Some constructions of affine structures on nilpotent Lie algebras

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Abstract

In this paper we are interested in affine structures on nilpotent algebras.

As a symplectic algebra $\mathfrak{g}$ can be equipped with an affine structure, we study how to extend this affine structure to central extension of $\mathfrak{g}$, in particular for the nilpotent contact Lie algebras. We apply this to construct affine structures on 7-dimensional contact Lie algebras.

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Introduction

The problem of existence of flat affine connections on a manifold can be studied in the case of Lie groups. If the affine connection is also left invariant, the covariant operator gives on the Lie algebra $\mathfrak{g}$ associated to the Lie group a structure called affine structure which corresponds to nullity of torsion and curvature tensors. The existence problem of such structures in the nilpotent case was introduced by J. Milnor.

Y. Benoist gave the first example of nilpotent Lie algebra which cannot be equipped with such a structure. This leads to characterize nilpotent Lie algebras which can be equipped with affine structure ([2]). An important class of Lie algebra endowed with such a structure are symplectic Lie algebras. We can observe easily that the 11-dimensional Benoist Lie algebra corresponds to a Lie algebra with a contact structure. In the nilpotent case, any Lie algebra equipped with a contact structure can be obtained as a one-dimensional central extension of a symplectic algebra. This leads to examinate how to extend the affine structure of the symplectic algebra to the contact algebra. One part of this work deals wih this problem.

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1 Affine structures on Lie algebras

1.1 Generalities: Affine connections

Definition 1 *An affine connection on a manifold M is an operator ∇ which for every vector field X defines an endomorphism ∇_X of the space $\mathcal{D}^1(M)$ of vector fields on M satisfying the two conditions*

$$\begin{array}{lrcl} (1) & \nabla_{fX+gY} & = & f\nabla_X + g\nabla_Y; \\ (2) & \nabla_X(fY) & = & f\nabla_X(Y) + (Xf)Y \end{array}$$

for $f, g \in C^\infty(M)$, $X, Y \in \mathcal{D}^1(M)$.

If M is an Lie group G, then an affine connection ∇ is called left invariant if the connection ∇' given by:

$$\nabla'_X(Y) = (\nabla_{X^\Phi}(Y^\Phi))^{\Phi^{-1}}, \qquad X, Y \in \mathcal{D}^1(G)$$

satisfies

$$\nabla' = \nabla$$

for every left translation Φ. It is equivalent to say that the left translations are affine maps on the affine Lie group (G, ∇).
Recall that the torsion of the affine connection ∇ is the tensor T defined by

$$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y].$$

The curvature of ∇ is given by the tensor C defined by

$$C(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

In this work we consider the left invariant connection satisfying $T = 0$, $C = 0$.

1.2 Affine structures on Lie algebras

Let $\mathfrak{g}$ be the Lie algebra of the affine group (G, ∇). Since the operator ∇ is left invariant, it induces a bilinear map, always noted ∇ . We will write $\nabla(X,Y) = \nabla_X Y$ to keep the classical notation of connection.

Remark. The multiplication ∇ given by $\nabla(X,Y) = \nabla_X Y$ for every $X, Y \in \mathfrak{g}$ satisfies the following identities:

$$\begin{array}{ll} 1) & \nabla(X, \nabla(Y,Z) - \nabla(Y, \nabla(X.Z)) = \nabla(\nabla((X,Y),Z) - \nabla(\nabla((Y,X),Z) \\ 2) & \nabla(X,Y) - \nabla(Y,X) = [X,Y] \end{array}$$

for every $X, Y, Z \in \mathfrak{g}$. This identities translate the fact that curvature and torsion are equal to zero.

An algebra $(\mathbb{R}^n, \nabla)$ with a multiplication satisfying 1) is a Vinberg algebra (also called left symmetric algebra). Any left invariant affine connection on $\mathfrak{g}$ defines on $A(\mathfrak{g})$ a structure of Vinberg algebra whose associator is a Lie bracket equal to the one of $\mathfrak{g}$. This leads to the following definition:

Definition 2 *An bilinear map ∇ on the Lie algebra $\mathfrak{g}$ satisfying*

$$1) \quad \nabla_X Y - \nabla_Y X = [X,Y]$$

$$2) \quad \nabla_X\nabla_Y - \nabla_Y\nabla_X = \nabla_{[X,Y]}$$

is called an affine structure on $\mathfrak{g}$.

Definition 3 *An affine structure on $\mathfrak{g}$ is complete if the following endomorphisms*

$$\begin{array}{cccc} \theta_X : & A(\mathfrak{g}) & \rightarrow & A(\mathfrak{g}) \\ & Y & \mapsto & Y + \nabla_Y X \end{array}$$

are bijective for all $X \in A(\mathfrak{g})$, where $A(\mathfrak{g})$ is the underlying vectorspace of $\mathfrak{g}$.

This is equivalent to the following properties ([11]):
a)

$$\begin{array}{cccc} R_X : & A(\mathfrak{g}) & \rightarrow & A(\mathfrak{g}) \\ & Y & \mapsto & \nabla_Y X \end{array}$$

is nilpotent for all $X \in A(\mathfrak{g})$
b) $tr(R_X) = 0$ for all $X \in A(\mathfrak{g})$.

