Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 305 (2005) 43–52
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Perturbations for linear difference equations

Tewfik Sari *, Tahia Zerizer

Laboratoire de Mathématiques, Université de Haute Alsace, 4, rue des Frères Lumière, 68093, Mulhouse, France

Received 10 February 2004
Available online 19 November 2004
Submitted by G. Ladas

Abstract

In this paper we study boundary value problems for perturbed second-order linear difference equations with a small parameter. The reduced problem obtained when the parameter is equal to zero is a first-order linear difference equation. The solution is represented as a convergent series in the small parameter, whose coefficients are given by means of solutions of the reduced problem.

Keywords: Difference equation; Perturbations

1. Introduction

This paper will focus on perturbations for linear difference equations. We are interested in the following boundary value problem:

$$\begin{aligned} &\varepsilon y_{k+2} + a_k y_{k+1} + b_k y_k = f_k, \quad 0 \leqslant k \leqslant N-2, \\ &y_0 = \alpha, \qquad y_N = \beta, \end{aligned} \tag{1}$$

where (a_k), (b_k), and (f_k), $0 \leqslant k \leqslant N-2$, are given finite sequences of real numbers, ε is a small parameter, and α and β are given constants. We study the existence and uniqueness

* Corresponding author.
E-mail addresses: t.sari@uha.fr (T. Sari), t.zerizer@uha.fr (T. Zerizer).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.10.012

of its solution and derive the asymptotic behavior of the solution when $\varepsilon \to 0$. We show how to represent the solution of problem (1) as a convergent series

$$y_k(\varepsilon) = \sum_{j=0}^{+\infty} \varepsilon^j y_k^{(j)}, \quad 0 \leqslant k \leqslant N. \tag{2}$$

Our method consists simply in writing problem (1) as a matrix equation of the form

$$(A_0 + \varepsilon U)y = f, \quad y = (y_0, \ldots, y_N)',$$

where A_0 is a nonsingular matrix and the prime denotes the transpose (see the proof of Theorem 1 for the details).

Problem (1) was considered by Comstock and Hsiao [1] in the homogeneous case $f_k = 0$. These authors developed a *singular perturbation method* for the study of the problem, by analogy with the case of ordinary differential equations. They wrote the solution as the sum of an *outer solution* and *a boundary layer correction* (see Section 2.4 for the details). Their results were proposed again by Holmes in his book (see [5, pp. 98–104]) since the problem under study is considered by this author as a typical problem to illustrate the subtle and interesting analogies between differential and difference equations.

Problem (1) has many great applications especially in the theory of discrete control systems [3,7,8]. So, Naidu and Rao (see [8, Chapter 1]) gave a lot of interest to problem (1) in the case of constant coefficient $a_k = a$, $b_k = b$ for all k, and extended the results of [1] to higher-order (see Section 3.1 for the details).

The paper is organized as follows. In Section 2 we prove our main result related to the existence and uniqueness of the solution of problem (1) as well as its representation as a convergent series (2). We also examine the case where problem (1) is replaced by

$$\begin{aligned} &a_k y_{k+2} + b_k y_{k+1} + \varepsilon y_k = f_k, \quad 0 \leqslant k \leqslant N-2, \\ &y_0 = \alpha, \qquad y_N = \beta. \end{aligned} \tag{3}$$

Problem (3) is called a *right end perturbation* in the literature [1,8]. In Section 3 we compare our results to those obtained by the procedure given by Naidu and Rao [8]. In particular, we show that the formal solution obtained by these authors is an asymptotic expansion of the solution of order $N-1$. In Section 4, we discuss the connection between both problem (1) and (3) and the difference equation obtained by discretization of a singularly perturbed boundary value problem associated with a second-order differential equation.

2. Main result

2.1. Formal solution

First, we seek a straightforward expansion of the form (2). Substituting (2) into (1) and equating coefficients term-wise then determines the coefficients of (2) successively. Thus, for power zero of ε we must have

$$a_k y_{k+1}^{(0)} + b_k y_k^{(0)} = f_k, \quad 0 \leqslant k \leqslant N-2, \qquad y_0^{(0)} = \alpha, \tag{4}$$

and

$$y_N^{(0)} = \beta. \tag{5}$$

Notice that (4) is an initial value problem. It defines the sequence $(y_0^{(0)}, \dots, y_{N-1}^{(0)})$ if and only if $a_k \neq 0$ for $0 \leqslant k \leqslant N-2$. The second boundary condition (5) defines $y_N^{(0)}$.

