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AVERAGING RESULTS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: Under rather general assumptions, we present some improved averaging results for functional differential equations. This is achieved with the help of nonstandard analysis and extends a similar result by the first author for a delay differential equation of a particular form.

Keywords: averaging method, functional differential equation, exponential stability, nonstandard analysis

1. Introduction

It is well known that the averaging method is a powerful tool for studying many perturbation problems in nonlinear oscillations, and some in celestial mechanics. There is a rich literature for ordinary differential equations (see [1–7] and the references given therein). The method is also extended to functional differential equations [8–13] of the form

$$\dot{x}(t) = \varepsilon f(t, x_t) \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter and $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. Assume that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\tau, u)\, d\tau = f^o(u) \tag{1.2}$$

exists, and consider the associated averaged ordinary differential equation

$$\dot{y}(t) = \varepsilon f^o(\tilde{y}) \tag{1.3}$$

where $\tilde{y}(\theta) = y$ for $\theta \in [-r, 0]$ and $y \in \mathbb{R}^d$. Under suitable conditions, it is shown that, if ε is sufficiently small then the difference between solutions x and y of (1.1) and (1.3), respectively, with the same initial conditions, is small over time $1/\varepsilon$.

Note that if we let $t \mapsto t/\varepsilon$ and $x(t/\varepsilon) = z(t)$ then equation (1.1) becomes

$$\dot{z}(t) = f\left(\frac{t}{\varepsilon}, z_{t,\varepsilon}\right) \tag{1.4}$$

where $z_{t,\varepsilon}(\theta) = z(t + \varepsilon\theta)$ for $\theta \in [-r, 0]$, which is an equation with a small delay.

In this article, we give justification of the averaging method for functional differential equations which can be brought into the form

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x_t\right). \tag{1.5}$$

That is, under rather general assumptions, we show that the solutions of (1.5) remain close to those of the averaged equation

$$\dot{y}(t) = f^o(y_t) \tag{1.6}$$

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where f^o is given in (1.2). Note that (1.6) is a functional differential equation and not an ordinary differential equation.

Among the articles justifying the averaging method for (1.5), we will cite the paper by Hale and Verduyn Lunel [11]. In this work, the authors introduce an extension of the averaging method to abstract evolutionary equations in Banach spaces. In particular, they rewrite (1.5) as an ordinary differential equation in an infinite-dimensional Banach space and proceed formally from there. Our approach is different since all analysis is kept in the associated natural phase space. Inspired with the work by the second author [7] where, by means of nonstandard analysis, the averaging method for ordinary differential equations is performed under weak assumptions, in [14] the first author presents a natural extension of the averaging method to delay differential equations of the particular form

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x(t-r)\right).$$

Using the same technique, the main result here generalizes a similar result of [14] to functional differential equations of the form (1.5) and then its proof is directly related to that proposed therein.

The rest of the paper is organized as follows: in Section 2, we present as Theorem 2.2 our main result concerning the closeness of solutions of (1.5) and (1.6) on finite time intervals, and in Theorem 2.5 we investigate the long time behavior of solutions of (1.5). This is done under the assumption that (1.6) has an exponentially stable equilibrium. For this case, the idea of the proof is the same as used for ordinary differential equations by Sanders and Verhulst [6]. The proofs of Theorems 2.2 and 2.5 are established in the framework of Robinson's *nonstandard analysis* [15] in its axiomatic approach, namely *Internal Set Theory*, by Nelson [16]. For the convenience of the reader, in Subsection 3.1 we provide a short tutorial on IST, and then in Subsection 3.2 we present the nonstandard translates in Theorems 3.6 and 3.7 of Theorems 2.2 and 2.5, respectively. Finally, in Section 4 we first begin with some preparatory lemmas and then give the proofs of Theorems 3.6 and 3.7.

2. Notations, Conditions, and Averaging Results

In this section, we introduce notations, state hypotheses and present our results on averaging for functional differential equations.

Let $r \geq 0$ be a given constant and $\mathscr{C}_o = \mathscr{C}([-r,0], \mathbb{R}^d)$, the Banach space of all continuous functions from $[-r,0]$ into $\mathbb{R}^d$ with the usual supremum norm

$$|\phi| = \sup_{-r\leq\theta\leq 0} |\phi(\theta)|, \quad \phi \in \mathscr{C}_o.$$

Even though single bars are used for norms in different spaces, no confusion should arise. Let $t_0 \in \mathbb{R}$ and $L > t_0$. If $x : [t_0 - r, L] \to \mathbb{R}^d$ is a continuous function and if $t \in [t_0, L]$ then $x_t \in \mathscr{C}_o$ is defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-r, 0].$$

We make the following assumptions:

(H1) The functional $f : \mathbb{R} \times \mathscr{C}_o \to \mathbb{R}^d$ in (1.5) is continuous and bounded.