1.3 Classical examples

Let us begin recalling classical examples of nilpotent Lie algebras equipped with an affine structure:

- Nilpotent Lie algebras of dimension less or equal to 7 are equipped with an affine structure ([3]).
- 2-step, 3-step and 4-step nilpotent Lie algebras can be endowed with an affine structure ([15], [4]).
- Symplectic Lie algebras:
  Let ω be a symplectic form on $\mathfrak{g}$, that is, a 2-cocycle $\omega \in Z^2(\mathfrak{g})$ of maximal rank. In this case the operator ∇_X defined by:
  $$\omega(adX(Y), Z) = -\omega(Y, \nabla_X Z)$$
  equippes $\mathfrak{g}$ with an affine structure.
- Lie algebras with a regular derivation:
  A Lie algebra $\mathfrak{g}$ having regular derivation is necessarily nilpotent, and in this case there is a regular diagonalisable derivation. Let f be such a derivation. For all $X \in \mathfrak{g}$, the operator
  $$\nabla_X = f^{-1} \circ adX \circ f.$$
  is an affine structure on $\mathfrak{g}$ [15].

1.4 Case of Lie algebras with no regular derivation

Theorem 4 *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a derivation f whose restriction to the derived subalgebra $\mathcal{C}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is nonsingular, then $\mathfrak{g}$ admits an affine structure.*

Proof. Let $\widetilde{f}$ be the restriction of f to $\mathcal{C}^1\mathfrak{g}$. Let $\widetilde{g}$ be the inverse mapping of $\widetilde{f}$. We can extend $\widetilde{g}$ to an endomorphism g of $\mathfrak{g}$ such that $\mathcal{C}^1\mathfrak{g}$ is an invariant subspace of g. Let us consider ∇ defined by

$$\nabla_X Y = g \circ adX \circ f(Y)$$

It satisfies

$$\begin{aligned}\nabla_X(Y) - \nabla_Y(X) &= g \circ adX \circ f(Y) - g \circ adY \circ f(X) \\ &= g(f[X,Y])\end{aligned}$$

because f is a derivation. As g and f are inverse one of the other on the derivated algebra, we deduce

$$\nabla_X(Y) - \nabla_Y(X) = [X,Y].$$

Likewise

$$\begin{aligned}\nabla_X\nabla_Y(Z) - \nabla_Y\nabla_X(Z) &= g[X,[Y,f(Z)]] - g[Y,[X,f(Z)]] \\ &= -g[f(Z),[X,Y]] \\ &= \nabla_{[X,Y]}(Z) \qquad \blacksquare\end{aligned}$$

Application: *Every 7-dimensional graded Lie algebra can be equipped with an affine structure.* In fact the classification of 7-dimensional nilpotent Lie algebras ([8]) shows that if $\mathfrak{g}$ admits a singular derivation f, the Kernel of f doesnot meet $[\mathfrak{g},\mathfrak{g}]$. Then all these algebras admits affine structures.
Remarks.
1. The first case of nilpotent Lie algebra whose derived algebra has only non invertible semi-simple derivations f such that $Kerf \cap \mathfrak{g} \neq \{0\}$ is 8-dimensional.
2. In [2] is stated that each 7-dimensional nilpotent Lie algebra can be equipped with an affine structure. The previous remark gives a part of the proof. We complete the proof in the last section, studying the non graded case.

Proposition 5 *Each n-dimensional graded filiform Lie algebra can be equipped with an affine structure*

Proof. From [7] We can see that the only non charateristically nilpotent Lie algebra is of type C_n. But in this case the hypothesis of the previous theorem is satisfied and an affine structure can be constructed on any of these algebras by the explained construction.

1.5 Remark: Complete and non complete structures

An affine structure is complete if the right translation endomorphisms R_X are nilpotent for all X in $\mathfrak{g}$. We know that in the nilpotent case the existence of a non complete affine structure induces the existence of a complete one ([2]). In the abelian case, the existence problem is trivial. Therefore we focus on the classification problem up to an affine transformation. The complete affine connections have been completely defined on $\mathbb{R}^2$ and $\mathbb{R}^3$ ([12], [5]) and classified. The noncomplete case, always for these classes, has been described in ([10]). The determination of affine structure on abelian Lie algebras consists in finding all associative commutative real Lie algebras with or without unit. The others results for greater dimensions concern the existence of an infinity of non affinely equivalent structures on abelian Lie algebras up to dimension 6. But does it exist a class of nilpotent Lie algebras where all affine structures would be complete? Some thought during a time that some classes of filiform Lie algebras could satisfy this property. In [9] we show up that graded filiform nilpotent Lie algebras can be provided with non complete affine structures. This leads in particular to the consequence that the 3-dimensional Heisenberg algebra can be equipped with a non-complete affine structure:

$$\nabla_{X_1} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \alpha & \beta & 0 \end{pmatrix}, \quad \nabla_{X_2} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \beta - 1 & \alpha + 1 & 0 \end{pmatrix}, \quad \nabla_{X_3} = 0.$$

