Averaging for Ordinary Differential Equations and Functional Differential Equations *

Tewfik Sari

Abstract

A nonstandard approach to averaging theory for ordinary differential equations and functional differential equations is developed. We define a notion of perturbation and we obtain averaging results under weaker conditions than the results in the literature. The classical averaging theorems approximate the solutions of the system by the solutions of the averaged system, for Lipschitz continuous vector fields, and when the solutions exist on the same interval as the solutions of the averaged system. We extend these results to perturbations of vector fields which are uniformly continuous in the spatial variable with respect to the time variable and without any restriction on the interval of existence of the solution.

Keywords Ordinary Differential Equation, Functional differential equations, averaging, stroboscopic method, nonstandard analysis.

2000 *Mathematics Subject Classification* 34C29, 34K25, 34E10, 34E18

1 Introduction

In the early seventies, Georges Reeb, who learned about Abraham Robinson's *Nonstandard Analysis* (NSA) [29], was convinced that NSA gives a language which is well adapted to the study of perturbation theory of differential equations (see [6] p. 374 or [25]). The axiomatic presentation *Internal Set Theory* (IST) [26] of NSA given by E. Nelson corresponded more to the Reeb's dream and was in agreement with his conviction *"Les entiers naïfs ne remplissent pas* $\mathbb{N}$*"*. Indeed, no formalism can recover exactly all the actual phenomena, and *nonstandard objects* which may be considered as a formalization of *non-naïve objects* are already elements of our usual (standard) sets. We do not need any use of *stars* and *enlargements*. Thus, the Reebian school adopted IST. For more informations about Reeb's dream and convictions see the Reeb's preface of Lutz and Goze's book [25], Stewart's book [40] p. 72, or Lobry's book [23].

The Reebian school of *nonstandard perturbation theory of differential equations* produced various and numerous studies and new results as attested by a

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lot of books and proceedings (see [2, 3, 4, 7, 8, 9, 10, 12, 23, 25, 30, 37, 42] and their references). It has become today a well-established tool in asymptotic theory, see, for instance, [17, 18, 20, 24, 39] and the five-digits classification 34E18 of the 2000 Mathematical Subject Classification. Canards and rivers (or Ducks and Streams [7]) are the most famous discoveries of the Reebian school. The classical perturbation theory of differential equations studies deformations, instead of perturbations, of differential equations (see Section 2.1). Classically the phenomena are described asymptotically, when the parameter of the deformation tends to some fixed value. The first benefit of NSA is a natural and useful notion of perturbation. A perturbed equation becomes a simple nonstandard object, whose properties can be investigated directly. This aspect of NSA was clearly described by E. Nelson in his paper *Mathematical Mythologies* [30], p. 159, when he said *"For me, the most exciting aspect of nonstandard analysis is that concrete phenomena, such as ducks and streams, that classically can only be described awkwardly as asymptotic phenomena, become mythologized as simple nonstandard objects."*

The aim of this paper is to present some of the basic nonstandard techniques for averaging in Ordinary Differential Equations (ODEs), that I obtained in [32, 36], and their extensions, obtained by M. Lakrib [19], to Functional Differential Equations (FDEs). This paper is organized as follows. In Section 2 we define the notion of *perturbation* of a vector field. The main problem of perturbation theory of differential equations is to describe the behavior of trajectories of perturbed vector fields. We define a standard topology on the set of vector fields, with the property that f is a perturbation of a standard vector field f_0 if and only if f is infinitely close to f_0 for this topology. In Section 3 we present the Stroboscopic Method for ODEs and we show how to use it in the proof of the averaging theorem for ODEs. In Section 4 we present the Stroboscopic Method for FDEs and we show how to use it in the proofs of the averaging theorem for FDEs. The nonstandard approaches of averaging are rather similar in structure both in ODEs and FDEs. It should be noticed that the usual approaches of averaging make use of different concepts for ODEs and for FDEs: compare with [5, 31] for averaging in ODEs and [13, 14, 15, 22] for averaging in FDEs.

2 Deformations and Perturbations

2.1 Deformations

The classical *perturbation theory of differential equations* studies families of differential equations

$$\dot{x} = F(x, \varepsilon), \tag{1}$$

where x belongs to an open subset U of $\mathbb{R}^n$, called *phase space*, and ε belongs to a subset B of $\mathbb{R}^k$, called *space of parameters*.

The family (1) of differential equations is said to be a *k-parameters deformation* of the vector field $F_0(x) = F(x, \varepsilon_0)$, where ε_0 is some fixed value of

ε. The main problem of the perturbation theory of differential equations is to investigate the behavior of the vector fields $F(x, \varepsilon)$ when ε tends to ε_0.

