Nonstandard Perturbation Theory of Differential Equations *

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> For me, the most exciting aspect of nonstandard analysis is that concrete phenomena, such as ducks and streams, that classically can only be described awkwardly as asymptotic phenomena, become mythologized as simple nonstandard objects.
> Edward Nelson, *Mathematical Mythologies* *(*[19], p. 159*)*

Abstract

A nonstandard perturbation theory of differential equation is developped. We discuss the major basic results of the theory : the Short Shadow Lemma, the Tikhonov's Theorem and the Averaging Theorem.
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1 Introduction

In the early seventies, Georges Reeb, who learnt about Abraham Robinson's *Nonstandard Analysis* (NSA) [18], was convinced that NSA gives a langage adapted perturbation theory of differential equations. The axiomatic presentation *Internal Set Theory* (IST) [16] of NSA given by E. Nelson corresponded

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more to the Reeb's dream [1] and was in agreement with his conviction "Les entiers naïfs ne remplissent pas **N**". Indeed, no formalism can recover exactly all the phenomena of reality, and *nonstandard objects* which may be considered as a formalization of *non-naïve objects* are already elements of our usual (standard) sets. We don't need any use of stars and enlargements. Thus the Reeb school adopted IST.

This school produced various and numerous studies and new results as attested by a lot of books and proceedings (see [14, 30, 7, 4, 8, 2, 6, 3, 19, 9, 5] and their references). Canards and rivers (or Ducks and Streams [5]) are the most famous discoveries of the Reebian school of *nonstandard perturbation theory of differential equations*. Our aim in this paper is to present some of the basic tools obtained by nonstandard analysis in perturbation theory of differential equations.

The classical perturbation theory of differential equations studies, instead of perturbations, deformations of differential equations (see Section 2.1). Classicaly the phenomena are described asymptotically, when the parameters of the deformation tends to some fixed value. The first benefit of NSA is a natural and useful notion of perturbation of a vector field. A perturbed equation becomes (*mythologized as*) a simple nonstandard object, whoses properties can be investigated directly.

This paper is organized as follows. In Section 2 we define the notion of a *perturbation* f of a standard vector field f_0. The main problem of perturbation theory of differential equation is to describe the behaviour of the trajectories of the perturbed vector vector field f. We define a standard topology on the set of vector fields, with the property that f is a perturbation of a standard vector field f_0 if and only if f is infinitely close to f_0 for this topology. In Section 3 we study the so-called *regular perturbations* and we give the Short Shadow Lemma which is the major basic result in perturbation theory of differential equations. We prove that the shadow of an orbit γ of a nearstandard point x, for the perturbed vector field f, contains the orbit γ_0 of the standard part x_0 of x, for the unperturbed standard vector field f_0 (Section 3.2). We investigate the long time behaviour when the unperturbed vector field has an asymptotically stable equilibrium point (Section 3.3). In Section 4 we study the so-called *singular perturbations*. We obtain a generalization of Tikhonov's Theorem which is the major basic result in singular perturbation theory. In Section 4.3 we discuss averaging and give an extention of the classical Averaging Theorem of Krylov Bogolioubov and Mitropolski.

[1]For more informations about the Reeb's dream and convictions see the Reeb's preface of Lutz and Goze's book [14], Stewart's book [25] p. 72 or Lobry's book [12].

2 Deformations and Perturbations

2.1 Deformations

The classical *perturbation theory of differential equations* studies families of differential equations

$$\dot{x} = F(x, \varepsilon), \tag{1}$$

where x belongs to an open subset U of $\mathbf{R}^n$, called *phase space*, and ε belongs to a subset B of $\mathbf{R}^k$, called *space of parameters*.

The family (1) of differential equations is said to be a *k-parameters deformation* of the vector field $F_0(x) := F(x, \varepsilon_0)$, where ε_0 is some fixed value of ε (often one takes $k = 1$, $B = [0, +\infty)$ and $\varepsilon_0 = 0$, so one restricts attention to 1-parameter deformations). The main problem of the perturbation theory of differential equation is to investigate the behaviour of the vector fields $F(x, \varepsilon)$ when ε tends to ε_0.