The affine representation is written:

$$\begin{pmatrix} a(x_1+x_2) & a(x_1+x_2) & 0 & x_1 \\ a(x_1+x_2) & a(x_1+x_2) & 0 & x_2 \\ \alpha x_1+(\beta-1)x_2 & \beta x_1+(\alpha+1)x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2 Nilpotent Lie algebras with a contact form

2.1 Nilpotent contact Lie algebras

Definition 6 *Let $\mathfrak{g}$ be an $(2p+1)$-dimensional algebra. A contact form on $\mathfrak{g}$ is a linear form $\omega \neq 0$ of $\mathfrak{g}^*$ such that $\omega \wedge (dw)^p \neq 0$. In this case $(\mathfrak{g}, \omega)$ or $\mathfrak{g}$ is called a contact Lie algebra.*

Proposition 7 ([6]) *Let $\mathfrak{g}$ be a contact nilpotent Lie algebra. Then the center $Z(\mathfrak{g})$ is one-dimensional.*

Proof. If $\mathfrak{g}$ is $(2p+1)$-dimensional and equipped with a contact form ω, $\dim Z(\mathfrak{g}) \leq 1$. This follows the fact that if we suppose that $\omega(Z(\mathfrak{g})) = 0$ then

$$\forall X \in Z(\mathfrak{g}) \quad d\omega(X,Y) = -\omega\,[X,Y] = 0.$$

Thus there exists X such that $\omega(X) = 0$ and $X \lrcorner d\omega = 0$, where $\lrcorner$ denotes the inner product. The vector X belongs to the characteristic subspace and $\omega \wedge d\omega^p = 0$. Thus $\omega(Z(\mathfrak{g})) \neq 0$ which proves that $\dim Z(\mathfrak{g}) \leq 1$. If moreover the Lie algebra $\mathfrak{g}$ is nilpotent then $\dim Z(\mathfrak{g}) = 1$ as the center of a nilpotent Lie algebra is never zero. ■

Corollary 8 *Let $\mathfrak{g}$ be a contact nilpotent Lie algebra. Then $\mathfrak{g}/Z(\mathfrak{g})$ is a symplectic Lie algebra.*

Thus any contact Lie algebra is a one-dimensional central extension of a symplectic Lie algebra:

$$0 \to V \to \mathfrak{g}_{2p+1} \to (\mathfrak{g}_{2p}, \theta) \to 0.$$

As any symplectic nilpotent Lie algebra can be equipped with an affine structure, we have

Corollary 9 *Any nilpotent contact Lie algebra is a one-dimensional central extension of an affine Lie algebra.*

2.2 Affines structures on nilpotent contact Lie algebras

Let $\widetilde{\mathfrak{g}}$ be a $(2p+1)$-dimensional nilpotent Lie algebra equipped with a contact form ω_c. Let $\mathfrak{g}$ be the symplectic algebra $\mathfrak{g} = \widetilde{\mathfrak{g}}/Z(\widetilde{\mathfrak{g}})$ and consider the set Φ of all 2-cocycle of $\mathfrak{g}$ of maximal rank. In particular $\pi^*(d\omega_c) = \theta_c \in \Phi$.

The Lie algebra $\widetilde{\mathfrak{g}}$ identified with $\mathfrak{g} \oplus V$ has the following brackets

$$[(X,\alpha),(Y,\lambda)]_{\widetilde{\mathfrak{g}}} = \Big([X,Y]_{\mathfrak{g}}, \theta_c(X,Y)\Big)$$

Let $\theta \in \Phi$ be a 2-cocycle of maximal rank and ∇ the affine structure coming from this symplectic form that is

$$\nabla_X Y = f(X)Y$$

where $f(X)$ is the following endomorphism:

$$\forall X, Y, Z \in \mathfrak{g} \qquad \theta\left(f(X)(Y), Z\right) = -\theta(Y, [X, Z]).$$

Let $\widetilde{\nabla} : \widetilde{\mathfrak{g}} \otimes \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ be an operator satisfying

$$(*) \begin{cases} \widetilde{\nabla}\left((X,0),(Y,0)\right) = \left(\nabla(X,Y), \varphi(X,Y)\right) \\ \widetilde{\nabla}\left((X,0),(0,\lambda)\right) = \widetilde{\nabla}\left((0,\lambda),(X,0)\right) \end{cases}$$

where φ is a bilinear map on $\mathfrak{g}$ such as

$$\varphi(X,Y) - \varphi(Y,X) = \theta_c(X,Y).$$

Lemma 10 *The operator $\widetilde{\nabla}$ satisfies the following identity:*

$$\widetilde{\nabla}\left((X,\alpha),(Y,\lambda)\right) - \widetilde{\nabla}\left((Y,\lambda),(X,\alpha)\right) = \left[(X,\alpha),(Y,\lambda)\right]_{\widetilde{\mathfrak{g}}}$$

