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B-Stability and Its Applications to the Tikhonov and Malkin–Gorshin Theorems

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1. INTRODUCTION

The notion of B-stability was introduced and comprehensively studied in [1–3]. This notion can be applied to analyzing the structure of a neighborhood of a compact invariant set M of a dynamical system (X, R, π). For the case in which the phase space X is locally compact, it was shown that B-stability is an intermediate property between stability and asymptotic stability. More precisely, if M is B-stable, then it is stable; if M is asymptotically stable, then it is B-stable. An elementary example of B-stability is given by the dynamical system described by the differential equation

$$\frac{dx}{dt} = \begin{cases} x^3 \sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

on the interval $X = [-1, 1] \subset R$, where the equilibrium $x = 0$ is taken as M. An analysis of the relationship between various notions of stability for compact invariant sets has shown that, under certain conditions, the requirement of asymptotic stability of M can be replaced by the requirement of B-stability with all desired results being preserved. In the present paper, we show that asymptotic stability can be replaced by B-stability in the Tikhonov theorem on the passage to the limit [4] as well as in the Malkin–Gorshin theorem on stability under permanent perturbations [5, 6]. To justify these assertions, we first prove an equivalent definition of B-stability of a set M.

Let us introduce the relevant definitions and notation. Let (X, R, π) be a dynamical system defined on a metric space (X, d) with distance $d : X \times X \to R^+$, where $\pi : (x, t) \to xt$ is a mapping of $X \times R$ into X, referred to as the *phase mapping*. By $L^+(x)$ [respectively, $L^-(x)$] we denote the set of all ω-limit points (respectively, α-limit points) of a point $x \in X$. If $I \subset R$ and $Y \subset X$, then we write $YI = \{x \in X : \ x = yt, \ y \in Y, \ t \in I\}$; next, $\bar{Y}$ is the closure of Y in X, $\operatorname{Fr} Y$ is the boundary of Y, and $\operatorname{int} Y$ is the interior of Y.

For each compact set M, we consider the following sets:

$$A^+(M) = \left\{x \in X : \ L^+(x) \neq \varnothing, \ L^+(x) \subset M\right\}, \qquad A^+_\omega(M) = \left\{x \in X : \ L^+(M) \cap M \neq \varnothing\right\},$$
$$B(M, a) = \{x \in X : \ d(M, x) < a\}, \qquad a > 0.$$

By $\gamma(x) = xR$ and $\gamma^\pm(x) = xR^\pm$ we denote the trajectory and the positive and negative half-trajectories, respectively, of an element $x \in X$. A set $Y \subset X$ is said to be *invariant* (respectively, *positively* or *negatively invariant*) if $YR = Y$ (respectively, $YR^\pm = Y$).

Recall (see [7]) that a point $x \in X$ is called a *weakly elliptic point* of a compact invariant set M if $L^+(x) \cap M \neq \varnothing$ and $L^-(x) \cap M \neq \varnothing$. By $E_\omega(M)$ we denote the set of all weakly elliptic points of M.

Let X be a locally compact metric space and M a compact invariant subset of X. One says that M is *stable* if each neighborhood of M contains a positively invariant neighborhood of M, *weakly attracting* if $A^+_\omega(M)$ is a neighborhood of M, *attracting* if $A^+(M)$ is a neighborhood of M, and *asymptotically stable* if it is stable and attracting [8, p. 58].

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2. B-STABILITY

We recall the following definitions.

Definition 1 [1–3]. Let M be a nonempty compact invariant subset of a locally compact metric space X. A neighborhood W of M is said to be *repelling for* $t < 0$ if, for each $x \in \operatorname{Fr} W$, there exists a $\tau < 0$ such that $x(\tau) \notin \bar{W}$.

Definition 2 [1–3]. A nonempty compact invariant subset M of a locally compact space X is said to be B-*stable* if every neighborhood of M contains a neighborhood that is repelling for $t < 0$.

The following property was studied in [9, 10]: a nonempty compact subset $M \subset X$ is called a *quasiattractor* if there exists a countable sequence (M_k) of compact invariant asymptotically stable sets such that $M = \bigcap M_k$.

The equivalence of the definitions of B-stability and of a quasiattractor was proved in [3].

The following criterion for B-stability holds.