(H2) The continuity of $f = f(\tau, u)$ in $u \in \mathscr{C}_o$ is uniform with respect to $\tau \in \mathbb{R}$.

(H3) For all $u \in \mathscr{C}_o$ there exists a limit

$$f^o(u) := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\tau, u)\, d\tau.$$

(H4) The averaged equation (1.6) enjoys uniqueness of solutions with the prescribed initial conditions.

For $t_0 \in \mathbb{R}$ and $\phi \in \mathscr{C}_0$, the solution of (1.6) such that $y_{t_0} = \phi$ is denoted by $y = y(\,\cdot\,; t_0, \phi)$. This solution is defined on $J = [t_0 - r, +\infty)$.

Remark 2.1. In assumption (H4) we anticipate the existence of solutions of (1.6). We will justify this a posteriori. Indeed, we will show in Lemma 4.1 below that f^o is continuous so that existence is guaranteed.

Under the above assumptions, we can establish the following main result on nearness of the solutions of (1.5) and (1.6) with the same initial conditions.

Theorem 2.2. *Assume that hypotheses* (H1)–(H4) *are satisfied. Let $t_0 \in \mathbb{R}$ and $\phi \in \mathscr{C}_0$. Let $y = y(\,\cdot\,; t_0, \phi)$ be the solution of* (1.6). *Then for any $L > t_0$ and any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, each solution x of* (1.5) *with $x_{t_0} = \phi$ is defined at least on $[t_0 - r, L]$ and the inequality $|x(t) - y(t)| < \delta$ holds for $t \in [t_0, L]$.*

It is possible to extend the validity of the averaging technique for all (future) time when the initial function of the solution of (1.6) lies in the domain of exponential stability of an exponentially stable equilibrium. For this, let us first recall the concept of *exponential stability* of equilibra.

Suppose that y_e is an equilibrium of (1.6), that is, $f^o(y_e) = 0$.

Definition 2.3. The equilibrium y_e is said to be exponentially stable if there exist b, K and $\lambda > 0$ such that, for any $t_0 \in \mathbb{R}$ and $\phi \in \mathscr{C}_o$, the solution $y = y(\,\cdot\,; t_0, \phi)$ of (1.6) for which $|\phi - y_e| < b$ is defined on $[t_0 - r, +\infty)$ and the inequality $|y(t) - y_e| \le K e^{-\lambda(t - t_0)} |\phi - y_e|$ holds for $t \ge t_0$.

Remark 2.4. The ball $\mathscr{B}$ with center y_e and radius b where the stability is exponential will be called the domain of exponential stability of y_e.

As a next result of this section, we will prove validity of the approximation of the solutions of (1.5) and (1.6) with the same initial conditions, for all time.

Theorem 2.5. *Assume that hypotheses* (H1)–(H4) *are satisfied. Let $t_0 \in \mathbb{R}$ and $\phi \in \mathscr{C}_0$. Let y_e be an equilibrium of* (1.6). *Assume further that*

(H5) *y_e is exponentially stable.*

(H6) *ϕ lies in $\mathscr{B}$.*

Let $y = y(\,\cdot\,; t_0, \phi)$ be the solution of (1.6). *Then for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, each solution x of* (1.5) *with $x_{t_0} = \phi$ is defined on $[t_0 - r, +\infty)$ and the inequality $|x(t) - y(t)| < \delta$ holds for $t \ge t_0$.*

3. Nonstandard Averaging Results

3.1. Internal Set Theory: A short tutorial. In IST (*Internal Set Theory*) we adjoin to ordinary mathematics (say ZFC) a new undefined unary predicate symbol "st" (read: "standard"). We call *internal* the formulas of IST without any occurrence of the predicate "st"; otherwise, we call them *external*. Thus, the internal formulas are formulas of ZFC. The axioms of IST are all axioms of ZFC restricted to internal formulas (in other words, IST is an extension of ZFC), plus three others that govern the use of the new predicate. Thus, *all theorems* of ZFC *remain valid* in IST. IST is a *conservative extension* of ZFC; that is, each internal theorem of IST is a theorem of ZFC. Some of the theorems that are proved in IST are external and can be reformulated so that they become internal. Indeed, there is an algorithm (a well-known *reduction algorithm*) to reduce any external formula $F(x_1, \dots, x_n)$ of IST without other free variables than $x_1, \dots, x_n$ to an internal formula $F'(x_1, \dots, x_n)$ with the same free variables, such that $F \equiv F'$, that is, $F \iff F'$ for all standard values of the free variables. In other words, each result which may be formalized within IST by a formula $F(x_1, \dots, x_n)$ is equivalent to the classical property $F'(x_1, \dots, x_n)$, provided the parameters $x_1, \dots, x_n$ are restricted to standard values. We give the reduction of the frequently occurring formula $\forall x\ (\forall^{\mathrm{st}} y\ A \Longrightarrow \forall^{\mathrm{st}} z\ B)$ where A and B are internal formulas