Proof. We have for all $X, Y \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{K}$

$$\begin{aligned} &\widetilde{\nabla}\left((X,\alpha),(Y,\lambda)\right) - \widetilde{\nabla}\left((Y,\lambda),(X,\alpha)\right) \\ =&\left(\nabla(X,Y), \varphi(X,Y)\right) + \lambda\widetilde{\nabla}\left((X,0),(0,1)\right) + \alpha\widetilde{\nabla}\left((0,1),(Y,0)\right) \\ &+\alpha\lambda\widetilde{\nabla}\left((0,1),(0,1)\right) - \left(\nabla(Y,X), \varphi(Y,X)\right) - \alpha\widetilde{\nabla}\left((Y,0),(0,1)\right) \\ &-\lambda\widetilde{\nabla}\left((0,1),(X,0)\right) - \lambda\alpha\widetilde{\nabla}\left((0,1),(0,1)\right) \\ =&\Big([X,Y]_{\mathfrak{g}}, \theta_c(X,Y)\Big) \end{aligned}$$

But

$$\left[(X,\alpha),(Y,\lambda)\right]_{\widetilde{\mathfrak{g}}} = \Big([X,Y]_{\mathfrak{g}}, \theta_c(X,Y)\Big),$$

which implies that

$$\widetilde{\nabla}\left((X,\alpha),(Y,\lambda)\right) - \widetilde{\nabla}\left((Y,\lambda),(X,\alpha)\right) = \left[(X,\alpha),(Y,\lambda)\right]_{\widetilde{\mathfrak{g}}}. \qquad \blacksquare$$

Thus the operator $\widetilde{\nabla}$ is associated to a flat torsionfree connection on $\widetilde{\mathfrak{g}}$.

We can note that if $\widetilde{\nabla'}$ is another bilinear map on $\widetilde{\mathfrak{g}}$ such that $\pi^*\widetilde{\nabla'} = \nabla$, the nullity of the torsion of the linear connection associated to $\widetilde{\nabla'}$ implies that $\widetilde{\nabla'}$ satisfies the same conditions $(*)$. This justifies the choice of the conditions $(*)$.

As we want that $\widetilde{\nabla}$ defines an affine structure on $\mathfrak{g}$, we introduce, in order to study the curvature of the linear connection associated to $\widetilde{\nabla}$, the following application:

$$\begin{aligned} &C(\ (X,\alpha),(Y,\lambda),(Z,\rho)) \\ =&\widetilde{\nabla}\Big((X,\alpha), \widetilde{\nabla}\left((Y,\lambda),(Z,\rho)\right)\Big) - \widetilde{\nabla}\Big((Y,\lambda), \widetilde{\nabla}\left((X,\alpha),(Z,\rho)\right)\Big) \\ &-\widetilde{\nabla}\Big(\left[(X,\alpha),(Y,\lambda)\right]_{\widetilde{\mathfrak{g}}}, (Z,\rho)\Big). \end{aligned}$$

This gives:

$$
\begin{aligned}
&C((X,\alpha),(Y,\lambda),(Z,\rho))\\
&=\widetilde{\nabla}((X,\alpha),(\nabla(Y,Z),\varphi(Y,Z))+\rho\widetilde{\nabla}((Y,0),(0,1))+\lambda\widetilde{\nabla}((0,1),(Z,0))+\\
&\lambda\rho\widetilde{\nabla}((0,1),(0,1)))-\widetilde{\nabla}((Y,\lambda),(\nabla(X,Z),\varphi(X,Z))+\rho\widetilde{\nabla}((X,0),(0,1))+\\
&\alpha\widetilde{\nabla}((0,1),(Z,0))+\alpha\rho\widetilde{\nabla}((0,1),(0,1)))-\widetilde{\nabla}(([X,Y]_{\mu},\theta_c(X,Y)),(Z,\rho))
\end{aligned}
$$

Lemma 11 *The operator* $\widetilde{\nabla}$ *satisfies:*

1)

$$
\begin{aligned}
&C((X,0),(Y,0),(Z,0))=(0,\varphi(X,\nabla(Y,Z))-\varphi(Y,\nabla(X,Z))-\varphi([X,Y]_{\mu},Z))\\
&+\varphi(Y,Z)\widetilde{\nabla}((X,0),(0,1))-\varphi(X,Z)\widetilde{\nabla}((Y,0),(0,1))-\theta_c(X,Y)\widetilde{\nabla}((Z,0),(0,1))
\end{aligned}
$$

2)

$$
\begin{aligned}
&C((X,0),(0,1),(Y,0))=\widetilde{\nabla}\Big((X,0),\widetilde{\nabla}((Y,0),(0,1))\Big)\\
&-\widetilde{\nabla}((\nabla(X,Y),0),(0,1))-\varphi(X,Y)\widetilde{\nabla}((0,1),(0,1))
\end{aligned}
$$

3)

$$
C((0,1),(Y,0),(0,1))=\widetilde{\nabla}\Big((0,1),\widetilde{\nabla}((Y,0),(0,1))\Big)-\widetilde{\nabla}\Big((Y,0),\widetilde{\nabla}((0,1),(0,1))\Big)
$$

This follows directly when we develop the expressions.