The intuitive notion of a *perturbation* of the vector field F_0 which would mean any vector field which is *close to* F_0 does not appear in the theory. The situation is similar in the theory of *almost periodic functions* which, classically, do not have *almost periods*. The nonstandard approach permits to give a very natural notion of almost period (see [16, 28, 33, 41]). The *classical perturbation theory of differential equations* considers *deformations* instead of *perturbations* and would be better called *deformation theory of differential equations*. Actually the vector field $F(x, \varepsilon)$ when ε is sufficiently close to ε_0 is called a perturbation of the vector field $F_0(x)$. In other words, the differential equation

$$\dot{x} = F_0(x) \tag{2}$$

is said to be the *unperturbed equation* and equation (1), for a fixed value of ε, is called the *perturbed equation*. This notion of *perturbation* is not very satisfactory since many of the results obtained for the family (1) of differential equations take place in all systems that are close to the unperturbed equation (2). Noticing this fact, V. I. Arnold (see [1], footnote page 157) suggested to study a neighborhood of the unperturbed vector field $F_0(x)$ in a suitable function space. For the sake of mathematical convenience, instead of neighborhoods, one considers deformations. According to V. I. Arnold, the situation is similar with the historical development of variational concepts, where the directional derivative (Gateaux differential) preceded the derivative of a mapping (Frechet differential). Nonstandard analysis permits to define a notion of perturbation. To say that a vector f is a *perturbation* of a standard vector field f_0 is equivalent to say that f is *infinitely close* to f_0 is a suitable function space, that is f *is in any standard neighborhood* of f_0. Thus, studying perturbations in our sense is nothing than studying neighborhoods, as suggested by V. I. Arnold.

2.2 Perturbations

Let X be a standard topological space. A point $x \in X$ is said to be infinitely close to a standard point $x_0 \in X$, which is denoted by $x \simeq x_0$, if x is in any standard neighborhood of x_0. Let A be a subset of X. A point $x \in X$ is said to be *nearstandard in* A if there is a standard $x_0 \in A$ such that $x \simeq x_0$. Let us denote by

$${}^{NS}A = \{x \in X : \exists^{st} x_0 \in A \; x \simeq x_0\},$$

the *external-set* of nearstandard points in A [34]. Let E be a standard uniform space. The points $x \in E$ and $y \in E$ are said to be infinitely close, which is denoted by $x \simeq y$, if (x, y) lies in every standard entourage. If E is a standard metric space, with metric d, then $x \simeq y$ is equivalent to $d(x, y)$ infinitesimal.

Definition 1 *Let X be a standard topological space X. Let E be a standard uniform space. Let D and D_0 be open subsets of X, D_0 standard. Let $f : D \to E$ and $f_0 : D_0 \to E$ be mappings, f_0 standard. The mapping f is said to be a* *perturbation of the mapping f_0, which is denoted by $f \simeq f_0$, if ${}^{NS}D_0 \subset D$ and $f(x) \simeq f_0(x)$ for all $x \in {}^{NS}D_0$.*

Let $\mathcal{F}_{X,E}$ be the set of mappings defined on open subsets of X to E :

$$\mathcal{F}_{X,E} = \{(f, D) : D \text{ open subset of } X \text{ and } f : D \to E\}.$$

Let us consider the topology on this set defined as follows. Let $(f_0, D_0) \in \mathcal{F}_{X,E}$. The family of sets of the form

$$\{(f, D) \in \mathcal{F}_{X,E} : K \subset D \ \forall x \in K \ (f(x), f_0(x)) \in U\},$$

where K is a compact subset of D_0, and U is an entourage of the uniform space E, is a basis of the system of neighborhoods of (f_0, D_0). Let us call this topology the *topology of uniform convergence on compacta.* If all the mappings are defined on the same open set D, this topology is the usual topology of uniform convergence on compacta on the set of functions on D to E.

Proposition 1 *Assume X is locally compact. The mapping f is a perturbation of the standard mapping f_0 if and only if f is infinitely close to f_0 for the topology of uniform convergence on compacta.*

Proof. Let $f : D \to E$ be a perturbation of $f_0 : D_0 \to E$. Let K be a standard compact subset of D_0. Let U be a standard entourage. Then $K \subset D$ and $f(x) \simeq f_0(x)$ for all $x \in K$. Hence $(f(x), f_0(x)) \in U$. Thus $f \simeq f_0$ for the topology of uniform convergence on compacta. Conversely, let f be infinitely close to f_0 for the topology of uniform convergence on compacta. Let $x \in {}^{NS}D_0$. There exists a standard $x_0 \in D_0$ such that $x \simeq x_0$. Let K be a standard compact neighborhood of x_0, such that $K \subset D_0$ (such a neighborhood exists since X is locally compact). Then $x \in K \subset D$ and $(f(x), f_0(x)) \in U$ for all standard entourages U, that is ${}^{NS}D_0 \subset D$ and $f(x) \simeq f_0(x)$ on ${}^{NS}D_0$. Hence f is a perturbation of f_0. ■

The notion of perturbation can be used to formulate Tikhonov's theorem on slow and fast systems whose fast dynamics has asymptotically stable equilibrium points [24], and Pontryagin and Rodygin's theorem on slow and fast systems whose fast dynamics has asymptotically stable cycles [39]. In the following section we use it to formulate the theorem of Krilov, Bogolyubov and Mitropolski of averaging for ODEs. All these theorems belong to the singular perturbation theory. In this paper, by a solution of an Initial Value Problem (IVP) associated to an ODE we mean a maximal (i.e. noncontinuable) solution. The fundamental nonstandard result of the regular perturbation theory of ODEs is called the Short Shadow Lemma. It can be stated as follows [36, 37] :

Let $g : D \to \mathbb{R}^d$ and $g_0 : D_0 \to \mathbb{R}^d$ be continuous vector fields, $D, D_0 \subset \mathbb{R}_+ \times \mathbb{R}^d$. Let a_0^0 and a^0 be initial conditions. Assume that g_0 and a_0^0 are standard. The IVP

$$dX/dT = g(T, X), \ \ X(0) = a^0 \tag{3}$$

is said to be a perturbation of the standard IVP

$$dX/dT = g_0(T, X), \ \ X(0) = a_0^0, \tag{4}$$

if $g \simeq g_0$ and $a^0 \simeq a_0^0$. To avoid inessential complications we assume that equation $dX/dT = g_0(T, X)$ has the uniqueness of the solutions. Let ϕ_0 be the solution of the IVP (4). Let I be its maximal interval of definition. Then, by the following theorem any solution of problem (3) also exist on I and is infinitely close to ϕ_0.