For higher powers of ε, we must have

$$a_k y_{k+1}^{(j)} + b_k y_k^{(j)} = -y_{k+2}^{(j-1)}, \quad 0 \leqslant k \leqslant N-2, \qquad y_0^{(j)} = 0, \tag{6}$$

and

$$y_N^{(j)} = 0, \tag{7}$$

for each $j \geqslant 1$. Notice that (6) is an initial value problem, which defines the sequence $(y_0^{(j)}, \dots, y_{N-1}^{(j)})$ if and only if $a_k \neq 0$ for $0 \leqslant k \leqslant N-2$. The second boundary condition (7) defines $y_N^{(j)}$.

We note that the terms $y_k^{(0)}$, $0 \leqslant k \leqslant N-1$, can be computed without any knowledge of the boundary condition $y_N = \beta$. By analogy with the case of differential equations, we say that there is a boundary layer at the right. Since $y_{N-1}^{(1)}$, depends on $y_N^{(0)} = \beta$, the higher-order terms $y_k^{(j)}$, $j \geqslant 1$, depend on the boundary condition $y_N = \beta$.

2.2. *Existence and convergence of the solution*

In this section we state conditions for problem (1) so it will have a unique solution and that the series (2) converges. Consider the norm

$$\|y\| = \max\big(|y_0|, \dots, |y_N|\big)$$

in $\mathbb{R}^{N+1}$ and, for a matrix $A = (a_{ij})$, the associated matrix norm

$$\|A\| = \sup_{\|y\|=1} \|Ay\| = \max_{i=0,\dots,N} \left(\sum_{j=0}^{N} |a_{ij}| \right).$$

We define

$$\varepsilon_0 := \frac{1}{\|UA_0^{-1}\|} \quad \text{and} \quad C := \big\|A_0^{-1}\big\| \|f\|. \tag{8}$$

Theorem 1. *Assume that $a_k \neq 0$ for $0 \leqslant k \leqslant N-2$ and $|\varepsilon| < \varepsilon_0$. Then the solution $(y_k(\varepsilon))$ of* (1) *exists and is unique and satisfies* (2) *uniformly for $0 \leqslant k \leqslant N$, where $y_k^{(0)}$ and $y_k^{(j)}$ are the solutions of* (4)(5) *and* (6)(7)*, respectively. More precisely, for all $n \geqslant 0$ and all $0 \leqslant k \leqslant N$, we have*

$$\left| y_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y_k^{(j)} \right| \leqslant C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0},$$

where ε_0 and C are given by (8).

Proof. We can write problems (4)(5), (6)(7), and (1) in the matrix forms

$$A_0 y^{(0)} = f, \qquad y^{(0)} = \left(y_0^{(0)}, \dots, y_N^{(0)}\right)', \tag{9}$$

$$A_0 y^{(j)} = -U y^{(j-1)}, \qquad y^{(j)} = \left(y_0^{(j)}, \dots, y_N^{(j)}\right)', \quad \text{and} \tag{10}$$

$$A_\varepsilon y = f, \qquad y = (y_0, \dots, y_N)', \tag{11}$$

respectively, where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & & & 0 \\ b_0 & a_0 & 0 & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & b_{N-2} & a_{N-2} & 0 \\ 0 & & & \cdots & 0 & 0 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 0 & 0 & \cdots & & & 0 \\ 0 & 0 & 1 & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & 0 & 0 & 1 \\ 0 & & & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$$A_\varepsilon = A_0 + \varepsilon U, \qquad f = (\alpha, f_0, \dots, f_{N-2}, \beta)'.$$

If matrix A_ε is nonsingular, then problem (11) has a unique solution $y(\varepsilon)$ which is given by

$$y(\varepsilon) = A_\varepsilon^{-1} f. \tag{12}$$

Let us compute the inverse of matrix A_ε. Since $a_k \neq 0$ for $0 \leqslant k \leqslant N-2$, matrix A_0 is nonsingular. Since $|\varepsilon| < \varepsilon_0$, we have $\|\varepsilon U A_0^{-1}\| < 1$. Thus

$$A_\varepsilon^{-1} = A_0^{-1}\left(I + \varepsilon U A_0^{-1}\right)^{-1} = A_0^{-1} \sum_{j=0}^{+\infty} \left(-\varepsilon U A_0^{-1}\right)^j. \tag{13}$$