Lemma 12 *If*

$$
C((X,0),(0,1),(Y,0))=0
$$

then $C((X,0),(Y,0),(0,1))=0$.

In fact

$$
\begin{aligned}
&C((X,0),(Y,0),(0,1))=\widetilde{\nabla}((X,0),\widetilde{\nabla}((Y,0),(0,1)))-\widetilde{\nabla}((Y,0),\widetilde{\nabla}((X,0),(0,1)))\\
&\qquad\qquad\qquad\qquad\quad-\widetilde{\nabla}(([X,Y],\theta_c(X,Y)),(0,1))\\
&=\widetilde{\nabla}((\nabla(X,Y),0),(0,1))+\varphi(X,Y)\widetilde{\nabla}((0,1),(0,1))-\widetilde{\nabla}((\nabla(Y,X),0),(0,1))\\
&\quad-\varphi(Y,X)\widetilde{\nabla}((0,1),(0,1))-\widetilde{\nabla}(([X,Y],\theta_c(X,Y)),(0,1))\\
&=\widetilde{\nabla}(([X,Y],0),(0,1))+\theta_c(X,Y)\widetilde{\nabla}((0,1),(0,1))-\widetilde{\nabla}([X,Y],0),(0,1))\\
&\quad-\theta_c(X,Y)\widetilde{\nabla}(0,1),(0,1))\\
&=0.
\end{aligned}
$$

Let us write some necessary conditions for the application C to be equal to zero. Let π be the canonical projection of $\widetilde{\mathfrak{g}}$ on $\mathfrak{g}$, that is :

$$
\pi(X,\alpha)=X.
$$

Let us identify $(X,0)$ with X which permits to consider $\mathfrak{g}$ as a vector subspace of $\widetilde{\mathfrak{g}}$. Let us denote V_X the vector definied by

$$
V_X=\pi(\widetilde{\nabla}((X,0),(0,1))).
$$

If $C=0$ we have:

$$
C((X,0),(Y,0),(Z,0))=0
$$

which implies that:

$$\varphi(Y,Z)\,V_X - \varphi(X,Z)\,V_Y - \theta_c(X,Y)\,V_Z = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

Let us suppose that we have $X \in \mathfrak{g}$ such that $V_X \neq 0$. Let us take a vector X satisfying this property. The orthogonal space for θ of the space $\mathbb{R}\{X\}$ is of codimension 1. Let $Y \in \mathbb{R}\{X\}^{\perp}$. Then $\theta_c(X,Y) = 0$ and

$$\varphi(Y,Z)\,V_X = \varphi(X,Z)\,V_Y, \quad Z \in \mathfrak{g}$$

If we can find a vector $Z \in \mathfrak{g}$ such as $\varphi(Y,Z) \neq 0$, we have that $V_Y \neq 0$ and in this case the non zero vectors V_X and V_Y are colinear. Let us assume that

$$V_X = \lambda_{X,Y} V_Y$$

then

$$\lambda_{X,Y}\varphi(Y,Z) = \varphi(X,Z)$$

for all $Z \in \mathfrak{g}$. Let us consider a basis $(X_1, ..., X_{2m})$ of $\mathfrak{g}$ in which the matrix of θ_c is reduced to the following form:

$$\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & 0 & \\ & & \ddots & & & \\ & & & & 0 & 1 \\ & 0 & & & -1 & 0 \end{pmatrix}$$

This shows that the matrix of φ is at least of rank 2 because $\varphi(X,Y) - \varphi(Y,X) = \theta_c(X,Y)$. For instance, if $\mathfrak{g}$ is 4-dimensional, the matrix of φ would be of the following form:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 - 1 & \alpha_6 & \alpha_7 & \alpha_8 \\ \alpha_3 & \alpha_7 & \alpha_{11} & \alpha_{12} \\ \alpha_4 & \alpha_8 & \alpha_{12} - 1 & \alpha_{16} \end{pmatrix}$$

We can suppose that $\varphi(X_1, \cdot) \neq 0$ (in other case, there would exist a vector X_i such that $\varphi(X_i, Z) \neq 0$ since φ is of non zero rank and it is sufficient to take a basis adapted to θ_c with X_i as first vector). Let Y belong to vector space $\{X_3, ..., X_{2n}\}$; we have that $\theta_c(X_1, Y) = 0$

$$\varphi(X_1, Z) = \lambda_{X_1,Y}\varphi(Y,Z) \qquad \text{for all } Z \in \mathfrak{g}$$

and

$$\lambda_{X_1,Y} \neq 0$$

Therefore the $(2m-2)$ last columns of the matrix of φ (associated to the chosen basis) are proportional to the first one. But we also have $\theta_c(X_2, X_3) = 0$ and

$$\varphi(X_2, Z) = \lambda_{X_2,X_3}\varphi(X_3, Z)$$

Similary we show that $\theta_c(X_1, X_3) = 0$ and

$$\varphi(X_1, Z) = \lambda_{X_1,X_3}\varphi(X_3, Z) \qquad \text{for all } Z \in \mathfrak{g}.$$

We deduce that $\varphi(X_2, Z) = \lambda\varphi(X_1, Z)$, which implies that φ is of rank 1. This is impossible and then we have $V_X = 0$.