$$\forall x (\forall^{\mathrm{st}} y\ A \Longrightarrow \forall^{\mathrm{st}} z B) \equiv \forall z\ \exists^{\mathrm{fin}} y' \forall x (\forall y \in y' A \Longrightarrow B). \tag{3.1}$$

The notations $\forall^{\mathrm{st}} X$ and $\exists^{\mathrm{fin}} X$ stand for $[\forall X, \mathrm{st}(X) \Longrightarrow \cdots]$ and $[\exists X, X \text{ finite} \,\&\, \dots]$, respectively.

A real number x is called *infinitesimal*, denoted by $x \simeq 0$, if its absolute value $|x|$ is smaller than each standard strictly positive real number, *limited* if its absolute value $|x|$ is smaller than some standard real number, *unlimited*, denoted by $x \simeq \pm\infty$, if it is not limited, and *appreciable* if it is neither unlimited nor infinitesimal. Two real numbers x and y are *infinitely close*, denoted by $x \simeq y$, if their difference $x - y$ is infinitesimal.

For x and y in a standard metric space E, the notation $x \simeq y$ means that the distance from x to y is infinitesimal. If there exists in that space a standard x_0 such that $x \simeq x_0$, the element x is called *nearstandard* in E and the standard point x_0 is called the *standard part* of x (it is unique) and is also denoted by ${}^o x$.

We must avoid external formulas in the axiom-schema of ZFC, in particular we must avoid external formulas in defining subsets. The notations $\{x \in \mathbb{R} : x \text{ is limited}\}$ or $\{x \in \mathbb{R} : x \simeq 0\}$ are not allowed. Moreover we can prove that

Lemma 3.1. *There do not exist subsets $\mathscr{L}$ and $\mathscr{I}$ of $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, x is in $\mathscr{L}$ if and only if x is limited, or x is in $\mathscr{I}$ if and only if x is infinitesimal.*

It happens sometimes in classical mathematics that a property is assumed, or proved, on a certain domain, and that afterwards we notice that the character of the property and the nature of the domain are incompatible. So actually the property must be valid on a larger domain. In the same manner, in nonstandard analysis, the result of Lemma 3.1 is frequently used to prove that the validity of a property exceeds the domain where it was established in a direct way. Suppose that we have shown that A holds for every limited x, then we know that A holds for some unlimited x, for otherwise we could let $\mathscr{L} = \{x \in \mathbb{R} : A\}$. This statement is called the *Cauchy principle*. It has the following frequently used application.

Lemma 3.2 (Robinson's lemma). *If g is a real function such that $g(t) \simeq 0$ for all limited $t \geq 0$, then there exists $\omega \simeq +\infty$ such that $g(t) \simeq 0$ for all $t \in [0, \omega]$.*

Proof. Indeed, the set of all $l \in \mathbb{R}$ such that $|g(t)| < 1/l$ for all $t \in [0, l]$ contains all limited l in $\mathbb{R}$, $l \geq 1$. By the Cauchy principle it must contain some unlimited ω. □

We conclude this section with two other applications of Cauchy's principle which will be used later.

Lemma 3.3. *If $\mathscr{P}(.)$ is an internal property such that $\mathscr{P}(a)$ holds for all appreciable real numbers $a > 0$, then there exists $0 < a_0 \simeq 0$ such that $\mathscr{P}(a_0)$ holds.*

Lemma 3.4. *Let $h : I \to \mathbb{R}$ be a function such that $h(t) \simeq 0$ for all $t \in I$. Then $\sup\{h(t) : t \in I\} \simeq 0$.*

Remark 3.5. For more information on the applications of nonstandard analysis to asymptotic theory of differential equations, the reader is referred to [17–24] and the references therein.

3.2. Averaging results. We first give nonstandard formulations of Theorems 2.2 and 2.5. Then, by use of the reduction algorithm, we show that the reduction of Theorems 3.6 and 3.7 below are Theorems 2.2 and 2.5, respectively.