Theorem 1. *A nonempty compact invariant subset M of a locally compact metric space X is B-stable if and only if the following conditions are satisfied*:

(1) *M is stable*;

(2) *every neighborhood of M contains a compact attracting subset containing M.*

Proof. Necessity. Let M be B-stable. Then [1–3] M is stable, and, by Theorem 1 in [3], every neighborhood of M contains a compact asymptotically stable neighborhood W. Since W is, in particular, attracting, we have the necessity of conditions (1) and (2).

Sufficiency. Let conditions (1) and (2) be satisfied. By Theorem 1 in [3], it suffices to show that every neighborhood of M contains a compact asymptotically stable neighborhood of M. Indeed, let $\varepsilon > 0$ be an arbitrary number such that the closure of $B(M, \varepsilon)$ is compact. Since M is stable, it follows that there exists a $\delta = \delta(\varepsilon) > 0$ such that $B(M, \delta)R^+ \subset B(M, \varepsilon)$. Likewise, for $\delta > 0$, there exists an $\eta = \eta(\delta)$ such that $B(M, \eta)R^+ \subset B(M, \delta)$. Moreover, it follows from condition (2) that there exists a compact attracting set K such that $M \subset K \subset B(M, \eta)$. This, together with Theorem 1 in [3], implies that the set $E_\omega(K)$ of all elliptic points of K is compactly invariant and asymptotically stable and $K \subset E_\omega(K)$; moreover, $A^+(E_\omega(K)) \equiv A^+(K)$. From the asymptotic stability of M, we find that $E_\omega(K) \subset B(M, \delta)$, since otherwise there is a trajectory issuing from the “ball” $B(M, \eta)$ and reaching the “sphere” $\operatorname{Fr} B(M, \delta)$, which contradicts the stability of M. We choose a number $\Delta > 0$ small enough to ensure that $B(E_\omega(K), \Delta) \subset B(M, \delta)$ and, at the same time,

$$B(E_\omega(K), \Delta) \subset A^+(E_\omega(K)) \equiv A^+(K)B(M, \delta).$$

We set $V = B(E_\omega(K), \Delta) \cup \left(\bigcup_{x \in \operatorname{Fr} B(E_\omega(K), \Delta)} \gamma^+(x)\right)$.

By definition, the set V is positively invariant, V is a neighborhood of M, and $V \subset A^+(E_\omega(K))$. In this case, the closure $W_1 = \bar{V}$ is also positively invariant; moreover, $W_1 \subset A^+(K) \equiv A^+(E_\omega(K))$. But $W = \bar{W}_1 \subset B(M, \varepsilon)$; therefore, W is a compact set. This, together with Lemma 2 in [3], implies that the set W is an asymptotically stable neighborhood of M.

In the same manner, taking the sequence $\varepsilon_n = \varepsilon \times 2^{-n}$, one can construct a sequence (W_n) of compact asymptotically stable neighborhoods satisfying assumptions (1) and (2) of Theorem 1 in [3]. Consequently, M is B-stable, and the proof of the theorem is complete.

3. THE TIKHONOV THEOREM

Consider the system of differential equations

$$\mu\, dx/dt = F(z, y, t, \mu), \qquad dy/dt = f(z, y, t, \mu), \tag{1}$$

where $z, F \in R^n$, $y, f \in R^m$, and $\mu > 0$, with the initial conditions

$$z(0, \mu) = z^0(\mu), \qquad y(0, \mu) = y^0(\mu). \tag{2}$$

The well-known Tikhonov theorem on the passage to the limit [4, p. 25] deals with a solution $z(t,\mu)$, $y(t,\mu)$ defined on an interval $[0,T]$ under the following assumptions.

I. The functions $F(z,y,t,\mu)$ and $f(z,y,t,\mu)$ are continuous and satisfy the Lipschitz condition with respect to z and y in some open domain G of the variables (z,y,t,μ).

II. The equation $F(z,y,t,\mu)=0$ has a root $z=\varphi(y,t)$ in some closed bounded domain $\bar{D}$ of the variables (y,t) such that $(y^0,0)\in\bar{D}$ and the following assertions are valid:

(1) the function $\varphi(y,t)$ is continuous in $\bar{D}$;

(2) $(\varphi(y,t),y,t,\mu)\in G$ for all $(y,t)\in\bar{D}$ and for sufficiently small $\mu>0$;

(3) the root $z=\varphi(y,t)$ is isolated in $\bar{D}$ in the sense that there exists a number $\eta>0$ such that, for all $(y,t)\in\bar{D}$, one has $F(z,y,t,\mu)\neq 0$ whenever $0<\|z-\varphi(y,t)\|\le\eta$.