Proposition 13 *Let $\widetilde{\mathfrak{g}}$ be a contact nilpotent Lie algebra. If the affine structure ∇ which is defined by a symplectic cocycle on $\mathfrak{g} = \widetilde{\mathfrak{g}}/Z(\widetilde{\mathfrak{g}})$ can be extended to an affine structure $\widetilde{\nabla}$ on $\widetilde{\mathfrak{g}}$, we have:*

$$\pi(\widetilde{\nabla}(X, T)) = 0$$

for all $X \in \mathfrak{g}$ and $T \in Z(\widetilde{\mathfrak{g}})$.

We have that for all vector X in $\mathfrak{g}$, $V_X = 0$ and $\widetilde{\nabla}((X, 0), (0, 1)) = (0, a_X)$. Then the equality $C((X, 0), (Y, 0), (Z, 0)) = 0$ implies that

$$\begin{aligned} &\varphi(X, \nabla(Y, Z)) - \varphi(Y, \nabla(X, Z)) - \varphi\left([X, Y]_\mu, Z\right) \\ &= -a_X\varphi(Y, Z) + a_Y\varphi(X, Z) + a_Z\theta_c(X, Y) \end{aligned}$$

Similarly $C((X, 0), (0, 1), (Y, 0)) = 0$ implies that

$$\widetilde{\nabla}((X, 0), (0, a_Y)) - \left(0, a_{\nabla(X,Y)}\right) - \varphi(X, Y)\,\widetilde{\nabla}((0, 1), (0, 1)) = 0.$$

This gives the following equation

$$\varphi(X, Y)\,\widetilde{\nabla}((0, 1), (0, 1)) = (a_Y a_X - a_{\nabla(X,Y)})(0, 1)$$

and

$$\varphi(Y, X)\,\widetilde{\nabla}((0, 1), (0, 1)) = (a_Y a_X - a_{\nabla(Y,X)})(0, 1)$$

if we permute the vectors X and Y. We combine this two equations to obtain:

$$\begin{aligned} \theta_c(X, Y)\widetilde{\nabla}((0, 1), (0, 1)) &= (a_{\nabla(Y,X)} - a_{\nabla(X,Y)})(0, 1) \\ &= a_{[X,Y]}(0, 1). \end{aligned}$$

This shows in particular that $\widetilde{\nabla}((0, 1), (0, 1)) = \rho(0, 1)$ and

$$\rho\theta_c(X, Y) = a_{[X,Y]}.$$

Finally $C((0, 1), (Y, 0), (0, 1)) = 0$ implies

$$a_Y\widetilde{\nabla}((0, 1), (0, 1)) = \widetilde{\nabla}\left((Y, 0), \widetilde{\nabla}((0, 1), (0, 1))\right)$$

thus

$$a_Y\widetilde{\nabla}((0, 1), (0, 1)) = \rho(0, a_Y).$$

This last equation is already satisfied.

Then let us suppose $\rho \neq 0$. In this case $a_{[X,Y]} \neq 0$ when $\theta_c(X, Y) \neq 0$. Let us take X in $Z(\mathfrak{g})$. As θ_c is of maximal rank, there is one Y such that $\theta_c(X, Y) \neq 0$. But $[X, Y] = 0$ implies $a_{[X,Y]} = 0$. This leads to contradiction.

Conclusion. As $\rho = 0$ we have that $\widetilde{\nabla}((0, 1), (0, 1)) = 0$. Then $a_{[X,Y]} = 0$ and the application $\alpha : \mathfrak{g} \to \mathbb{R}$ defined by $\alpha(X) = a_X$ gives an one-dimensional linear representation of $\mathfrak{g}$. We deduce

Theorem 14 *Let* $(\widetilde{\mathfrak{g}}, \omega_c)$ *be a nilpotent contact Lie algebra and* ∇ *an affine structure on the symplectic algebra* $\widetilde{\mathfrak{g}}/Z(\widetilde{\mathfrak{g}})$. *Let* $\widetilde{\nabla}$ *be the corresponding operator on* $\widetilde{\mathfrak{g}}$ *and* $\alpha : \mathfrak{g} \to \mathbb{R}$ *the deduced one-dimensional linear representation of* $\mathfrak{g}$.

When α *is the trivial representation,* $\widetilde{\nabla}$ *is an affine structure if and only if*

1) $\widetilde{\nabla}(U, (0,1)) = 0$ *for all* $U \in \widetilde{\mathfrak{g}}$.

2) φ *satisfies* $\varphi(X, \nabla(Y,Z)) - \varphi(Y, \nabla(X,Z)) - \varphi\left([X,Y]_\mu, Z\right) = 0$, *i.e. if it is a 2-cocycle for cohomology of the Vinberg algebra associated to* ∇ *with values in a trivial module.*

When α *is a non-trivial representation,* $\widetilde{\nabla}$ *is an affine structure if and only if*

1) $\widetilde{\nabla}((0,1),(0,1)) = 0$, $\widetilde{\nabla}((X,0),(0,1)) = 0$ *for all* $X \in Ker\alpha$.

