simple central algebras. The matrix algebras $M_n(\mathbb{C})$ are Poisson algebras. Then, considering the classical equivalence relation for define the Brauer Group, the class of matrix algebra constitutes an unity. Now the opposite algebra A^{op} also is a Poisson algebra. In fact the associative product is given by $a._{op}b = ba$ and the Lie bracket by $[a,b]_{op} = ba - ab$. Thus

$$[a, b._{op}c]_{op} = [a, cb]_{op} = cba - acb$$

$$b._{op}[a,c]_{op} + [a,b]_{op}._{op}c = cab - acb + cba - cab = cba - acb$$

this gives the Poisson structure of A^{op}. The opposite algebra A^{op} is, modulo the equivalence relation, the inverse of A.

5 Deformations of non associative algebras

5.1 Lie-admissible algebras

In [R], special classes of non-associative algebras whose laws give a Lie bracket by anticommutation are presented. If A is a $\mathbb{K}$-algebra, we'll note by a_μ the associator of its law μ:

$$a_\mu(X,Y,Z) = \mu(\mu(X,Y),Z) - \mu(X,\mu(Y,Z)).$$

Let Σ_n be the n-symmetric group.

Definition 5 *An algebra A is Lie-admissible if*

$$\sum_{\sigma \in \Sigma_3} a_\mu \circ \sigma = 0,$$

with $\sigma(X_1, X_2, X_3) = (X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, X_{\sigma^{-1}(3)})$.

Let G be a sub-group of Σ_3. The Lie-admissible algebra (A, μ) is called G-associative if

$$\sum_{\sigma \in G} a_\mu \circ \sigma = 0.$$

Let us note that this last identity implies the Lie-admissible identity. If G is the trivial sub-group, then the corresponding class of G-associative algebras is nothing other that the associative algebras but for all other sub-group, we have non-assotiative algebras. For example, if $G =< Id, \tau_{23} >$ we obtain the pre-Lie algebras ([G]). If $G =< Id, \tau_{12} >$ the corresponding algebras are the Vinberg algebras.

5.2 External tensor product

It is easy to see that each one of the categorie of G-associative algebras is not tensorial exept for $G =< Id >$. But in [G.R] we have proved the following result:

Theorem 8 *For every sub-group G of Σ_3, let $\mathcal{G} - \mathcal{A}ss$ be the associated operad and $\mathcal{G} - \mathcal{A}ss^!$ its dual operad. For every $\mathcal{G} - \mathcal{A}ss$-algebra A and $\mathcal{G} - \mathcal{A}ss^!$-algebra B, $A \otimes B$ is a $\mathcal{G} - \mathcal{A}ss$-algebra.*

Recall the structure of $\mathcal{G} - \mathcal{A}ss^!$-algebras.

-If $G =< Id >$, then the $\mathcal{G} - \mathcal{A}ss^!$ are the associative algebras,

-If $G =< Id, \tau_{12} >$, then the $\mathcal{G} - \mathcal{A}ss^!$ are the associative algebras satisfying $abc = bac$,

-If $G =< Id, \tau_{23} >$, then the $\mathcal{G} - \mathcal{A}ss^!$ are the associative algebras satisfying $abc = acb$,

-If $G =< Id, \tau_{13} >$, then the $\mathcal{G} - \mathcal{A}ss^!$ are the associative algebras satisfying $abc = cab$,

-If $G = \mathcal{A}_{\ni}$, then the $\mathcal{G} - \mathcal{A}ss^!$ are the associative algebras satisfying $abc = bac = cab$,

-If $G = \Sigma_3$, then the $\mathcal{G} - \mathcal{A}ss^!$ are the 3-commutative associative algebras ([R]).

5.3 Valued deformation of G-associative algebras

Definition 6 *Let $\mathcal{A}$ be a $\mathcal{G} - \mathcal{A}ss$ algebra and A a valued $\mathcal{G} - \mathcal{A}ss^!$ algebra. Let μ be the law of the A-algebra $\mathcal{A} \otimes \mathcal{A}$. A deformation of $\mathcal{A}$ of basis A is a A-algebra whose law μ' satisfies:*

$$\mu'(X, Y) - \mu(X, Y) \in \mathcal{A} \otimes \mathfrak{m}$$

for every $X, Y \in \mathcal{A}$, where $\mathfrak{m}$ is the maximal two-sided ideal of A.

If A is a commutative associative algebra, then it's a $\mathcal{G} - \mathcal{A}ss^!$-algebra for every G. We can study the valued deformations in this particular case.

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References

[A] Ancochea Bermudez J.M,. On the rigidity of solvableLie algebras in *Deformation Theiry of Algebras and Structures and Applications*, Kluwer Acad. Dordrecht (1988), 403-445.

[C] Carles R., Sur la structure des algèbres de Lie rigides. *Ann. Inst. Fourier*, **34** (1984), 65-82.

[Ca] Campoamor R., Invariants of solvable rigid Lie algebras up to dimension 8. *J. Phys. A: Math Gen.*, **35** (2002), 6293-6306.

[F] Fialowski A., Post, G., Versal deformation of the Lie algebra L_2. *J. Algebra* **236**,1, (2001), 93-109.

[G] Gerstenhaber M., On the deformations of ring and algebras, *Ann. Math.* **74**, 11, (1964), 59-103.

[G.R] Goze M., Remm E., On the categories of Lie-admissible. Preprint Mulhouse (2001) xxx RA/0210291 algebras,

[L.S] Lichtenbaum S., Schlessinger M., The cotangent complex of a morphism. Trans. Amer. Math. Soc. 128 1967 41-70.

[M.S] Markl, Martin; Stasheff, James D. Deformation theory via deviations. *J. Algebra* **170**, 1, (1994), 122-155.

[P] Petit T., Algèbres enveloppantes des algèbres de Lie rigides. Thèse Mulhouse 2001.

[R] Remm E., Opérades Lie-admissibles. *C.R.Acad. Sci. Paris*, ser I 334 (2002) 1047-1050.

[S] Schlessinger M., Functors of Artin rings. Trans. Amer. Math. Soc. 130 1968 208-222.