Theorem 1 (Short Shadow Lemma) *Let problem (3) be a perturbation of problem (4). Every solution ϕ of problem (3) is a perturbation of the solution ϕ_0 of problem (4), that is, for all nearstandard t in I, $\phi(t)$ is defined and satisfies $\phi(t) \simeq \phi_0(t)$.*

Let us consider the restriction ψ of ϕ to ${}^{NS}I$. By the Short Shadow Lemma, for standard $t \in I$, it takes nearstandard values $\psi(t) \simeq \phi_0(t)$. Thus its shadow, which is the unique standard mapping which associate to each standard t the standard part of $\psi(t)$, is equal to ϕ_0. In general the shadow of ϕ is not equal to ϕ_0. Thus, the Short Shadow Lemma describes only the "short time behaviour" of the solutions.

3 Averaging in Ordinary Differential Equations

The method of averaging is well-known for ODEs. The fundamental result of this theory asserts that, for small $\varepsilon > 0$, the solutions of a nonautonomous system

$$\dot{x} = f\left(t/\varepsilon, x, \varepsilon\right), \qquad \text{where } \dot{x} = dx/dt, \tag{5}$$

are approximated by the solutions of the averaged autonomous system

$$\dot{y} = F\left(y\right), \qquad \text{where } F(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t, x, 0)dt. \tag{6}$$

The approximation of the solutions of (5) by the solutions of (6) means that if $x(t, \varepsilon)$ is a solution of (5) and $y(t)$ is the solution of the averaged equation (6) with the same initial condition, which is assumed to be defined on some interval $[0, T]$, then for $\varepsilon \simeq 0$ and for all $t \in [0, T]$, we have $x(t, \varepsilon) \simeq y(t)$.

The change of variable $z(\tau) = x(\varepsilon\tau)$ transforms equation (5) into equation

$$z' = \varepsilon f\left(\tau, z, \varepsilon\right), \qquad \text{where } z' = dz/d\tau. \tag{7}$$

Thus, the method of averaging can be stated for equation (7), that is, if ε is infinitesimal and $0 \leq \tau \leq T/\varepsilon$ then $z(\tau, \varepsilon) \simeq y(\varepsilon\tau)$.

Classical results were obtained by Krilov, Bogolyubov, Mitropolski, Eckhaus, Sanders, Verhulst (see [5, 31] and the references therein). The theory is very delicate. The dependence of $f(t, x, \varepsilon)$ in ε introduces many complications in the formulations of the conditions under which averaging is justified. In the classical approach, averaging is justified for systems (5) for which the vector field f is Lipschitz continuous in x. Our aim in this section is first to formulate this problem with the concept of perturbations of vector fields and then to give a theorem of averaging under hypothesis less restrictive than the usual hypothesis. In our approach, averaging is justified for all perturbations of a continuous vector field which is continuous in x uniformly with respect to t. This assumption is of course less restrictive than Lipschitz continuity with respect to x.

3.1 KBM vector fields

Definition 2 *Let U_0 be an open subset of $\mathbb{R}^d$. The continuous vector field $f_0 : \mathbb{R}_+ \times U_0 \to \mathbb{R}^d$ is said to be a Krilov-Bogolyubov-Mitropolski (KBM) vector field if it satisfies the following conditions*

1. *The function $x \to f_0(t,x)$ is continuous in x uniformly with respect to the variable t.*
2. *For all $x \in U_0$ the limit $F(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f_0(t,x)dt$ exists.*
3. *The averaged equation $\dot{y}(t) = F(y(t))$ has the uniqueness of the solution with prescribed initial condition.*

Notice that, in the previous definition, conditions (1) and (2) imply that the function F is continuous, so that the averaged equation considered in condition (3) is well defined. In the case of non autonomous ODEs, the definition of a perturbation given in Section 2 must be stated as follows.

Definition 3 *Let U_0 and U be open subsets of $\mathbb{R}^d$. A continuous vector field $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$ is said to be a perturbation of the standard continuous vector field $f_0 : \mathbb{R}_+ \times U_0 \to \mathbb{R}^d$ if U contains all the nearstandard points in U_0, and $f(s,x) \simeq f_0(s,x)$ for all $s \in \mathbb{R}_+$ and all nearstandard x in U_0.*

Theorem 2 *Let $f_0 : \mathbb{R}_+ \times U_0 \to \mathbb{R}^d$ be a standard KBM vector field and let $a_0 \in U_0$ be standard. Let $y(t)$ be the solution of the IVP*

$$\dot{y}(t) = F(y(t)), \qquad y(0) = a_0, \tag{8}$$

defined on the interval $[0,\omega[$, $0 < \omega \leq \infty$. Let $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$ be a perturbation of f_0. Let $\varepsilon > 0$ be infinitesimal and $a \simeq a_0$. Then every solution $x(t)$ of the IVP

$$\dot{x}(t) = f\left(t/\varepsilon, x(t)\right), \qquad x(0) = a, \tag{9}$$

is a perturbation of $y(t)$, that is, for all nearstandard t in $[0,\omega[$, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$.