From (12) and (13) we have

$$y(\varepsilon) = \sum_{j=0}^{+\infty} \varepsilon^j y^{(j)}, \quad \text{where } y^{(j)} = A_0^{-1}\left(-U A_0^{-1}\right)^j f. \tag{14}$$

Since $A_0 y^{(0)} = f$ and $A_0 y^{(j)} = -U y^{(j-1)}$, for $j \geqslant 1$ we deduce that $y^{(0)}$ and $y^{(j)}$ are the solutions of (9) and (10), respectively. Hence, $y_k^{(0)}$ and $y_k^{(j)}$ are the solutions of (4)(5) and (6)(7), respectively. This proves the first part of the theorem. Let us evaluate now the reminder of the series. We have

$$\begin{aligned} \left\| A_\varepsilon^{-1} - A_0^{-1} \sum_{j=0}^{n} \left(-\varepsilon U A_0^{-1}\right)^j \right\| &\leqslant \left\|A_0^{-1}\right\| \sum_{j=n+1}^{+\infty} \left\|\varepsilon U A_0^{-1}\right\|^j \\ &= \frac{\|A_0^{-1}\| \, \|\varepsilon U A_0^{-1}\|^{n+1}}{1 - \|\varepsilon U A_0^{-1}\|} \leqslant \left\|A_0^{-1}\right\| \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}. \end{aligned} \tag{15}$$

From (14) and (15) we deduce that

$$\left\| y(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y^{(j)} \right\| \leqslant \left\| A_\varepsilon^{-1} - A_0^{-1} \sum_{j=0}^{n} \left(-\varepsilon U A_0^{-1}\right)^j \right\| \|f\|$$
$$\leqslant \left\| A_0^{-1} \right\| \|f\| \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}.$$

This completes the proof of the theorem. □

2.3. *Right end perturbation*

Let $z_k = y_{N-k}$, for $0 \leqslant k \leqslant N$. Then Eq. (3) becomes

$$\varepsilon z_{k+2} + b_{N-k-2} z_{k+1} + a_{N-k-2} z_k = f_{N-k-2},$$

which is of the form (1). From Theorem 1 we deduce that for right end perturbations the boundary value problem has a unique solution if and only if $b_k \neq 0$ for $0 \leqslant k \leqslant N-2$. The solution is the sum of a convergent series. The boundary layer is located at the left.

2.4. *Comstock–Hsiao's approach*

The homogeneous case

$$\begin{aligned} &\varepsilon y_{k+2} + a_k y_{k+1} + b_k y_k = 0, \quad 0 \leqslant k \leqslant N-2, \\ &y_0 = \alpha, \qquad y_N = \beta \end{aligned} \tag{16}$$

was considered by Comstock and Hsiao [1]. These authors developed a *singular perturbation method* for the study of (16). They gave an asymptotic approximation of the solution of (16) when $\varepsilon \to 0$. They did not study the problem of the existence and uniqueness of this solution. Let z_k be the solution of the initial value problem (the *reduced problem*)

$$\begin{aligned} &a_k z_{k+1} + b_k z_k = 0, \quad 0 \leqslant k \leqslant N-2, \\ &z_0 = \alpha. \end{aligned} \tag{17}$$

Let w_k be the solution of the final value problem (the *boundary layer equation*)

$$\begin{aligned} &w_{k+2} + a_k w_{k+1} = 0, \quad N-2 \geqslant k \geqslant 0, \\ &w_N = \beta - z_N. \end{aligned} \tag{18}$$

The main result of [1] is that the solution $y_k(\varepsilon)$ of (16) has the representation form

$$y_k(\varepsilon) = z_k + \varepsilon^{N-k} w_k + O(\varepsilon), \tag{19}$$

as $\varepsilon \to 0$, uniformly for $0 \leqslant k \leqslant N$ (see [1, Theorem 2]).