III. The system $d\bar{y}/dt=f(\varphi(\bar{y},t),\bar{y},t,0)$, $\bar{y}(0)=y^0$ has a unique solution $\bar{y}(t)$ on the interval $[0,T]$; moreover, $(\bar{y}(t),t)\in D$ for all $t\in[0,T]$, where D is the interior of $\bar{D}$. We also assume that the function $f(\varphi(y,t),y,t)$ satisfies the Lipschitz condition with respect to the variable y in the domain $\bar{D}$.

IV. The equilibrium $\tilde{z}=\varphi(y,t)$ of the system of differential equations

$$d\tilde{z}/d\tau=F(\tilde{z},y,t,\mu),\qquad \tau\ge 0, \tag{3}$$

where y and t are treated as parameters, is an asymptotically stable fixed point uniformly with respect to $(y,t)\in\bar{D}$. This means that, for any $\varepsilon>0$, there exists a $\bar{\delta}(\varepsilon)>0$ [independent of $(y,t)\in\bar{D}$] such that if $\|\tilde{z}(0)-\varphi(y,t)\|<\bar{\delta}(\varepsilon)$, then $\|\tilde{z}(\tau)-\varphi(y,t)\|<\bar{\varepsilon}$ for all $\tau\ge 0$, and moreover, $\tilde{z}(\tau)\to\varphi(y,t)$ as $\tau\to+\infty$.

V. The initial condition z^0 belongs to the domain of asymptotic stability of the fixed point $\tilde{z}_0=\varphi(y^0,0)$, i.e., the solution $\tilde{z}(\tau)$ of Eq. (3) with the initial condition $z(0)=z^0$ satisfies the condition $(\tilde{z}(\tau),y^0,0)\in G$ for all $\tau\ge 0$ and $\tilde{z}(\tau)\to\varphi(y^0,0)$ as $\tau\to+\infty$.

Theorem 2 [4, p. 25]. *If assumptions* I–V *are valid, then there exists a* $\mu>0$ *such that for* $0<\mu\le\mu_0$, *the solution* $z(t,\mu)$, $y(t,\mu)$ *of problem* (1), (2) *exists on the interval* $[0,T]$, *is unique, and satisfies the conditions*

$$\lim_{\mu\to 0} y(t,\mu)=\bar{y}(t)\qquad \textit{if}\quad 0\le t\le T, \tag{4}$$

$$\lim_{\mu\to 0} z(t,\mu)=\varphi(\bar{y}(t),t)\qquad \textit{if}\quad 0<t\le T. \tag{5}$$

The proof of this theorem is based on the following auxiliary assertion.

Lemma 1 [4, p. 26]. *Suppose that assumptions* I–V *are valid. Let* $\varepsilon>0$ *be a number such that* $\bar{U}=\{(z,y,t):\ \|z-\varphi(y,t)\|\le\varepsilon,\ (y,t)\in D\}\subset G$. *Then there exist* $\delta=\delta(\varepsilon)>0$ *and* $\mu_0=\mu_0(\varepsilon)$ *such that if* $0<\mu<\mu_0$, *then the solution* $(z(t,\mu),y(t,\mu))$ *of system* (1) *with the initial condition* $(z(t_0,\mu)),y(t_0,\mu))\in U_\delta$ *exists and does not leave* U_ε *for all* $t\ge t_0$ *at least while* $(y(t,\mu),t)\in D$.

One can weaken condition IV of uniform asymptotic stability by replacing it by the following condition.

IV′. The fixed point $\tilde{z}=\varphi(y,t)$ of system (3) is B-stable uniformly with respect to $(y,t)\in\bar{D}$.

This means that, for every $\varepsilon>0$, there exists a $\bar{\delta}(\varepsilon)>0$ [independent of $(y,t)\in\bar{D}$] such that, for all $(y,t)\in\bar{D}$, there exists a compact set $K(y,t)$ satisfying the relations

$$\varphi(y,t)\in K(y,t)\subset B\left(\varphi(y,t),\bar{\delta}(\varepsilon)\right)$$

and such that if $\|\tilde{z}(0)-\varphi(y,t)\|<\bar{\delta}(\varepsilon)$, then

$$\|\tilde{z}(\tau)-\varphi(y,t)\|\le\varepsilon$$

for all $\tau\ge 0$ and $L^+(\tilde{z}(\tau))\subset K(y,t)$.