The proof, in the particular case of almost periodic vector fields, is given in Section 3.4. The proof in the general case is given in Section 3.5.

3.2 Almost solutions

The notion of almost solution of an ODE is related to the classical notion of ε-*almost solution.*

Definition 4 *A function $x(t)$ is said to be an* almost solution *of the standard differential equation $\dot{x} = G(t,x)$ on the standard interval $[0,L]$ if there exists a finite sequence $0 = t_0 < \cdots < t_{N+1} = L$ such that for $n = 0,\cdots,N$ we have*

$$t_{n+1} \simeq t_n,\ x(t) \simeq x(t_n) \text{ for } t \in [t_n, t_{n+1}],\ \text{and } \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq G(t_n, x(t_n)).$$

The aim of the following result is to show that an almost solution of a standard ODE is infinitely close to a solution of the equation. This result which was first established by J. L. Callot (see [11, 27]) is a direct consequence of the nonstandard proof of the existence of solutions of continuous ODEs [26].

Theorem 3 *If $x(t)$ is an almost solution of the standard differential equation $\dot{x} = G(t,x)$ on the standard interval $[0,L]$, $x(0) \simeq y_0$, with y_0 standard, and the IVP $\dot{y} = G(t,y)$, $y(0) = y_0$, has a unique solution $y(t)$, then $y(t)$ is defined at least on $[0,L]$ and we have $x(t) \simeq y(t)$, for all $t \in [0,L]$.*

Proof. See [11, 36] ■

Let us apply this theorem to obtain an averaging result for an ODE which does not satisfy all the hypothesis of Theorem 2. Consider the ODE (see [11, 27, 36])

$$\dot{x}(t) = \sin \frac{tx}{\varepsilon}. \tag{10}$$

The conditions (2) and (3) in Definition 2 are satisfied with $F(x) = 0$. Thus, the solutions of the averaged equation are constant. But condition (1) of the definition is not satisfied, since the function $f(t,x) = \sin(tx)$ is not continuous in x uniformly with respect to t. Hence Theorem 2 does not apply. In fact the solutions of (10) are not nearly constant and we have the following result :

Proposition 2 *If $\varepsilon > 0$ is infinitesimal then, in the region $t \geq x > 0$ the solutions of (10) are infinitely close to hyperbolas $tx =$ constant. In the region $x > t \geq 0$, they are infinitely close to the solutions of the ODE*

$$\dot{x} = G(t,x), \qquad \text{where } G(t,x) = \frac{\sqrt{x^2 - t^2} - x}{t}. \tag{11}$$

Proof. The isocline curves $I_k = \{(t,x) : tx = 2k\pi\varepsilon\}$ and $I'_k = \{(t,x) : tx = \left(2k + \frac{3}{2}\right)\pi\varepsilon\}$ define, in the region $t \geq x > 0$, tubes in which the trajectories are trapped. Thus for $t \geq x > 0$ the solutions are infinitely close to the hyperbolas $tx =$ constant. This argument does not work for $x > t \geq 0$. In this region, we consider the microscope

$$T = \frac{t - t_k}{\varepsilon}, \qquad X = \frac{x - x_k}{\varepsilon},$$

where (t_k, x_k) are the points where a solution $x(t)$ of (10) crosses the curve I_k. Then we have

$$\frac{dX}{dT} = \sin\left(x_k T + t_k X + \varepsilon T X\right), \qquad X(0) = 0.$$

By the Short Shadow Lemma (Theorem 1), $X(T)$ is infinitely close to a solution of $dX/dT = \sin\left(x_k T + t_k X\right)$. By straightforward computations we have

$$\frac{x_{k+1} - x_k}{t_{k+1} - t_k} \simeq G(t_k, x_k).$$

Hence, in the region $x > t \geq 0$, the function $x(t)$ is an almost solution of the ODE (11). By Theorem 3, the solutions of (10) are infinitely close to the solutions of (11). ■

3.3 The Stroboscopic Method for ODEs

In this section we denote by $G : \mathbb{R}_+ \times D \to \mathbb{R}^d$ a standard continuous function, where D is a standard open subset of $\mathbb{R}^d$. Let $x : I \to \mathbb{R}^d$ be a function such that $0 \in I \subset \mathbb{R}_+$.

Definition 5 *We say that x satisfies the* Strong Stroboscopic Property *with respect to G if there exists $\mu > 0$ such that for every positive limited $t_0 \in I$ with $x(t_0)$ nearstandard in D, there exists $t_1 \in I$ such that $\mu < t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, and*

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} \simeq G(t_0, x(t_0)).$$

The real numbers t_0 and t_1 are called *successive instants of observation* of the stroboscopic method.

Theorem 4 (Stroboscopic Lemma for ODEs) *Let $a_0 \in D$ be standard. Assume that the IVP $\dot{y}(t) = G\left(t, y(t)\right)$, $y(0) = a_0$, has a unique solution y defined on some standard interval $[0, L]$. Assume that $x(0) \simeq a_0$ and x satisfies the Strong Stroboscopic Property with respect to G. Then x is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

Proof. Since x satisfies the Strong Stroboscopic Property with respect to G, it is an almost solution of the ODE $\dot{x} = G(t, x)$. By Theorem 3 we have $x(t) \simeq y(t)$ for all $t \in [0, L]$. The details of the proof can be found in [36]. ■

The Stroboscopic Lemma has many applications in the perturbation theory of differential equations (see [11, 32, 35, 36, 38, 39]). Let us use this lemma to obtain a proof of Theorem 2.