The intuitive notion of a *perturbation* of the vector field F_0 which would mean any vector field which is *close to* F_0 does not appear in the theory. Thus, the *classical perturbation theory of differential equations* considers *deformations* instead of *perturbations* and would be better called *deformation theory of differential equations* [2] . Actually the vector field $F(x, \varepsilon)$ when ε is sufficiently close to ε_0 is called a perturbation of the vector field $F_0(x)$. In other words, the differential equation

$$\dot{x} = F_0(x), \tag{2}$$

is said to be the *unperturbed equation* and equation (1), for a fixed value of ε, is called the *perturbed equation*. Thus the classical notions of *deformation* and *perturbation* are practically the same.

This classical notion of perturbation is not very satisfactory since many of the results obtained for the family (1) of differential equations take place in all systems that are close to the unperturbed equation (2). Arnold (see [1], footnote page 157) suggest to simply study a neighbourhood of the unperturbed vector field $F_0(x)$ in a suitable function space. For the sake of mathematical convenience, instead of neighbourhoods, one consider deformations. According to Arnold, the situation is similar with the historical development of variational concepts, where the directional derivative (Gateaux differential) preceeded the derivative of a mapping (Frechet differential). Nonstandard analysis permits to define a notion of perturbation. To say that a vector f is a *perturbation* of a standard vector field f_0 is equivalent to say that f is *infinitely close* to f_0 is a suitable function space, that is f *is in any standard neighbourhoud* of f_0. Thus,

[2]The situation is similar in perturbation theory of linear operators and in all other perturbation theories of algebraic or geometric structures : theses theories consider only deformations and do not have convenient notions of perturbations (see Makhlouf [15]). Another example is given by *almost periodic functions* which do not have *almost periods*. The nonstandard approch permits to give a very natural notion of almost period (see Robinson [17], Klugler [11], Stroyan and Luxemburg [26], Sari [20]).

studying perturbations in our sense is nothing than studying neighbourhoud, as suggested by Arnold.

2.2 Perturbations

Let X be a standard topological space. A point $x \in X$ is said to be infinetly close to a standard point $x_0 \in X$, which is denoted by $x \simeq x_0$, if x is in any standard neighbourhood of x_0. Let A be a subset of X. A point $x \in X$ is said to be *nearstandard in* A if there is a standard $x_0 \in A$ such that $x \simeq x_0$. Let us denote by

$$^{NS}A = \{x \in X : \exists^{st} x_0 \in A\ x \simeq x_0\}$$

the *external-set* of nearstandard points in A. The shadow of A is denoted by ^oA. It is the standard subset of X whose standard elements are those elements x_0 of X such that there exits $x \in A$ satisfying $x \simeq x_0$. Hence

$$^oA = {}^S\{x_0 \in X : \exists x \in A\ x \simeq x_0\} = {}^S\{x_0 \in X : \mathrm{hal}(x_0) \cap A \neq \emptyset\}.$$

Let E be a standard uniform space. The points $x \in E$ and $y \in E$ are said to be infinetely close, which is denoted by $x \simeq y$, if (x, y) lies in every standard entourage. If E is a standard metric space, with metric d, then $x \simeq y$ is nothing that $d(x, y)$ infinitesimal.