Theorem 3.6. *Let $f : \mathbb{R} \times \mathscr{C}_o \to \mathbb{R}^d$ be standard. Assume that all hypotheses in Theorem 2.2 are satisfied. Let $t_0 \in \mathbb{R}$ and $\phi \in \mathscr{C}_o$ be standard. Let $y = y(\cdot; t_0, \phi)$ be the solution of (1.6). Let $\varepsilon > 0$ be infinitesimal. Then for all standard $L > t_0$, each solution x of (1.5) with $x_{t_0} = \phi$ is defined at least on $[t_0 - r, L]$ and satisfies $x(t) \simeq y(t)$ for $t \in [t_0, L]$.*

Theorem 3.7. *Let $f : \mathbb{R} \times \mathscr{C}_o \to \mathbb{R}^d$ be standard. Let $t_0 \in \mathbb{R}$ and $\phi \in \mathscr{C}_o$ be standard. Let y_e be a standard equilibrium of (1.6). Assume that all hypotheses in Theorem 2.5 are satisfied. Let $y = y(\cdot; t_0, \phi)$ be the solution of (1.6). Let $\varepsilon > 0$ be infinitesimal. Then each solution x of (1.5) with $x_{t_0} = \phi$ is defined on $[t_0 - r, +\infty)$ and satisfies $x(t) \simeq y(t)$ for $t \geq t_0$.*

The proofs of Theorems 3.6 and 3.7 are postponed to Section 4. Theorems 3.6 and 3.7 are external statements, hereafter we show that the reduction of Theorem 3.6 (Theorem 3.7) is Theorem 2.2 (respectively, Theorem 2.5).

Reduction of Theorem 3.6. Without loss of generality, let $t_0 = 0$. Let $L > 0$ be standard. We adopt the following abbreviation: B is the formula "If $\delta > 0$ then each solution x of (1.5) with $x_{t_0} = \phi$ is defined at least on $[-r, L]$ and the inequality $|x(t) - y(t)| < \delta$ holds for $t \in [0, L]$." Then to say that "each solution x of (1.5) with $x_{t_0} = \phi$ is defined at least on $[-r, L]$ and satisfies $x(t) \simeq y(t)$ for $t \in [0, L]$" is the same as saying $\forall^{\mathrm{st}}\delta B$. By Theorem 3.6, we have

$$\forall\varepsilon(\forall^{\mathrm{st}}\eta\varepsilon < \eta \Longrightarrow \forall^{\mathrm{st}}\delta B). \tag{3.2}$$

In this formula L is standard; and ε, η, and δ range over the strictly positive real numbers. By (3.1), formula (3.2) is equivalent to

$$\forall\delta\exists^{\mathrm{fin}}\eta'\ \forall\varepsilon(\forall\eta \in \eta'\varepsilon < \eta \Longrightarrow B). \tag{3.3}$$

For η' a finite set, $\forall\eta \in \eta'\varepsilon < \eta$ is the same as $\varepsilon < \varepsilon_0$ for $\varepsilon_0 = \min\eta'$, and so formula (3.3) becomes

$$\forall\delta\exists\varepsilon_0\forall\varepsilon\ (\varepsilon < \varepsilon_0 \Longrightarrow B).$$

That is, the statement of Theorem 2.2 holds. By transfer, it holds for any $L > 0$. $\square$

The reduction of Theorem 3.7 to Theorem 2.5 follows almost verbatim the reduction of Theorem 3.6 to Theorem 2.2 and is left to the reader.

4. Proofs of Theorems 3.6 and 3.7

4.1. Preliminaries. 1. In what follows we will prove some results we need for the proof of Theorem 3.6.

Let $\varepsilon > 0$ be infinitesimal. Let $f : \mathbb{R} \times \mathscr{C}_o \to \mathbb{R}^d$ be standard and assume that all assumptions in Theorem 3.6 hold. The external formulations of conditions (H1), (H2), and (H3) are as follows:

(H1$'$) $\forall^{\mathrm{st}}\tau \in \mathbb{R}\ \forall^{\mathrm{st}}u \in \mathscr{C}_o\ \forall\tau' \in \mathbb{R}\ \forall u' \in \mathscr{C}_o\ (\tau' \simeq \tau, u' \simeq u \Longrightarrow f(\tau, u') \simeq f(\tau, u))$ and there exists a standard constant M such that: $|f(\tau, u)| \leq M$, $\forall^{\mathrm{st}}\tau \in \mathbb{R}$, $\forall^{\mathrm{st}}u \in \mathscr{C}_o$ (and by transfer the inequality holds for all $\tau \in \mathbb{R}$ and all $u \in \mathscr{C}_o$).