3.4 Proof of Theorem 2 for almost periodic vector fields

Suppose that f_0 is almost periodic in t then any of its translates $f_0(s+\cdot, x_0)$ is a nearstandard function, and f_0 has an average F which satisfies [16, 28, 33, 41]

$$F(x) = \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} f_0(t,x)dt,$$

uniformly with respect to $s \in \mathbb{R}_+$. Since F is standard and continuous, we have

$$F(x) \simeq \frac{1}{T} \int_s^{s+T} f_0(t,x)dt, \tag{12}$$

for all $s \in \mathbb{R}_+$, all $T \simeq \infty$ and all nearstandard x in U_0. Let $x : I \to U$ be a solution of problem (9). Let t_0 be an instant of observation : t_0 is limited in I, and $x_0 = x(t_0)$ is nearstandard in U_0. The change of variables

$$X = \frac{x\left(t_0 + \varepsilon T\right) - x_0}{\varepsilon},$$

transforms (9) into

$$dX/dT = f(s+T, x_0 + \varepsilon X), \qquad \text{where } s = t_0/\varepsilon.$$

By the Short Shadow Lemma (Theorem 1), applied to $g(T,X) = f(s+T, x_0 + \varepsilon X)$ and $g_0(T,X) = f_0(s+T, x_0)$, for all limited $T > 0$, we have $X(T) \simeq \int_0^T f_0(s+r, x_0)dr$. By Robinson's Lemma this property is true for some unlimited T which can be chosen such that $\varepsilon T \simeq 0$. Define $t_1 = t_0 + \varepsilon T$. Then we have

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = \frac{X(T)}{T} \simeq \frac{1}{T} \int_0^T f_0(s+r, x_0)dr = \frac{1}{T} \int_s^{s+T} f_0(t, x_0)dt \simeq F(x_0).$$

Thus x satisfies the Strong Stroboscopic Property with respect to F. Using the Stroboscopic Lemma for ODEs (Theorem 4) we conclude that $x(t)$ is infinitely close to a solution of the averaged ODE (8).

3.5 Proof of Theorem 2 for KBM vector fields

Let f_0 be a KBM vector field. From condition (2) of Definition 2 we deduce that for all $s \in \mathbb{R}_+$, we have $F(x) = \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} f_0(t,x)dt$, but the limit is not uniform on s. Thus for unlimited positive s, the property (12) does not hold for all unlimited T, as it was the case for almost periodic vector fields. However, using also the uniform continuity of f_0 in x with respect to t we can show that (12) holds for some unlimited T which are not very large. This result is stated in the following technical lemma [36]

Lemma 1 *Let* $g : \mathbb{R}_+ \times \mathcal{M} \to \mathbb{R}^d$ *be a standard continuous function where* $\mathcal{M}$ *is a standard metric space. We assume that* g *is continuous in* $m \in \mathcal{M}$ *uniformly* *with respect to* $t \in \mathbb{R}_+$ *and that* g *has an average* $G(m) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(t,m)dt$. *Let* $\varepsilon > 0$ *be infinitesimal. Let* $t \in \mathbb{R}_+$ *be limited. Let* m *be nearstandard in* $\mathcal{M}$. *Then there exists* $\alpha > \varepsilon$, $\alpha \simeq 0$ *such that, for all limited* $T \geq 0$ *we have*

$$\frac{1}{S}\int_s^{s+TS} g(r,m)dr \simeq TG(m), \quad \textit{where} \quad s = t/\varepsilon, \quad S = \alpha/\varepsilon.$$

The proof of Theorem 2 needs another technical lemma whose proof can be found also in [36].

Lemma 2 *Let* $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $h : \mathbb{R}_+ \to \mathbb{R}^d$ *be continuous functions. Suppose that* $g(T,X) \simeq h(T)$ *holds for all limited* $T \in \mathbb{R}_+$ *and all limited* $X \in \mathbb{R}^d$, *and* $\int_0^T h(r)dr$ *is limited for all limited* $T \in \mathbb{R}_+$. *Then, any solution* $X(T)$ *of the IVP* $dX/dT = g(T,X)$, $X(0) = 0$, *is defined for all limited* $T \in \mathbb{R}_+$ *and satisfies* $X(T) \simeq \int_0^T h(r)dr$.

Proof of Theorem 2. Let $x : I \to U$ be a solution of problem (9). Let $t_0 \in I$ be limited, such that $x_0 = x(t_0)$ is nearstandard in U_0. By Lemma 1, applied to $g = f_0$, $G = F$ and $m = x(t_0)$, there is $\alpha > 0$, $\alpha \simeq 0$ such that for all limited $T \geq 0$ we have

$$\frac{1}{S}\int_s^{s+TS} f_0(r,x_0)dr \simeq TF(x_0), \quad \text{where } s = t_0/\varepsilon, \ S = \alpha/\varepsilon. \tag{13}$$