2) $\varphi(X, \nabla(Y,Z)) - \varphi(Y, \nabla(X,Z)) - \varphi\left([X,Y]_\mu, Z\right) = \alpha(Z)\theta_c(X,Y)$ *for all* $X, Y \in Ker(\alpha)$.

3 Application: Affine structures on 7-dimensional characteristically nilpotent algebras equipped with a contact structure.

We use the list of 7-dimensional nilpotent Lie algebras given in [8]. From the previous section any graded 7-dimensional nilpotent Lie algebra admits an affine structure. Here we apply the construction of affine structure on contact Lie algebra for discribe such structure for any 7-dimensional characteristically nilpotent contact Lie algebra. From the classification [8], these Lie algebras are isomorphic to η_7^4, $\eta_7^{12}(\lambda)$, η_7^{14},η_7^{19} (this algebra is isomorphic to $\eta_7^{12}(\lambda = 0)$, this is a little mistake of [8]) , η_7^{21} and η_7^{28}. Then for each of these algebras we give the symplectic form on the quotient used in the construction of the affine structure and we describe the affine structure.

3.1 η_7^4

This algebra is equipped with the following affine structure which comes from the extension of the symplectic 6-dimensional algebra $\widetilde{\mathfrak{g}}/\mathbb{C}\{X_7\}$ whose symplectic form is:

$$\theta = 9(\omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_4) + 8(\omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4).$$

The affine structure is defined from

$$\nabla_{X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{81}{64} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{9}{8} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{81}{64} & -\frac{9}{8} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & \frac{1}{8} & 0 \end{pmatrix}, \nabla_{X_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{81}{64} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{8}{9} & 0 & 0 & 0 \\ a_2 & 0 & 0 & \frac{3}{8} & \frac{1}{3} & 0 & 0 \end{pmatrix},$$

We obtain the other ∇_{X_i} for $i = 2, .., 7$ from the relations $\nabla_{X_i} = [\nabla_{X_1}, \nabla_{X_{i-1}}]$.

3.2 $\eta_7^{12}(\lambda)$

Here the center is X_6.

For $\lambda \neq 0$, the symplectic form used is:

$$\theta = \lambda(\omega_1 \wedge \omega_7 + \omega_2 \wedge \omega_4) + 4(\omega_3 \wedge \omega_4 - \omega_2 \wedge \omega_5)$$

This algebra is equipped with the following affine structure:

$$\nabla_{X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{4} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda^2}{16} & \frac{\lambda}{4} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda^2}{8} & a_2 & \frac{\lambda^2}{8} & \frac{\lambda}{4} & 1 & 0 & 0 \end{pmatrix}, \quad \nabla_{X_2} = \begin{pmatrix} 0 & \frac{4}{\lambda} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda}{4} & -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda^2}{16} & -\frac{\lambda}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{\lambda} & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 & 0 & \frac{\lambda}{4} & 0 \end{pmatrix},$$

$\nabla_{X_i} = [\nabla_{X_1}, \nabla_{X_{i-1}}], i = 3, 4, 5, 6$ and $\nabla_{X_7} = [\nabla_{X_2}, \nabla_{X_3}] - \nabla_{X_5}$.

For $\lambda = 0$, the symplectic form used is:

$$\theta = 4\omega_1 \wedge \omega_4 + 2\sqrt{5}(\omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_4 + \omega_3 \wedge \omega_7) + \omega_3 \wedge \omega_4 - \omega_2 \wedge \omega_5$$

This algebra is equipped with the following affine structure:

$$\nabla_{X_1} = \begin{pmatrix} -\frac{\sqrt{5}}{12} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{6} & \frac{\sqrt{5}}{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{5}}{12} & 0 & -\frac{5}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{\sqrt{5}}{3} & 0 \\ \frac{2}{\sqrt{5}} & 1 & \frac{1}{2\sqrt{5}} & \frac{1}{24} & 0 & \frac{\sqrt{5}}{12} & 0 \\ -\frac{192}{35} & 0 & 0 & -\frac{2\sqrt{5}}{105} & \frac{5}{21} & 0 & 0 \end{pmatrix},$$

$$\nabla_{X_2} = \begin{pmatrix} \frac{1}{24} & -\frac{1}{48\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{5}}{12} & -\frac{1}{24} & 0 & 0 & 0 & 0 & 0 \\ -1 & \frac{1}{2\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{24} & 0 & \frac{\sqrt{5}}{12} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{5}}{12} & 0 & \frac{5}{6} & 0 \\ 1 & -\frac{1}{2\sqrt{5}} & 1 & -\frac{1}{48\sqrt{5}} & 0 & -\frac{1}{24} & 0 \\ 0 & 0 & -\frac{24\sqrt{5}}{175} & \frac{34}{105} & \frac{8}{21\sqrt{5}} & -\frac{4\sqrt{5}}{35} & 0 \end{pmatrix},$$

$\nabla_{X_i} = [\nabla_{X_1}, \nabla_{X_{i-1}}], i = 3, 4, 5, 6$ and $\nabla_{X_7} = [\nabla_{X_2}, \nabla_{X_3}] - \nabla_{X_5}$.