(H2$'$) $\forall^{\mathrm{st}}u \in \mathscr{C}_o\ \forall u' \in \mathscr{C}_o\ \forall\tau \in \mathbb{R}\ (u' \simeq u \Longrightarrow f(\tau, u') \simeq f(\tau, u))$.

(H3$'$) There exists a standard function $f^o : \mathscr{C}_o \to \mathbb{R}^d$ such that

$$f^o(u) \simeq \frac{1}{T}\int_0^T f(\tau, u)\, d\tau, \quad \forall^{\mathrm{st}}u \in \mathscr{C}_o,\ \forall T \simeq +\infty.$$

The following lemmas are crucial in the proof of Theorem 3.6.

Lemma 4.1. *The functional f^o is continuous and satisfies*

$$f^o(u) \simeq \frac{1}{T}\int_0^T f(\tau, u)\, d\tau$$

for all $u \in \mathscr{C}_o$, u nearstandard, and all $T \simeq +\infty$.

Proof. See [7, Lemma 4, p. 106].

Lemma 4.2. *There exists $\mu > 0$ with the property that for all limited $t \geq 0$ and all nearstandard $u \in \mathscr{C}_o$, there exists $\alpha > 0$ such that $\mu < \alpha \simeq 0$ and*

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, u)\, d\tau \simeq f^o(u).$$

Proof. See [25, Lemma 4.2, p. 7] and [7, Lemma 5, p. 107]. $\square$

Lemma 4.3. *Let $\phi \in \mathscr{C}_o$ be standard. Let x be a solution of* (1.5) *with $x_0 = \phi$, and let I be its maximal interval of definition. Let $L_1 > 0$ be standard such that $[0, L_1] \subset I$. Then x is S-continuous and nearstandard on $[0, L_1]$, and there exist $N_o \in \mathbb{N}$ and an infinitesimal partition $\{t_n : n = 0, \dots, N_o + 1\}$ of $[0, L_1]$ such that $t_0 = 0$, $t_{N_o} < L_1 \le t_{N_o+1}$, and for $n \in \{0, \dots, N_o\}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$ and*

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f(\tau, x_{t_n})\, d\tau \simeq f^o(x_{t_n}). \tag{4.1}$$

Proof. It will be done in two steps.

Step 1. Clearly, x is S-continuous on $[0, L_1]$. Indeed, let M be a standard bound of f on $\mathbb{R} \times \mathscr{C}_o$, and let $t, t' \in [0, L_1]$ be such that $t \simeq t'$, then

$$|x(t) - x(t')| \le \int_{t'}^{t} \left| f\left(\frac{\tau}{\varepsilon}, x_\tau\right) \right| d\tau \le M|t - t'| \simeq 0.$$

Furthermore, taking into account that $\phi(0)$ is limited, it is easy to see that $x(t)$ is nearstandard for all $t \in [0, L_1]$.

Note that, from what precedes, it is not difficult to deduce that x_t is nearstandard (in $\mathscr{C}_o$) for all $t \in [0, L_1]$.

Step 2. Let $\mu > 0$ be given as in Lemma 4.2, and let $S_\mu = \{\lambda \in \mathbb{R} / \forall t \in [0, L_1]\ \exists \alpha \in \mathbb{R} : \mathscr{P}_\mu(t, \alpha, a)\}$ where

$$\mathscr{P}_\mu(t, \alpha, a) \equiv \mu < \alpha < a, \quad \left| \frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x_t)\, d\tau - f^o(x_t) \right| < a.$$

Observe, by Lemma 4.2, that $a \in S_\mu$ for all appreciable real numbers $a > 0$. Hence, by Lemma 3.3, there exists $0 < a_0 \simeq 0$ such that $a_0 \in S_\mu$, i.e., a_0 is such that for all $t \in [0, L_1]$ there exists $\alpha \in \mathbb{R}$ such that $\mathscr{P}_\mu(t, \alpha, a_0)$ holds. By the axiom of choice there exists a function $c : [0, L_1] \to \mathbb{R}$ such that $c(t) = \alpha$, i.e., $\mathscr{P}_\mu(t, c(t), a_0)$ holds for all $t \in [0, L_1]$. As $c(t) > \mu$ for all $t \in [0, L_1]$, it suffices to let $t_0 = 0 < t_1 < \dots < t_{N_o} \le L_1 < t_{N_o+1}$ with $t_{n+1} = t_n + c(t_n)$ for $n \in \{0, \dots, N_o\}$, to complete the proof of the lemma. □