The change of variables

$$X(T) = \frac{x(t_0 + \alpha T) - x_0}{\alpha}$$

transforms (9) into

$$dX/dT = f(s + ST, x_0 + \alpha X).$$

By Lemma 2, applied to $g(T,X) = f(s+ST, x_0+\alpha X)$ and $h(T) = f_0(s+ST, x_0)$, and (13), for all limited $T > 0$, we have

$$X(T) \simeq \int_0^T f_0(s + Sr, x_0)dr = \frac{1}{S}\int_s^{s+TS} f_0(r,x_0)dr \simeq TF(x_0).$$

Define the successive instant of observation of the stroboscopic method t_1 by $t_1 = t_0 + \alpha$. Then we have

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(1) \simeq F(x_0).$$

Since $t_1 - t_0 = \alpha > \varepsilon$ and $x(t) - x(t_0) = \alpha X(T) \simeq 0$ for all $t \in [t_0, t_1]$, we have proved that the function x satisfies the Strong Stroboscopic Property with respect to F. By the Stroboscopic Lemma, for any nearstandard $t \in [0, \omega[$, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$. ■

4 Functional Differential Equations

Let $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^d)$, where $r > 0$, denote the Banach space of continuous functions with the norm $\|\phi\| = \sup\{\|\phi(\theta)\| : \theta \in [-r, 0]\}$, where $\|\cdot\|$ is a norm of $\mathbb{R}^d$. Let $L \geq t_0$. If $x : [-r, L] \to \mathbb{R}^d$ is continuous, we define $x_t \in \mathcal{C}$ by setting $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$ for each $t \in [0, L]$. Let $g : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$, $(t, u) \mapsto g(t, u)$, be a continuous function. Let $\phi \in \mathcal{C}$ be an initial condition. A Functional Differential Equation (FDE) is an equation of the form

$$\dot{x}(t) = g(t, x_t), \qquad x_0 = \phi.$$

This type of equation includes differential equations with delays of the form

$$\dot{x}(t) = G(t, x(t), x(t - r)),$$

where $G : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Here we have $g(t, u) = G(t, u(0), u(-r))$.

The method of averaging was extended [13, 22] to the case of FDEs of the form

$$z'(\tau) = \varepsilon f(\tau, z_\tau), \tag{14}$$

where ε is a small parameter. In that case the averaged equation is the ODE

$$y'(\tau) = \varepsilon F(y(\tau)), \tag{15}$$

where F is the average of f. It was also extended [14] to the case of FDEs of the form

$$\dot{x}(t) = f(t/\varepsilon, x_t). \tag{16}$$

In that case the averaged equation is the FDE

$$\dot{y}(t) = F(y_t). \tag{17}$$

Notice that the change of variables $x(t) = z(t/\varepsilon)$ does not transform equation (14) into equation (16), as it was the case for ODEs (7) and (5), so that the results obtained for (14) cannot be applied to (16). In the case of FDEs of the form (14) or (16), the classical averaging theorems require that the vector field f is Lipschitz continuous in x uniformly with respect to t. In our approach, this condition is weakened and we only assume that the vector field f is continuous in x uniformly with respect to t. Also in the classical averaging theorems it is assumed that the solutions $z(\tau, \varepsilon)$ of (14) and $y(\tau)$ of (15) exist in the same interval $[0, T/\varepsilon]$ or that the solutions $x(t, \varepsilon)$ of (16) and $y(t)$ of (17) exist in the same interval $[0, T]$. In our approach, we assume only that the solution of the averaged equation is defined on some interval and we give conditions on the vector field f so that, for ε sufficiently small, the solution $x(t, \varepsilon)$ of the system exists at least on the same interval.

4.1 Averaging for FDEs in the form $z'(\tau) = \varepsilon f(\tau, z_\tau)$

We consider the IVP, where ε is a small parameter

$$z'(\tau) = \varepsilon f(\tau, z_\tau), \qquad z_0 = \phi.$$

The change of variable $x(t) = z(t/\varepsilon)$ transforms this equation in

$$\dot{x}(t) = f(t/\varepsilon, x_{t,\varepsilon}), \qquad x(t) = \phi(t/\varepsilon), \quad t \in [-\varepsilon r, 0], \tag{18}$$

where $x_{t,\varepsilon} \in \mathcal{C}$ is defined by $x_{t,\varepsilon}(\theta) = x(t + \varepsilon\theta)$ for $\theta \in [-r, 0]$.

Let $f : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$ be a standard continuous function. We assume that

(H1) The function $f : u \mapsto f(t, u)$ is continuous in u uniformly with respect to the variable t.

(H2) For all $u \in \mathcal{C}$ the limit $F(u) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t, u)dt$ exists.

We identify $\mathbb{R}^d$ to the subset of constant functions in $\mathcal{C}$, and for any vector $c \in \mathbb{R}^d$, we denote by the same letter, the constant function $u \in \mathcal{C}$ defined by $u(\theta) = c$, $\theta \in [-r, 0]$. Averaging consists in approximating the solutions $x(t, \varepsilon)$ of (18) by the solution $y(t)$ of the averaged ODE

$$\dot{y}(t) = F(y(t)), \qquad y(0) = \phi(0). \tag{19}$$

According to our convention, $y(t)$, in the right-hand side of this equation, is the constant function $u^t \in \mathcal{C}$ defined by $u^t(\theta) = y(t)$, $\theta \in [-r, 0]$. Since F is continuous, this equation is well defined. We assume that

(H3) The averaged ODE (19) has the uniqueness of the solution with prescribed initial condition.