Notice that the reduced problem (17) defines z_k only for $0 \leqslant k \leqslant N-1$. We do not have the value of z_N to our disposal, so that problem (18) is not well-defined. The results of [1] were discussed by Holmes (see [5, p. 98]). Also, this author did not clarified the problem with the definition of z_N.

3. The constant coefficients case

The representation (19) of the solution was suggested by the special case of (16) where a_k and b_k are constant:

$$\begin{aligned} &\varepsilon y_{k+2} + a y_{k+1} + b y_k = 0, \quad 0 \leqslant k \leqslant N-2, \\ &y_0 = \alpha, \qquad y_N = \beta. \end{aligned} \tag{20}$$

In this case problem (17) may be solved until $k = N - 1$, so that z_N is defined and the boundary layer equation (18) is well-defined. The representation (19) of the solution was extended to higher-order by Naidu and Rao [8].

3.1. *Naidu–Rao's expansion*

In [8, Section 1.2], Naidu and Rao represented the solution $y_k(\varepsilon)$, $0 \leqslant k \leqslant N$, of (20), as the sum of an *outer series* and a *correction series*:

$$y_k(\varepsilon) = \sum_{j=0}^{+\infty} \varepsilon^j z_k^{(j)} + \varepsilon^{N-k} \sum_{j=0}^{+\infty} \varepsilon^j w_k^{(j)}. \tag{21}$$

The coefficients $z_k^{(0)}$ and $z_k^{(j)}$, $j \geqslant 1$, of the outer series, are the solutions of the initial value problems

$$a z_{k+1}^{(0)} + b z_k^{(0)} = 0, \qquad z_0^{(0)} = \alpha, \tag{22}$$

and

$$a z_{k+1}^{(j)} + b z_k^{(j)} = -z_{k+2}^{(j-1)}, \qquad z_0^{(j)} = 0, \tag{23}$$

respectively. The coefficients $w_k^{(0)}$ and $w_k^{(j)}$, $j \geqslant 1$, of the correction series are the solutions of the final value problems

$$w_{k+2}^{(0)} + a w_{k+1}^{(0)} = 0, \qquad w_N^{(0)} = \beta - z_N^{(0)}, \tag{24}$$

and

$$w_{k+2}^{(j)} + a w_{k+1}^{(j)} = -b w_k^{(j-1)}, \qquad w_N^{(j)} = -z_N^{(j)}, \tag{25}$$

respectively. This formal procedure was not justified in [8]. We prove that the series (21) is an asymptotic expansion of the solution of order $N - 1$ (see Proposition 3 below). It is not an asymptotic expansion of order N as shown in the following proposition.

Proposition 2. *The series* (21) *is not an asymptotic expansion of the solution* $y_k(\varepsilon)$ *of order* N.

Proof. Since $z_0^{(0)} = \alpha$ and $z_0^{(j)} = 0$ for $j \geqslant 1$, from (21) we get

$$y_0(\varepsilon) = \sum_{j=0}^{+\infty} \varepsilon^j z_0^{(j)} + \varepsilon^N \sum_{j=0}^{+\infty} \varepsilon^j w_0^{(j)} = \alpha + \sum_{j=0}^{+\infty} \varepsilon^{j+N} w_0^{(j)}. \tag{26}$$

Since in general $w_0^{(j)} \neq 0$, (26) is not an asymptotic expansion of $y_0(\varepsilon) = \alpha$ of order N. □

3.2. There is no need of correction series

According to Theorem 1, the solution $y_k(\varepsilon)$, $0 \leqslant k \leqslant N$, of (20) has the representation (2) where $y_k^{(0)}$ and $y_k^{(j)}$, $j \geqslant 1$, $0 \leqslant k \leqslant N$, are the solutions of problems

$$\begin{aligned} &a y_{k+1}^{(0)} + b y_k^{(0)} = 0, \quad 0 \leqslant k \leqslant N-2, \\ &y_0^{(0)} = \alpha \quad \text{and} \quad y_N^{(0)} = \beta, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &a y_{k+1}^{(j)} + b y_k^{(j)} = -y_{k+2}^{(j-1)}, \quad 0 \leqslant k \leqslant N-2, \\ &y_0^{(j)} = 0 \quad \text{and} \quad y_N^{(j)} = 0, \end{aligned} \tag{28}$$

respectively. Notice that problems (27) and (28) differ from problems (22) and (23), respectively, only by the fact that the first order difference equation is used to compute the solution only for $0 \leqslant k \leqslant N-2$. Let us compare the expansion (21) of Naidu and Rao and our expansion (2). From (21) we see that

$$y_k(\varepsilon) = \sum_{j=0}^{+\infty} \varepsilon^j \tilde{y}_k^{(j)},$$

where

$$\tilde{y}_k^{(j)} = \begin{cases} z_k^{(j)}, & \text{if } k + j \leqslant N-1, \\ z_k^{(j)} + w_k^{(j+k-N)}, & \text{if } k + j \geqslant N. \end{cases} \tag{29}$$