Definition 1. *Let $f : D \to E$ and $f_0 : D_0 \to E$ be mappings, from the open subsets D, and D_0 of the standard topological space X, to the uniform space E, f_0 standard. The mapping f is said to be a perturbation of the mapping f_0, which is denoted by $f \simeq f_0$, if $^{NS}D_0 \subset D$ and $f(x) \simeq f_0(x)$ for all $x \in {}^{NS}D_0$.*

We can adopt this definition, because D_0 being a standard open subset of X, $^{NS}D_0 \subset D_0$, so $f(x)$ and $f_0(x)$ are both defined for all $x \in {}^{NS}D_0$. Let $\mathcal{C}_{X,E}$ be the set of mappings defined on open subsets of X to E :

$$\mathcal{C}_{X,E} = \{(f, D) : D \text{ open subset of } X \text{ and } f : D \to E\}.$$

Let us consider the topology on this set defined as follows. Let $(f_0, D_0) \in \mathcal{C}_{X,E}$. The family of sets of the form

$$\{(f, D) \in \mathcal{C}_{X,E} : K \subset D\ \forall x \in K\ (f(x), f_0(x)) \in U\}$$

where K is a compact subset of D_0 and U is an entourage of the uniform space E is a basis of the system of neighbourhoods of (f_0, D_0). Let us call this topology the *topology of uniform convergence on compacta.* If all the mappings are defined on the same open set D, this topology is the usual topology of uniform convergence on conpacta on the set of functions on D to E.

Proposition 2.1. *Asume X is locally compact. The mapping f is a perturbation of the standard mapping f_0 if and only if f is infinitely close to f_0 for the topology of uniform convergence on compacta.*

Proof. Let $f : D \to E$ be a perturbation of $f_0 : D_0 \to E$. Let K be a standard compact subset of D_0. Let U be a standard entourage. Then $K \subset D$ and $f(x) \simeq f_0(x)$ for all $x \in K$. Hence $(f(x), f_0(x)) \in U$. Thus $f \simeq f_0$ for the topology of uniform convergence on compacta. Conversly let f be infinitely close to f_0 for the topology of uniform convergence on compacta. Let $x \in {}^{NS}D_0$. There exists a standard $x_0 \in D_0$ such that $x \simeq x_0$. Let K be a standard compact neighbourhood of x_0, such that $K \subset D_0$ (such a neighbourhoud exists since X is locally compact). Then $x \in K \subset D$ and $(f(x), f_0(x)) \in U$ for all standard entourage U, that is ${}^{NS}D_0 \subset D$ and $f(x) \simeq f_0(x)$ on ${}^{NS}D_0$. Hence f is a perturbation of f_0. This completes the proof.

2.3 Perturbations of Vector Fields

Let n be a standard positive integer. Let $\|x\|$ be a standard norm on $\mathbf{R}^n$. A continuous function $f : D \to \mathbf{R}^n$ where D is an open subset of $\mathbf{R}^n$ is called a vector field. Let $\mathcal{X}$ be the set of vector fields defined on open subsets of $\mathbf{R}^n$:

$$\mathcal{X} = \{(f, D) : D \text{ open subset of } \mathbf{R}^n \text{ and } f : D \to \mathbf{R}^n \text{ continuous}\}$$

According to Proposition 2.1, a vector field $f : D \to \mathbf{R}^n$ is a perturbation of the standard vector field $f_0 : D_0 \to \mathbf{R}^n$, if and only if f is infitely close to f_0 for the topology of *uniform convergence on compacta* on the set $\mathcal{X}$. That is, for any standard compact subset $K \subset D_0$ and any standard $a > 0$, we have $K \subset D$ and $\sup_{x \in K} \|f(x) - f_0(x)\| < a$.

Main Problem. *Let f be a perturbation of the standard vector field f_0. The main problem of perturbation theory of differential equations is to study the shadows of the orbits of f and to desribe them by using the orbits of f_0.*

3 Regular Perturbations

When the parameters are involved in the differential equations in such a way that the usual theory of continuous dependance of the solutions with respect to the parameters can be applied, the problems are known in the litterature as *regular perturbations*, whereas when the parameters are involved in the differential equations in such a way that the usual theory of continuous dependance of the solutions with respect to the parameters cannot be applied, the problems are called *singular perturbations.* Singularly perturbed systems possess properties which are basically differents from the ones of the regularly perturbed systems. They will be considered in the Section 4. The Short Shadow Lemma considers regular perturbations and is one of the first results obtained in nonstandard perturbation theory of differential equations ([14, 6]).