Lemma 4.4. *Let $\phi \in \mathscr{C}_o$ be standard. Let x be a solution of* (1.5) *with $x_0 = \phi$, and let I be its maximal interval of definition. Let $L_1 > 0$ be standard such that $[0, L_1] \subset I$. Then for all $t \in [0, L_1]$*

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau \simeq \int_0^t f^o(x_\tau)\, d\tau.$$

Proof. By Lemma 4.3, there exists an infinitesimal partition $\{t_n : n = 0, \dots, N_o + 1\}$ of $[0, L_1]$ such that $t_0 = 0$, $t_{N_o} < L_1 \le t_{N_o+1}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$ and

$$\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f(\tau, x_{t_n})\, d\tau \simeq f^o(x_{t_n}). \tag{4.2}$$

Let $t \in [0, L_1]$ and let $N \in \mathbb{N}$ be such that $t_N < t \le t_{N+1}$. We have

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau)\, d\tau = \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau) \right) d\tau$$

$$= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau) \right) d\tau + \int_{t_N}^{t} \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau) \right) d\tau. \tag{4.3}$$

Let $\alpha = \max\{\alpha_n : 0 \le n \le N-1\}$. By Lemma 3.4, we have $\alpha \simeq 0$. Let M be a standard bound for f and then for f^o too. Then

$$\left|\int_{t_N}^{t} \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau)\right) d\tau\right| \le \int_{t_N}^{t} \left(\left|f\left(\frac{\tau}{\varepsilon}, x_\tau\right)\right| + |f^o(x_\tau)|\right) d\tau \le 2M\alpha \simeq 0.$$

Thus, from (4.3) we obtain the estimate

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau)\,d\tau \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f^o(x_\tau)\right) d\tau. \tag{4.4}$$

By Lemma 4.3, we have $x_\tau \simeq x_{t_n}$ for $\tau \in [t_n, t_{n+1}]$ and x_{t_n} is nearstandard so that by condition (H2′) and Lemma 4.1 (the continuity of f^o) it follows respectively that

$$f\left(\frac{\tau}{\varepsilon}, x_\tau\right) = f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) + \gamma_n(\tau), \quad f^o(x_\tau) = f^o(x_{t_n}) + \delta_n(\tau)$$

with $\gamma_n(\tau) \simeq 0 \simeq \delta_n(\tau)$. Hence, from (4.4) it follows that

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau)\,d\tau \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n}) + \eta_n(\tau)\right) d\tau$$
$$= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n})\right) d\tau + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \eta_n(\tau)\,d\tau$$

where $\eta_n(\tau) = \gamma_n(\tau) + \delta_n(\tau)$, and therefore

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau)\,d\tau \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n})\right) d\tau$$

since

$$\left|\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \eta_n(\tau)\,d\tau\right| \le \overline{\eta} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} d\tau = \overline{\eta}.t_N$$

where $\overline{\eta} = \max\{\sup\{|\eta_n(\tau)| : t_n \le \tau \le t_{n+1}\} : 0 \le n \le N-1\}$ is, by Lemma 3.4, infinitesimal and so is $\overline{\eta}.t_N$.

Let $n \in \{0, \ldots, N-1\}$. By means of (4.2), we have

$$\int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) - f^o(x_{t_n})\right) d\tau = \int_{t_n}^{t_n+\alpha_n} f\left(\frac{\tau}{\varepsilon}, x_{t_n}\right) d\tau - \alpha_n.f^o(x_{t_n})$$
$$= \varepsilon \int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f(\tau, x_{t_n})\,d\tau - \alpha_n.f^o(x_{t_n}) = \alpha_n\left(\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f(\tau, x_{t_n})\,d\tau - f^o(x_{t_n})\right) = \alpha_n.\beta_n$$

with $\beta_n \simeq 0$.

Let $\bar{\beta} = \max\{|\beta_n| : 0 \leq n \leq N-1\}$. By Lemma 3.4, $\bar{\beta} \simeq 0$ and then $\bar{\beta}.t_N \simeq 0$. It follows that

$$\left|\sum_{n=0}^{N-1} \alpha_n.\beta_n\right| \leq \bar{\beta} \sum_{n=0}^{N-1} \alpha_n = \bar{\beta} \sum_{n=0}^{N-1} (t_{n+1} - t_n) = \bar{\beta}.t_N \simeq 0.$$

This implies that

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau - \int_0^t f^o(x_\tau)\, d\tau \simeq \sum_{n=0}^{N-1} \alpha_n.\beta_n \simeq 0$$

and completes the proof of Lemma 4.4. □

Lemma 4.5. *Let $\phi \in \mathscr{C}_o$ be standard. Let x be a solution of* (1.5) *with $x_0 = \phi$, and let I be its maximal interval of definition. Let $L_1 > 0$ be standard such that $[0, L_1] \subset I$. Then the shadow of x on $[0, L_1]$ coincides with the solution y of* (1.6) *on this interval so that $x(t) \simeq y(t)$ for all $t \in [0, L_1]$.*