Proposition 3. *If* $0 \leqslant j \leqslant N-1$ *and* $0 \leqslant k \leqslant N$, *then we have* $\tilde{y}_k^{(j)} = y_k^{(j)}$, *that is to say, the series* (21) *is an asymptotic expansion of* $y_k(\varepsilon)$ *of order* $N-1$.

Proof. The proof is by induction on j. Let us prove the property for $j = 0$. From (29), (22), and (24) we get

$$\begin{aligned} &a\tilde{y}_{k+1}^{(0)} + b\tilde{y}_k^{(0)} = a z_{k+1}^{(0)} + b z_k^{(0)} = 0, \quad \text{if } k \leqslant N-2, \\ &\tilde{y}_0^{(0)} = z_0^{(0)} = \alpha, \qquad \tilde{y}_N^{(0)} = z_N^{(0)} + w_N^{(0)} = \beta. \end{aligned}$$

Thus $(\tilde{y}_0^{(0)}, \dots, \tilde{y}_N^{(0)})$ satisfies (27). By the uniqueness of the solution of (27), we conclude that $\tilde{y}_k^{(0)} = y_k^{(0)}$ for $0 \leqslant k \leqslant N$. Let j be such that $1 \leqslant j \leqslant N-1$ and

$$\tilde{y}_k^{(j-1)} = y_k^{(j-1)}, \quad \text{for } 0 \leqslant k \leqslant N. \tag{30}$$

From (29) we get

$$a\tilde{y}_{k+1}^{(j)} + b\tilde{y}_k^{(j)} = \begin{cases} a z_{k+1}^{(j)} + b z_k^{(j)}, & \text{if } k + j \leqslant N-2, \\ a z_{k+1}^{(j)} + b z_k^{(j)} + a w_{k+1}^{(0)}, & \text{if } k + j = N-1, \\ a z_{k+1}^{(j)} + b z_k^{(j)} + a w_{k+1}^{(j+k+1-N)} & \\ \quad + b w_k^{(j+k-N)}, & \text{if } k + j \geqslant N. \end{cases}$$

In the first case, $k + j \leqslant N - 2$, from (23) and (29) we get

$$a\tilde{y}_{k+1}^{(j)} + b\tilde{y}_k^{(j)} = -z_{k+2}^{(j-1)} = -\tilde{y}_{k+2}^{(j-1)}.$$

In the second case, $k + j = N - 1$, from (23), (24), and (29) we get

$$a\tilde{y}_{k+1}^{(j)} + b\tilde{y}_k^{(j)} = -z_{k+2}^{(j-1)} - w_{k+2}^{(0)} = -\tilde{y}_{k+2}^{(j-1)}.$$

In the third case, $k + j \geqslant N$, from (23), (25), and (29) we get

$$a\tilde{y}_{k+1}^{(j)} + b\tilde{y}_k^{(j)} = -z_{k+2}^{(j-1)} - w_{k+2}^{(j+k+1-N)} = -\tilde{y}_{k+2}^{(j-1)}.$$

Thus, we have shown that

$$a\tilde{y}_{k+1}^{(j)} + b\tilde{y}_k^{(j)} = -\tilde{y}_{k+2}^{(j-1)}, \quad k \geqslant 0.$$

Using the induction assumption (30), we get

$$a\tilde{y}_{k+1}^{(j)} + b\tilde{y}_k^{(j)} = -y_{k+2}^{(j-1)}, \quad 0 \leqslant k \leqslant N - 2. \tag{31}$$

Since $j \leqslant N - 1$, from (29), (23), and (25) we get

$$\tilde{y}_0^{(j)} = z_0^{(j)} = 0, \qquad \tilde{y}_N^{(j)} = z_N^{(j)} + w_N^{(j)} = 0. \tag{32}$$