3.1 Short Shadow Lemma

Let $f\ : D \to \mathbf{R}^n$, $f_0\ : D_0 \to \mathbf{R}^n$ be vector fields, $x_0^0 \in D_0$ and $x^0 \in D$, f_0 and x_0^0 standard. The initial value problem

$$\dot{x} = f(x),\ \ x(0) = x^0, \tag{3}$$

is said to be a perturbation of the (standard) initial value problem

$$\dot{x} = f_0(x),\ \ x(0) = x_0^0, \tag{4}$$

if $f \simeq f_0$ and $x^0 \simeq x_0^0$. To avoid inessentials complications we assume in all this section that equation $\dot{x} = f_0(x)$ has the unicity of the solutions. Let ϕ_0 be the noncontinuable solution of initial value problem 4. Let I be its maximal interval of definition. Will any solution of problem (3) also exist on I and be close to ϕ_0 ? This question is answered by the following theorem (see [22]).

Theorem 3.1. (Short Shadow Lemma). *Let problem (3) be a perturbation of problem (4). Every solution ϕ of probem (5) is a perturbation of the solution ϕ_0 of problem (4), that is, for all nearstandard t in I $\phi(t)$ is defined and satisfies $\phi(t) \simeq \phi_0(t)$.*

Let us consider the external-mapping $\phi : {}^{NS}I \to D$. By the Short Shadow Lemma, it takes nearstandard values $\phi(t) \simeq \phi_0(t)$ for standard t, and its shadow is the unique standard mapping which associate to each standard t the standard part of $\phi(t)$, that is $\phi_0(t)$. Hence the shadow of ϕ restricted to ${}^{NS}I$ is equal to ϕ_0. In general the shadow of ϕ is not equal to ϕ_0. Thus, the Short Shadow Lemma investigate only the short time behaviour of the solutions.

3.2 Semicontinuity Properties of Orbits

Corollary 3.2. *Let problem (3) be a perturbation of problem (4). Let γ_0 be the orbit through x_0^0 for the vector field f_0. Let γ be an any orbit through x^0 for the vector field f. Then the shadow of γ contains γ_0. Its contains also the limit sets $\alpha(\gamma_0)$ and $\omega(\gamma_0)$.*

Proof. Let x be standard in γ_0. There is a standard $t \in I$ such that $x = \phi_0(t)$. Thus $\phi(t) \simeq x$. Hence $x \in {}^o\gamma$. We have shown that every standard point in γ_0 is in ${}^o\gamma$, so by transfer every point in γ_0 is in ${}^o\gamma$. Thus $\gamma_0 \subset {}^o\gamma$. The shadow ${}^o\gamma$ of the orbit γ is a closed set. Since it contains γ_0, it contains its closure. Hence the limits sets $\alpha(\gamma_0)$ and $\omega(\gamma_0)$ are conained in ${}^o\gamma$. This completes the proof.

A classical consequence of this result is the following Let $\mathcal{X}(D)$ be the set of all vector fields $f : D \to \mathbf{R}^n$ having the unicity of the solutions. The set valued mapping of $\mathcal{X}(D) \times D$ into the power set $\mathcal{P}(D)$ which carries (f, x^0) into the orbit $\gamma_f(x^0)$ of f through x^0 is lower semicontinuous when $\mathcal{X}(D)$ is endowed with the topology of uniform convergence on compacta (see [22, 23] for the details).