3.3 η_7^{14}

This center of this algebra is $\mathbb{C}\{X_6\}$. The symplectic form used to construct the affine structure on η_7^{14} is

$$\theta = \omega_1 \wedge \omega_5 - \frac{1}{6}\omega_1 \wedge \omega_7 + \frac{1}{6}\omega_2 \wedge \omega_4 + \omega_3 \wedge \omega_4 + \omega_3 \wedge \omega_7$$

$$\nabla_{X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & -6 & 0 \\ 0 & \frac{1}{36} & \frac{1}{6} & 1 & 0 & 1 & 0 \\ 0 & \frac{1}{6} & 1 & 6 & 0 & 6 & 0 \\ a_1 & a_2 & 0 & \frac{2}{3} & -3 & \frac{1}{2} & 0 \end{pmatrix}, \ \nabla_{X_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{36} & \frac{1}{36} & 0 & 1 & 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & \frac{2}{9} & -\frac{1}{2} & 0 & -\frac{4}{3} & 0 \end{pmatrix},$$

$\nabla_{X_i} = [\nabla_{X_1}, \nabla_{X_{i-1}}], i = 3, 4, 5, 6$ and $\nabla_{X_7} = [\nabla_{X_2}, \nabla_{X_3}] - \nabla_{X_4}$.

3.4 η_7^{21}

The center is $\mathbb{C}\{X_6\}$ and the symplectic form used is

$$\theta = \omega_1 \wedge \omega_5 - \frac{6}{11}\omega_1 \wedge \omega_7 + \frac{5}{11}\omega_2 \wedge \omega_4 + \frac{5}{11}\omega_2 \wedge \omega_5 + \frac{6}{11}\omega_3 \wedge \omega_4.$$

The affine structure is given by:

$$\nabla_{X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{11}{6} & -\frac{5}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5}{6} & -1 & -\frac{6}{5} & 0 & 0 & 0 \\ 0 & -\frac{55}{36} & -\frac{11}{6} & -\frac{11}{5} & 0 & 0 & 0 \\ a_1 & a_2 & -\frac{197}{72} & -\frac{37}{15} & \frac{19}{30} & -\frac{4}{5} & 0 \end{pmatrix}, \ \nabla_{X_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{11}{6} & -\frac{5}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{6} & \frac{5}{3} & 1 & 0 & 0 & 0 & 0 \\ -\frac{5}{6} & -\frac{5}{3} & -1 & 0 & 0 & 0 & 0 \\ -\frac{55}{36} & -\frac{55}{18} & -\frac{11}{6} & -1 & 0 & 0 & 0 \\ a_2 & b_2 & -\frac{251}{72} & 0 & -\frac{1}{6} & 0 & 0 \end{pmatrix}.$$

We obtain the other ∇_{X_i} for $i = 3, 4, 5, 6$ from the relation $\nabla_{X_i} = [\nabla_{X_1}, \nabla_{X_{i-1}}]$ and $\nabla_{X_7} = [\nabla_{X_2}, \nabla_{X_3}] - \nabla_{X_4} - \nabla_{X_5}$.

3.5 η_7^{28}

This algebra is equipped with the following affine structure which comes from the extension of the symplectic 6-dimensional algebra $\widetilde{\mathfrak{g}}/\mathbb{C}\{X_6\}$ whose symplectic form is:

$$\theta = 4(\omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_4 + \omega_3 \wedge \omega_4 + \omega_3 \wedge \omega_7) - 3\omega_2 \wedge \omega_7.$$

The connection is given by

$$\nabla_{X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{7} & \frac{3}{7} & -\frac{4}{7} & 0 & -\frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{7} & \frac{4}{7} & \frac{4}{7} & 0 & \frac{4}{7} & 0 \\ a_1 & a_2 & 0 & 0 & \frac{2}{7} & \frac{1}{4} & 0 \end{pmatrix}, \ \nabla_{X_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \frac{4}{7} & \frac{4}{7} & \frac{4}{7} & 0 & 0 & 0 & 0 \\ a_2 & -\frac{1}{7} & -\frac{1}{7} & \frac{3}{7} & 0 & \frac{5}{28} & 0 \end{pmatrix},$$

$\nabla_{X_i} = [\nabla_{X_1}, \nabla_{X_{i-1}}], i = 3, 4, 5, 6$ and

$$\nabla_{X_7} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{5}{28} & -\frac{4}{7} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark. There exists also 7-dimensional Lie algebras without semisimple derivations and which are not contact algebras. They are also central extensions of symplectic 6-dimentional Lie algebras. But in this case the cocycle θ used for the extension is degenerated and the 7-dimensional corresponding Lie algebras have no contact form. In this case we can take again the previous construction, but the vector V_X for $X \in \mathfrak{g}$ is not trivial in general. In these particular cases we can find affine structures.

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