Proof. From Lemmas 4.3 and 4.4, we have that x is S-continuous and nearstandard on $[0, L_1]$, and satisfies

$$x(t) \simeq \phi(0) + \int_0^t f^o(x_\tau)\, d\tau, \quad \forall t \in [0, L_1].$$

Then, if ${}^o x$ is the shadow of x on $[0, L_1]$, it can easily be shown that the standard function $z : [0, L_1] \to \mathbb{R}$ defined by

$$z(t) = \begin{cases} {}^o x(t), & t \in [0, L_1] \\ \phi(t), & t \in [-r, 0] \end{cases}$$

is a solution of (1.6). From (H4) we deduce that z and y coincide on $[-r, L_1]$ so that $x(t) \simeq {}^o x(t) = z(t) = y(t)$ for $t \in [0, L_1]$, which finishes the proof. □

Lemma 4.6. *Let $\phi \in \mathscr{C}_o$ be standard. Let x be a solution of* (1.5) *with $x_0 = \phi$, and let I be its maximal interval of definition. Let $L_1 > 0$ be limited such that $[0, L_1] \subset I$. Then $x(t) \simeq y(t)$ for all $t \in [0, L_1]$.*

Proof. If $L_1 \simeq 0$ there is nothing to prove. Suppose that L_1 is not infinitesimal. By Lemma 4.5, we have $x(t) \simeq y(t)$ for all $t \in [0, a]$ and all standard a such that $0 < a \leq L_1$. By the permanence principle, the approximation holds also for some $a \simeq L_1$. Since $x(t) \simeq x(a)$ and $y(t) \simeq y(a)$ for all $t \in [a, L_1]$, we have $x(t) \simeq y(t)$ for all $t \in [0, L_1]$. This completes the proof of the lemma. □

2. In this part, we give in Lemma 4.7 below the external formulation of an equilibrium exponential stability definition. This result is needed for the proof of Theorem 3.7.

Lemma 4.7. *The equilibrium y_e of* (1.6) *is exponentially stable if and only if it admits a standard domain of exponential stability, that is, there exist standard b, K and $\lambda > 0$ such that, for any standard $t_0 \in \mathbb{R}$ and $\phi \in \mathscr{C}_o$, the solution $y = y(\,\cdot\,; t_0, \phi)$ of* (1.6) *for which $|\phi - y_e| < b$ is defined on $[t_0 - r, +\infty)$ and the inequality $|y(t) - y_e| \leq K e^{-\lambda(t - t_0)} |\phi - y_e|$ holds for $t \geq t_0$.*

Proof. The conclusion of the lemma is obtained by successive use of the transfer principle. □

4.2. Proof of Theorem 3.6. Let $L > t_0$ be standard and let K be a standard tubular neighborhood around $\Gamma = y([t_0, L])$. Let x be a solution of (1.5) with $x_{t_0} = \phi$ and let I be its maximal interval of definition. Define the set $S = \{L_1 \in I \cap [t_0, L] \;/ x([t_0, L_1]) \subset K\}$. Clearly, S is nonempty ($t_0 \in S$) and bounded above by L. Let L_0 be a lower upper bound of S and let $L_1 \in S$ be such that $L_0 - \varepsilon < L_1 \leq L_0$. By continuation, there exists an appreciable L_2, such that x remains defined on $[t_0, L_1 + \varepsilon L_2]$. By Lemma 4.6 we have $x(t) \simeq y(t)$ for $t \in [t_0, L_1 + \varepsilon L_2]$. Suppose $L_1 + \varepsilon L_2 \leq L$. Then $[t_0, L_1 + \varepsilon L_2] \subset I$ and $x([t_0, L_1 + \varepsilon L_2]) \subset K$ imply that $L_1 + \varepsilon L_2 \in S$, which is a contradiction with $L_1 + \varepsilon L_2 > L_0$. Thus, $L_1 + \varepsilon L_2 > L$, that is, we have $x(t) \simeq y(t)$ for all $t \in [t_0, L] \subset [t_0, L_1 + \varepsilon L_2]$. □

4.3. Proof of Theorem 3.7. Let x be a solution of (1.5) with $x_{t_0} = \phi$. On $[t_0 - r, t_0]$ we have $x(t) = y(t) = \phi(t)$ and therefore the conclusion of the theorem holds. By Theorem 3.6, the approximation $x(t) \simeq y(t)$ is satisfied for all $t \in [t_0, L]$, $L > t_0$, with L standard. Let $t_1 > t_0$, t_1 standard. The instant of time t_1 will be chosen suitably later.