From (31) and (32) we see that $(\tilde{y}_0^{(j)}, \ldots, \tilde{y}_N^{(j)})$, satisfies (28). By the uniqueness of the solution of (28), we conclude that $\tilde{y}_k^{(j)} = y_k^{(j)}$ for $0 \leqslant k \leqslant N$. □

4. Connection with differential equations

The connection between both Eqs. (1) and (3) and the numerical solutions of the singularly perturbed second-order boundary value problem

$$\begin{aligned} &\varepsilon y'' + p(x)y' + q(x)y = f(x), \quad 0 < x < 1, \\ &y(0) = \alpha, \qquad y(1) = \beta \end{aligned} \tag{33}$$

was clarified by Holmes [5]. It is well known (see [9, Section 3.A]) that if $p(x) < 0$, the solution of (33) is approximated on $[0, 1)$ by the solution of the initial value problem

$$p(x)\dot{z} + q(x)z = f(x), \qquad z(0) = \alpha,$$

and has a terminal layer near $x = 1$. If, instead, $p(x) > 0$ for $0 \leqslant x \leqslant 1$, the solution $y(x, \varepsilon)$ would be approximated on $(0, 1]$ by the solution of the final value problem

$$p(x)\dot{z} + q(x)z = f(x), \qquad z(1) = \beta,$$

and the boundary layer is near $x = 0$. The location of the boundary layer depends on the sign of the coefficient of the first derivative. If, instead, $p(x)$ has a zero in the interval [0, 1], we have a turning-point problem, whose analysis is more delicate (see [9, Section 3.E]).

We solve numerically (33) on a uniform grid of size $h = 1/(N + 1)$. Let $x_n = nh$, $p_n = p(x_n)$, $q_n = q(x_n)$, $f_n = f(x_n)$, and $y_n(\varepsilon) = y(x_n, \varepsilon)$. The standard centered difference approximation

$$y_n'' = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

will be used for the second derivative. Using the backward-difference approximation for the first derivative

$$y'_n = \frac{y_n - y_{n-1}}{h}, \tag{34}$$

we get from (33) that

$$\begin{aligned} &\varepsilon y_{n+1} + (a_n - 2\varepsilon) y_n + (b_n + \varepsilon) y_{n-1} = h^2 f_n, \quad 1 \leqslant n \leqslant N-1, \\ &y_0 = \alpha, \qquad y_N = \beta, \end{aligned} \tag{35}$$

where $a_n = hp_n + h^2 q_n$ and $b_n = -hp_n$. This equation differs from (1) only in the dependence in ε of the coefficients. We have the following result whose proof is similar to the proof of Theorem 1 and is left to the reader.

Theorem 4. *Assume that $a_n \neq 0$ for $1 \leqslant n \leqslant N-1$. There exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$, the solution $(y_0(\varepsilon), \dots, y_N(\varepsilon))$ of* (35) *exists, is unique, and is the sum of a convergent series*

$$y_n(\varepsilon) = \sum_{j=0}^{+\infty} \varepsilon^j y_n^{(j)}, \quad 0 \leqslant n \leqslant N,$$

where $(y_0^{(0)}, \dots, y_N^{(0)})$ is the solution of problem

$$\begin{aligned} &a_n y_n^{(0)} + b_n y_{n-1}^{(0)} = h^2 f_n, \quad 1 \leqslant n \leqslant N-1, \\ &y_0^{(0)} = \alpha \quad \textit{and} \quad y_N^{(0)} = \beta, \end{aligned}$$

and, for each $j \geqslant 1$, $(y_0^{(j)}, \dots, y_N^{(j)})$ is the solution of problem

$$\begin{aligned} &a_n y_n^{(j)} + b_n y_{n-1}^{(j)} = 2y_n^{(j-1)} - y_{n-1}^{(j-1)} - y_{n+1}^{(j-1)}, \quad 1 \leqslant n \leqslant N-1, \\ &y_0^{(j)} = 0 \quad \textit{and} \quad y_N^{(j)} = 0. \end{aligned}$$

We note that the condition $a_n \neq 0$ is satisfied if $p_n \neq 0$ and $h \neq 0$ is small enough, that is, if $p(x) \neq 0$ holds throughout the interval $[0, 1]$. For turning point problems, it is possible that the condition $a_n \neq 0$ be not satisfied, so that the solution of the difference equation (35) does not exist.