By $L^{+}(\tilde{z}(\tau))$ we denote the set of all ω-limit points of the solution $\tilde{z}(\tau)$.

Condition V can be replaced by one of the following conditions:

(1) $\tilde{z}_0 = \varphi(y^0, 0)$ is asymptotically stable, and z^0 belongs to its attraction domain;

(2) $z^0 = \varphi(y^0, 0)$.

Let us show that Lemma 1 is valid under assumptions I–III, IV′, (1), and (2). Suppose that, for $\varepsilon > 0$, there exists a $\bar{\delta}(\varepsilon) > 0$ satisfying condition IV′.

We set $\delta = \bar{\delta}(\varepsilon/2)$. The existence and uniqueness of solutions is provided by condition I. Solutions can be continued with the preservation of uniqueness until they leave U_ε. It remains to prove the existence of a number $\mu_0 = \mu_0(\varepsilon) > 0$ such that if $0 < \mu \leq \mu_0$, then the solution does not leave U_ε while $(y(t,\mu), t) \in D$.

Suppose the contrary: there is no μ_0 with this property. Then there exists a sequence (μ_n) converging to zero and such that the corresponding sequence $(z(t,\mu_n), y(t,\mu_n))$ of solutions with the initial data $(z(t_n^0,\mu_n), y(t_n^0,\mu_n), t_n^0) \in U_\delta$ has the following property: there exists a $\bar{t}_n > t_n^0$ such that $\|z(t,\mu_n) - \varphi(y(t,\mu_n),t)\| < \varepsilon$, $(y(t,\mu_n),t) \in D$ if $t_n^0 \leq t < \bar{t}_n$, and moreover,

$$\|z(\bar{t}_n,\mu_n) - \varphi(y(\bar{t}_n,\mu_n),\bar{t})\| = \varepsilon, \qquad (y(\bar{t}_n,\mu_n),\bar{t}_n) \in D. \tag{6}$$

By t_n we denote the maximum of numbers $t \in [t_0^n, \bar{t}_n]$ for which the solution $(z(t,\mu_n), y(t,\mu_n))$ intersects the boundary of $\bar{U}_\delta$, i.e., $\|z(t_n,\mu_n) - \varphi(y(t_n,\mu_n),t_n)\| = \delta$. Then

$$\delta < \|z(t,\mu_n) - \varphi(y(t,\mu_n),t)\| < \varepsilon \tag{7}$$

for $t_n < t < \bar{t}_n$. From the sequence $(z(t_n,\mu_n), y(t_n,\mu_n), t_n) \in \bar{U}_\delta$, one can extract a convergent subsequence. Suppose that the sequence itself is convergent. Then

$$(z(t_n,\mu_n), y(t_n,\mu_n), t_n) \to \left(\breve{z}, \breve{y}, \breve{t}\right)$$

as $n \to +\infty$; moreover, $\left(\breve{z}, \breve{y}, \breve{t}\right) \in \bar{U}_\delta$, i.e., $\left\|\breve{z} - \varphi\left(\breve{y}, \breve{t}\right)\right\| = \delta$.

If $\mu = \mu_n$, then, in system (1), we perform the change of variables $\tau = (t - t_n)/\mu_n$. Then we obtain

$$dz/d\tau = F(z, y, t_n + \mu_n\tau), \qquad dy/d\tau = \mu_n f(z, y, t_n + \mu_n\tau). \tag{8}$$

The solutions $z(t,\mu_n) = z(t_n + \mu_n\tau, \mu_n)$, $y(t,\mu_n) = y(t_n + \mu_n\tau, \mu_n)$ treated as functions of the variable τ satisfy system (8) with the initial data $z(\tau = 0) = z(t_n,\mu_n)$, $y(\tau = 0) = y(t_n,\mu_n)$.