Now, for $n \in \mathbb{N}$, let $I_n = [t_0 + nt_1, t_0 + (n+1)t_1]$. The collection $\{I_n\}_{n\geq 0}$ is a partition of the infinite time interval $[t_0, +\infty)$ so that $[t_0, +\infty) = \bigcup_{n\geq 0} I_n$. On each interval I_n, $n \geq 1$, we define y_n as the solution of (1.6) with initial function $y_n(t) = x(t)$ for $t \in [t_0 + nt_1 - r, t_0 + nt_1]$. By Theorem 3.6, the approximation $x(t) \simeq y_n(t)$ holds for all $t \in I_n$, $n \geq 1$.

Let $n \geq 1$ and $t \geq t_0 + nt_1$. From the definition of exponential stability and its properties we have

$$|y(t) - y_n(t)| \leq Ke^{-\lambda(t-t_0-nt_1)} \sup_{s\in[t_0+nt_1-r,t_0+nt_1]} |y(s) - y_n(s)| \tag{4.5}$$

where $K \geq 1$ and $\lambda > 0$ are standard. Using the triangle inequality, we have for $s \in [t_0 + nt_1 - r, t_0 + nt_1]$

$$|y(s) - y_n(s)| \leq |y(s) - y_{n-1}(s)| + |y_n(s) - y_{n-1}(s)|. \tag{4.6}$$

However, by Theorem 3.6, we have $y_n(s) = x(s) \simeq y_{n-1}(s)$ for all $s \in [t_0 + nt_1 - r, t_0 + nt_1]$ and then, by Lemma 4.6 $\alpha := \max_{n\geq 0} \sup_{s\in[t_0+nt_1-r,t_0+nt_1]} |y_n(s) - y_{n-1}(s)| \simeq 0$. Take $t_1 \geq r - t_0$. From (4.5) and (4.6), it follows that

$$|y(t) - y_n(t)| \leq Ke^{-\lambda(t-t_0-nt_1)} \left(\sup_{s\in[t_0+nt_1-r,t_0+nt_1]} |y(s) - y_{n-1}(s)| + \alpha \right) \tag{4.7}$$

so that

$$\begin{aligned} &\sup_{s\in[t_0+(n+1)t_1-r,t_0+(n+1)t_1]} |y(s) - y_n(s)| \\ \leq K &\sup_{s\in[t_0+(n+1)t_1-r,t_0+(n+1)t_1]} e^{-\lambda(s-t_0-nt_1)} (\sup_{s\in[t_0+nt_1-r,t_0+nt_1]} |y(s) - y_{n-1}(s)| + \alpha) \\ &= Ke^{-\lambda(t_0+t_1-r)} (\sup_{s\in[t_0+nt_1-r,t_0+nt_1]} |y(s) - y_{n-1}(s)| + \alpha) \end{aligned}$$

or equivalently

$$|y - y_n|_n \leq Ke^{-\lambda(t_0+t_1-r)} \left(|y - y_{n-1}|_{n-1} + \alpha\right), \qquad n = 1, 2, \ldots,$$

where $|y - y_n|_n := \sup_{s\in[t_0+(n+1)t_1-r,t_0+(n+1)t_1]} |y(s) - y_n(s)|$. Choose t_1 such that $Ke^{-\lambda(t_0+t_1-r)} < 1$. Since $|y - y_0|_0 = 0$ we deduce that

$$|y - y_n|_n \leq \frac{Ke^{-\lambda(t_0+t_1-r)}}{1 - Ke^{-\lambda(t_0+t_1-r)}}\alpha.$$

Return now to (4.7). The above inequality implies, for $t \in I_n$, $n \geq 0$, that

$$|y(t) - y_n(t)| \leq Ke^{-\lambda(t-t_0-nt_1)} \left(\frac{Ke^{-\lambda(t_0+t_1-r)}}{1 - Ke^{-\lambda(t_0+t_1-r)}} + 1 \right) \alpha \leq \frac{K\alpha}{1 - Ke^{-\lambda(t_0+t_1-r)}}.$$

That is, $y(t) \simeq y_n(t)$ on I_n.

Thus, for $t \in I_n$, $n \geq 0$,

$$(x(t) \simeq y_n(t), y(t) \simeq y_n(t)) \Longrightarrow x(t) \simeq y(t).$$

As n is chosen arbitrarily, this completes the proof. □

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