We note that the coefficients $y_n^{(0)}$, $0 \leqslant n \leqslant N-1$, can be computed without any knowledge of the final boundary condition. Thus, the solution of (35), where h is kept fixed and $\varepsilon \to 0$ has a boundary layer which is located at the right. Following Holmes (see [5, p. 102]), we claim that this observation is a strong evidence that one should use (34) to solve problem (33) only in the case $p(x) < 0$, for which the boundary layer is located at $x = 1$. In the case $p(x) > 0$ the boundary layer is located at $x = 0$ and one should use the forward-difference approximation for the first derivative

$$\dot{y}_n = \frac{y_{n+1} - y_n}{h}.$$

We get from (33) that

$$(a_n+\varepsilon)y_{n+1}+(b_n-2\varepsilon)y_n+\varepsilon y_{n-1}=h^2 f_n, \quad 1\leqslant n\leqslant N-1,$$
$$y_0=\alpha, \qquad y_N=\beta, \tag{36}$$

where $a_n = hp_n$ and $b_n = -hp_n + h^2 q_n$. This equation differs from (3) only in the dependence in ε of the coefficients. Now, the boundary layer is located at left, that is, in the same location as the associated differential equation. Actually, the difference equation (36) was considered by Farrell and Shishkin [2] to approximate problem (33). These authors used the iterative Gauss Siedel process for (36) and obtained theoretical results on its convergence.

We note that using the centered difference approximation of the first derivative

$$y_n' = \frac{y_{n+1}-y_{n-1}}{2h},$$

we get from (33) that

$$a_n y_{n+1}+b_n y_n+c_n y_{n-1}=h^2 f_n, \quad 1\leqslant n\leqslant N-1,$$
$$y_0=\alpha, \qquad y_N=\beta, \tag{37}$$

where $a_n = hp_n/2+\varepsilon$, $b_n = h^2q_n - 2\varepsilon$, and $c_n = -hp_n/2+\varepsilon$. This equation is not of the form (1) or (3).

It is worthwhile to notice that Eqs. (35)–(37) contain a second small parameter: the step size h. This parameter h should be relatively small for Eq. (35), (36) or (37) to be an accurate approximation of Eq. (33). Actually, we must keep $\varepsilon > 0$ small and fixed and study the asymptotic solutions of (35)–(37) when $h \to 0$. We will not pursue this discussion, but the interested readers should consult the literature concerning numerical schemes for solving stiff differential equations [4,6].

References

[1] C. Comstock, G.C. Hsiao, Singular perturbations for difference equations, Rocky Mountain J. Math. 6 (1976) 561–567.

[2] P.A. Farrell, G.I. Shishkin, On the convergence of iterative methods for linear systems arising from singularly perturbed equations, in: Proceedings of the Copper Mountain Conference on Iterative Methods, 1998 (see http://trident.mcs.kent.edu/~farrell/papers/1998/cm98.pdf).

[3] Z. Gajic, M.-T. Lim, Optimal Control of Singularly Perturbed Linear Systems and Applications: High-Accuracy Techniques, Dekker, New York, 2001.

[4] E. Haier, G. Wanner, Solving Ordinary Differential Equations II, Stiff and Differential-Algebraic Problems, Springer Ser. Comput. Math., vol. 14, Springer-Verlag, Berlin, 1991.

[5] M.H. Holmes, Introduction to Perturbation Methods, Texts Appl. Math., vol. 20, Springer-Verlag, New-York, 1998.

[6] W.L. Miranker, Numerical Methods for Stiff Equations and Singular Perturbation Problems, Reidel, Dordrecht, 1981.

[7] D.S. Naidu, A.K. Rao, A singular perturbation method for initial value problems with inputs in discrete control systems, Internat J. Control 33 (1981) 953–965.

[8] D.S. Naidu, A.K. Rao, Singular perturbation analysis of discrete control systems, Lecture Notes in Math., vol. 1154, Springer-Verlag, Berlin, 1985.

[9] R.E. O'Malley Jr., Singular Perturbation Methods for Ordinary Differential Equations, Springer-Verlag, New York, 1991.