(H4) The function f is quasi-bounded in the variable u uniformly with respect to the variable t, that is, for every $t \in \mathbb{R}_+$ and every limited $u \in \mathcal{C}$, $f(t, u)$ is limited in $\mathbb{R}^d$.

Notice that conditions (H1), (H2) and (H3) are similar to conditions (1), (2) and (3) of Definition 2. In the case of FDEs we need also condition (H4). In classical words, the *uniform quasi boundedness* means that for every bounded subset B of $\mathcal{C}$, $f(\mathbb{R}_+ \times B)$ is a bounded subset of $\mathbb{R}^d$. This property is strongly related to the continuation properties of the solutions of FDEs (see Sections 2.3 and 3.1 of [15]).

Theorem 5 *Let $f : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$ be a standard continuous function satisfying the conditions (H1)-(H4). Let ϕ be standard in $\mathcal{C}$. Let $L > 0$ be standard and let $y : [0, L] \to \mathbb{R}^d$ be the solution of (19). Let $\varepsilon > 0$ be infinitesimal. Then every solution $x(t)$ of the problem (18) is defined at least on $[-\varepsilon r, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

4.2 The Stroboscopic Method for ODEs revisited

In this section we give another formulation of the stroboscopic method for ODEs which is well adapted to the proof of Theorem 5. Moreover, this formulation of the Stroboscopic Method will be easily extended to FDEs (see Section 4.4). We denote by $G : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, a standard continuous function. Let $x : I \to \mathbb{R}^d$ be a function such that $0 \in I \subset \mathbb{R}_+$.

Definition 6 *We say that x satisfies the* Stroboscopic Property *with respect to G if there exists $\mu > 0$ such that for every positive limited $t_0 \in I$, satisfying $[0, t_0] \subset I$ and $x(t)$ is limited for all $t \in [0, t_0]$, there exists $t_1 \in I$ such that $\mu < t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, and*

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} \simeq G\left(t_0, x(t_0)\right).$$

The difference with the Strong Stroboscopic Property with respect to G considered in Section 3.3 is that now we assume that the successive instant of observation t_1 exists only for those values t_0 for which $x(t)$ is limited for all $t \in [0, t_0]$. In Definition 5, in which we take $D = \mathbb{R}^d$, we assumed the stronger hypothesis that t_1 exists for all limited t_0 for which $x(t_0)$ is limited.

Theorem 6 (Second Stroboscopic Lemma for ODEs) *Let $a_0 \in D$ be standard. Assume that the IVP $\dot{y}(t) = G\left(t, y(t)\right)$, $y(0) = a_0$, has a unique solution y defined on some standard interval $[0, L]$. Assume that $x(0) \simeq a_0$ and x satisfies the Stroboscopic Property with respect to G. Then x is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$.*

Proof. Since x satisfies the Stroboscopic Property with respect to G, it is an almost solution of the ODE $\dot{x} = G(t, x)$. By Theorem 3 we have $x(t) \simeq y(t)$ for all $t \in [0, L]$. The details of the proof can found in [19] or [21]. ■

Proof of Theorem 5. Let $x : I \to \mathbb{R}^d$ be a solution of problem (18). Let $t_0 \in I$ be limited, such that $x(t)$ is limited for all $t \in [0, t_0]$. By Lemma 1, applied to $g = f$, $G = F$ and the constant function $m = x(t_0)$, there is $\alpha > 0$, $\alpha \simeq 0$ such that for all limited $T \geq 0$ we have

$$\frac{1}{S}\int_s^{s+TS} f(r, x(t_0))dr \simeq TF(x(t_0)), \quad \text{where } s = t_0/\varepsilon,\ S = \alpha/\varepsilon. \tag{20}$$

Using the uniform quasi boundedness of f we can show (for the details see [19] or [21]) that $x(t)$ is defined and limited for all $t \simeq t_0$. Hence the function

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \varepsilon\theta) - x(t_0)}{\alpha}, \quad \theta \in [-r, 0], \quad T \in [0, 1],$$

is well defined. In the variable $X(\cdot, T)$ system (18) becomes

$$\frac{\partial X}{\partial T}(0, T) = f(s + ST, x(t_0) + \alpha X(\cdot, T)).$$

Using assumptions (H1) and (H4) together with (20), we obtain after some computations that for all $T \in [0, 1]$, we have

$$X(0, T) \simeq \int_0^T f(s + Sr, x(t_0))dr = \frac{1}{S}\int_s^{s+TS} f(r, x(t_0))dr \simeq TF(x(t_0)).$$

Define the successive instant of observation of the stroboscopic method t_1 by $t_1 = t_0 + \alpha$. Then we have

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(0, 1) \simeq F(x(t_0)).$$

Since $t_1 - t_0 = \alpha > \varepsilon$ and $x(t) - x(t_0) = \alpha X(0, T) \simeq 0$ for all $t \in [t_0, t_1]$, we have proved that the function x satisfies the Stroboscopic Property with respect to F. By the Second Stroboscopic Lemma for ODEs, for any $t \in [0, L]$, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$. ■

4.3 Averaging for FDEs in the form $\dot{x}(t) = f(t/\varepsilon, x_t)$

We consider the IVP, where ε is a small parameter

$$\dot{x}(t) = f(t/\varepsilon, x_t), \qquad x_0 = \phi, \tag{21}$$

We assume that f satisfies conditions (H1), (H2) and (H4) of Section 4.1. Now, the averaged equation is not the ODE (19), but the FDE

$$\dot{y}(t) = F(y_t), \qquad y_0 = \phi. \tag{22}$$

Averaging consists in approximating the solutions $x(t, \varepsilon)$ of (21) by the solution $y(t)$ of the averaged FDE (22). Condition (H3) in Section 4.1 must be restated as follows

(H3) The averaged FDE (22) has the uniqueness of the solution with prescribed initial condition.