By the theorem on the continuous dependence of solutions on parameters, on each compact set $0 \leq \tau \leq \tau_0$, the limit relations

$$\lim_{n\to+\infty} z(t_n + \mu_n\tau, \mu_n) = \tilde{z}(\tau), \qquad \lim_{n\to+\infty} y(t_n + \mu_n\tau, \mu_n) = \tilde{y}(\tau) \tag{9}$$

hold uniformly with respect to $\tau \in [0, \tau_0]$, where $\tilde{z}(\tau)$, $\tilde{y}(\tau)$ is the solution of the problem

$$d\tilde{z}/d\tau = F\left(\tilde{z}, \tilde{y}, \breve{t}\right), \qquad d\tilde{y}/d\tau = 0, \qquad \tilde{z}(0) = \breve{z}, \qquad \tilde{y}(0) = \breve{y}.$$

Obviously, $\tilde{y}(\tau) = \breve{y}$, $\tilde{z}(\tau)$ is a solution of this problem, which, at the same time, is a solution of the problem $d\tilde{z}/d\tau = F\left(\tilde{z}, \breve{y}, \breve{t}\right)$, $\tilde{z}(0) = \breve{z}$. Since $\left\|\breve{z} - \varphi\left(\breve{y}, \breve{t}\right)\right\| = \delta = \bar{\delta}(\varepsilon/2)$, it follows from condition IV′ that $\left\|\tilde{z}(\tau) - \varphi\left(\breve{y}, \breve{t}\right)\right\| < \varepsilon/2$ and, at the same time, $L^{+}(\tilde{z}(\tau)) \subset K\left(\breve{y}, \breve{t}\right)$.

Therefore, $\left\|\tilde{z}(\tau_0) - \varphi\left(\breve{y}, \breve{t}\right)\right\| < \delta$ for sufficiently large $\tau = \tau_0$, since the ball $B\left(\varphi\left(\breve{y}, \breve{t}\right), \delta\right)$ is an open neighborhood of the compact set $K\left(\breve{y}, \breve{t}\right)$. Consequently, using the limits (9), we obtain

$$\|z(t,\mu_n) - \varphi(y(t,\mu_n),t)\| < \varepsilon/2 \tag{10}$$

for a sufficiently large positive integer n_0 provided that $t_n \leq t \leq t_n + \mu_n\tau_0$ and $n \geq n_0$, and

$$\|z(t,\mu_n) - \varphi(y(t,\mu_n),t)\| < \delta \tag{11}$$

for $t = t_n + \mu n\tau_0$. If $\bar{t}_n \le t_n + \mu_n\tau_0$, then inequality (10) contradicts (6). But if $\bar{t}_n > t_n + \mu_n\tau_0$, then inequality (11) contradicts the left inequality in (7). The proof of the lemma is complete.

In the proof of the lemma, we have used the same scheme as in the proof of Lemma 1 in [11]. One can readily see what modifications are needed to justify the Tikhonov theorem on the passage to the limit [4, p. 25] with uniform asymptotic stability replaced by B-stability. Thus, we claim that the following assertion is valid.

Theorem 3. *Suppose that the right-hand sides of system* (1) *do not explicitly depend on* t. *If conditions* I–III, IV′, (1), *and* (2) *are satisfied, then there exists a* μ_0 *such that, for* $0 < \mu \le \mu_0$, *problem* (1), (2) *has a unique solution* $z(t,\mu)$, $y(t,\mu)$ *defined on the interval* $[0,T]$ *and satisfying the limit relations* (4) *and* (5).

4. STABILITY UNDER PERMANENT PERTURBATIONS

Let us consider the system of differential equations

$$\dot{x} = f(x), \tag{12}$$

where f is a continuous function such that the solution is unique in some domain $D \subset R^n$ containing the origin. Let $f(0) = 0$. We analyze the stability of the zero solution of system (12) not only under perturbations of the initial conditions $x(0) = x_0$, but also under perturbations of the right-hand sides of the system. In other words, along with (12), we consider the system of equations

$$\dot{y} = f(y) + g(t,y), \tag{13}$$

where $g : R^+ \times D \to R^n$ is a continuous function. The function g is treated as a permanently acting perturbation. Here we do not assume that $g(t,0) = 0$ for all $t \ge 0$, i.e., the origin is not necessarily an equilibrium of system (13). We only assume the uniqueness of solutions of system (13) in the domain $R^n \times D$.

Definition 3 [5, 6]. The solution $x = 0$ of system (12) is said to be *stable under permanent perturbations* if, for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that the solution $y(t,y_0)$ of system (13) has the property $y(t,y_0) \in B_\varepsilon$, $t \ge 0$, for an arbitrary point $y_0 \in B_\delta$ and for an arbitrary function g satisfying the condition $\|g\|_{B_\varepsilon} \le \delta$.

Here the norm of a function g ensuring the above-mentioned properties of solutions of system (13) is defined as $\|g\|_{B_\varepsilon} = \sup_{t\ge 0,\, y\in B_\varepsilon} \|g(t,y)\|$.

Theorem 4. *If the zero solution of system* (12) *is* B*-stable, then it is also stable under permanent perturbations.*

Remark. This result generalizes the well-known Malkin–Gorshin theorem [5, 6], in which the uniform asymptotic stability of the origin is assumed. Massera [12, 13] constructed examples showing that the stability of the zero solution under permanent perturbations does not imply the uniform asymptotic stability. We can readily see that the origin is B-stable in all of these examples (we speak of the autonomous case). Note that the theorems in [5, 6] also pertain to systems of nonautonomous differential equations. Unlike the original proof, which was based on the existence of a Lyapunov function for system (12), our proof of Theorem 4 is performed by methods of qualitative theory of dynamical systems.

Proof of Theorem 4. The proof is by contradiction. Suppose that the zero solution of system (12) is not stable under permanent perturbations. Then there exists an $\varepsilon > 0$ such that, for every $\delta > 0$, there exists a point $y_0 \in B_\delta$ and a function g such that $\|g\|_{B_\varepsilon} \le \delta$ and the solution of system (13) satisfies the condition $y(t,y_0) \notin B_\varepsilon$ for some $t > 0$. By virtue of the B-stability of the origin, there exists an $\eta > 0$ such that $B_\eta \subset B_\varepsilon$ and B_η is a repelling neighborhood of the point $x = 0$ for $t < 0$. Since the solution $y(t,y_0)$ leaves B_ε, we see that so much the more it leaves B_η. Let t be the first positive instant of time for which $y(t,y_0)$ leaves B_η; thus, $y(s,y_0) \in B_\eta$

for all $s \in [0, t[$ and $\|y(t, y_0)\| = \eta$. We set $\delta = 1/n$. Then there exists a sequence $(y_0^{(n)})$ of initial states converging to zero as $n \to +\infty$, a sequence (g_n) of functions such that $\|g_n\|_{B_\varepsilon} \leq 1/n$, and a sequence $\left(t^{(n)}\right)$ of nonnegative instants of time such that the points $z_n = y(t^{(n)}, y_0^{(n)})$ satisfy the condition $\|z_n\| = \eta$. Obviously, the sequence (z_n) contains a convergent subsequence. Therefore, without loss of generality, we can assume that there exists a limit $\lim_{n\to+\infty} z_n = \tilde{z}$. Obviously, the limit point satisfies the relation $\|\bar{z}\| = \eta$. By the theorem on the continuous dependence of solutions on parameters and initial data, we have

$$\lim_{n\to+\infty} y\left(t^{(n)} + s, y_0^{(n)}\right) = z(s), \tag{14}$$

where $z(s)$ is the unique solution of system (12). Furthermore, $z(0) = \bar{z}$. More precisely, if the solution $z(s)$ is defined and does not leave the closure of the ball B_ε for $s \in [\tau, 0]$, then, for a sufficiently large positive integer n, the function $y(t^0 + s, y_0^{(n)})$ is also defined on the interval $[\tau, 0]$; therefore, the limit (14) is uniform on the set $[\tau, 0]$. By virtue of B-stability, the negative half-trajectory of the point $\bar{z}$ cannot entirely lie in the ball $\bar{B}_\eta$; consequently, there exists a $\tau_0 < 0$ such that $z(\tau_0) \notin \bar{B}_\eta$. We take a $\tau_0 < 0$ such that $z([\tau_0, 0]) \subset \bar{B}_\varepsilon$.

Let us show that (t_0^n) is an unbounded sequence. Indeed, otherwise it would follow from the theorem on the continuous dependence of solutions that $\lim_{n\to+\infty} y(t, y_0^{(n)}) = 0$ for $t \in \left[0, t^{(n)}\right]$, which contradicts the identity $\|y(t^0, y_0^{(n)})\| = \eta$.

Therefore, we can claim that $t^0 + \tau_0 \geq 0$ for a sufficiently large positive integer n. By definition, $y(t^{(n)} + \tau_0, y_0^{(n)}) \in B_\eta$ for a sufficiently large index n, and at the same time, $z(\tau_0) \notin \bar{B}_\eta$. We have arrived at a contradiction; consequently, the zero solution of system (12) is stable under permanent perturbations. The proof of the theorem is complete.

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