Theorem 7 *Let $f : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$ be a standard continuous function satisfying the conditions (H1)-(H4). Let ϕ be standard in $\mathcal{C}$. Let $L > 0$ be standard and let $y : [0, L] \to \mathbb{R}^d$ be the solution of problem (22). Let $\varepsilon > 0$ be infinitesimal. Then every solution $x(t)$ of the problem (21) is defined at least on $[-r, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [-r, L]$.*

4.4 The Stroboscopic Method for FDEs

Since the averaged equation (22) is an FDE, we need an extension of the stroboscopic method for ODEs given in Section 4.2. In this section we denote by $G : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^d$, a standard continuous function. Let $x : I \to \mathbb{R}^d$ be a function such that $[-r, 0] \subset I \subset \mathbb{R}_+$.

Definition 7 *We say that x satisfies the* Stroboscopic Property *with respect to G if there exists $\mu > 0$ such that for every positive limited $t_0 \in I$, satisfying $[0, t_0] \subset I$ and $x(t)$ and $G(t, x_t)$ are limited for all $t \in [0, t_0]$, there exists $t_1 \in I$ such that $\mu < t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, and*

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} \simeq G\left(t_0, x_{t_0}\right).$$

Notice that now we assume that the successive instant of observation t_1 exists for those values t_0 for which both $x(t)$ and $G(t, x_t)$ are limited for all $t \in [0, t_0]$. In the limit case $r = 0$, the Banach space $\mathcal{C}$ is identified with $\mathbb{R}^d$ and the function x_t is identified with $x(t)$ so that, $G(t, x_t)$ is limited, for all limited $x(t)$. Hence the "Stroboscopic Property with respect to G" considered in the previous definition is a natural extension to FDEs of the "Stroboscopic Property with respect to G" considered in Definition 6.

Theorem 8 (Stroboscopic Lemma for FDEs) *Let $\phi \in \mathcal{C}$ be standard. Assume that the IVP $\dot{y}(t) = G\left(t, y_t\right)$, $y_0 = \phi$, has a unique solution y defined on some standard interval $[-r, L]$. Assume that the function x satisfies the Stroboscopic Property with respect to G and $x_0 \simeq \phi$. Then x is defined at least on $[-r, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [-r, L]$.*

Proof. Since x satisfies the Stroboscopic Property with respect to G, it is an almost solution of the FDE $\dot{x} = G(t, x_t)$. For FDEs, we have to our disposal an analog of Theorem 3. Thus $x(t) \simeq y(t)$ for all $t \in [0, L]$. The details of the proof can found in [19] or [21]. ■

Proof of Theorem 7. Let $x : I \to \mathbb{R}^d$ be a solution of problem (21). Let $t_0 \in I$ be limited, such that both $x(t)$ and $F(x_t)$ are limited for all $t \in [0, t_0]$. ¿From the uniform quasi boundedness of f we deduce that $x(t)$ is S-continuous on $[0, t_0]$. Thus x_t is nearstandard for all $t \in [0, t_0]$. By Lemma 1, applied to $g = f$, $G = F$ and $m = x_{t_0}$, there is $\alpha > 0$, $\alpha \simeq 0$ such that for all limited $T \geq 0$ we have

$$\frac{1}{S} \int_s^{s+TS} f(r, x_{t_0}) dr \simeq TF\left(x_{t_0}\right), \quad \text{where } s = t_0/\varepsilon,\ S = \alpha/\varepsilon. \tag{23}$$

Using the uniform quasi boundedness of f we can show (for the details see [19] or [21]) that $x(t)$ is defined and limited for all $t \simeq t_0$. Hence the function

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \theta) - x(t_0 + \theta)}{\alpha}, \quad \theta \in [-r, 0], \quad T \in [0, 1],$$

is well defined. In the variable $X(\cdot, T)$ system (21) becomes

$$\frac{\partial X}{\partial T}(0, T) = f(s + ST, x_{t_0} + \alpha X(\cdot, T)).$$

Using assumptions (H1) and (H4) together with (23), we obtain that for all $T \in [0, 1]$, we have

$$X(0, T) \simeq \int_0^T f\left(s + Sr, x_{t_0}\right) dr = \frac{1}{S} \int_s^{s+TS} f\left(r, x_{t_0}\right) dr \simeq TF\left(x_{t_0}\right).$$

Define the successive instant of observation of the stroboscopic method t_1 by $t_1 = t_0 + \alpha$. Then we have

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(0,1) \simeq F(x_{t_0})$$

Since $t_1 - t_0 = \alpha > \varepsilon$ and $x(t) - x(t_0) = \alpha X(0,T) \simeq 0$ for all $t \in [t_0, t_1]$, we have proved that the function x satisfies the Stroboscopic Property with respect to F. By the Stroboscopic Lemma for FDEs, for any $t \in [0, L]$, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$. ■

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Laboratoire de Mathématiques, Université de Haute Alsace,
4 rue des Frères Lumière, F68093 Mulhouse, France.
E-mail address : Tewfik.Sari@uha.fr