3.3 Asymptotic Stability and Perturbations

According to the Short Shadow Lemma, a solution $\phi(t)$ of the perturbed problem (3) follows the solution $\phi_0(t)$ of the unperturbed problem (4) until $\phi_0(t)$ reaches its limit sets. The solution of $\phi_0(t)$ may behave in one of several ways : it may be unbounded as $t \to \infty$, it may tend toward an equilibrium point, or it may approach a more complex attractor. Obviously, if the unperturbed equation has multiple stable equilibria, the asymptotic behaviour of a solution is determined by its initial value. Assume the second case occurs, that is the solution $\phi_0(t)$ tends toward an equilibrium $x = \xi$. When this equilibrium is asymptotically stable, the approximate given by the Short Shadow Lemma holds for all positive values of time. More precisely (see [21, 22])

Theorem 3.3. *Let problem (3) be a perturbation of problem (4). Assume there exists a standard isolated root $x = \xi$ of equation $f_0(x) = 0$, the equilibrium point $x = \xi$ of equation (4) is asymptotically stable and x_0^0 lies in its basin of attraction. Then every solution ϕ of probem (5) is defined for all $t \geq 0$ and satisfies $\phi(t) \simeq \phi_0(t)$.*

3.4 Shadows of Orbits of Perturbed Vector Fields

Theorem 3.4. *Let f be a perturbation of the standard vector field f_0. Let γ be an orbit of f. Let Γ be the shadow of γ. Then $\Gamma \cap D_0$ is a closed invariant subset of f_0.*

Proof. When γ has no nearstandard points in D_0, $\Gamma \cap D_0 = \emptyset$ and there is nothing to prove. Let x_0 be a standard point in $\Gamma \cap D_0$. There exists $x \in \gamma$ such that $x \simeq x_0$. By the corollary of the Short Shadow Lemma Γ contains the orbit γ_0 of f_0 through x_0. We have shown that Γ contains the orbit of f_0 through every standard point x_0 in Γ, so by transfer Γ contains the orbit of f_0 through every point x_0 in Γ. Thus $\Gamma \cap D_0$ is invariant. It is closed since it is a shadow. This conpletes the proof.

This description of shadows of orbits of the perturbed vector field f is far from being complete or usefull, whithout any additionnal hypothesis on the size of the perturbation. However the shadow Γ of an orbit of the perturbed vector field f is in fact a *chaining of orbits* of f_0 (see [21] for the details). For example if the vector field f is a perturbation of the trivial vector field $f_0 = 0$ then Theorem 3.4 asserts only that Γ is a closed set. When we assume that f is of the form εF where ε is infinitesimal and F satisfies suitable conditions, then more informations can be obtained on Γ (see Sections 4.2 and 4.3).

4 Singular Perturbations

4.1 Slow and Fast Vectors Fields

Let us consider the initial value problem

$$\begin{cases} \dfrac{dx}{d\tau} = f(x,y) & x(0) = x^0 \\ \dfrac{dy}{d\tau} = h(x,y) & y(0) = y^0 \end{cases} \tag{5}$$

where x and $f : D \to \mathbf{R}^n$ are n-dimensional, y and $h : D \to \mathbf{R}^m$ are m-dimensional and D is an open subset of $\mathbf{R}^{n+m}$. Assume that

ASSUMPTION (A). *The mapping f is a perturbation of a standard continuous mapping $f_0 : D_0 \to \mathbf{R}^n$, $h(x,y)$ is infinitesimal for all nearstandard (x,y) in D_0 and the initial condition (x^0, y^0) is nearstandard in D_0. Let $(x_0^0, y_0^0) \in D_0$ be standard such that $(x^0, y^0) \simeq (x_0^0, y_0^0)$.*

The vectors $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ are the fast and slow components of the system. System (5) is called a *fast and slow system.* This system is a perturbation of system

$$\begin{cases} \dfrac{dx}{d\tau} = f_0(x,y) & x(0) = x_0^0 \\ \dfrac{dy}{d\tau} = 0 & y(0) = y_0^0 \end{cases} \tag{6}$$

Equation

$$\frac{dx}{d\tau} = f_0(x,y) \tag{7}$$

where y is considered as a parameter is called the *fast equation.*

ASSUMPTION (B). *The fast equation (7) has the unicity of the solutions, for all values of y.*

We refer to problem

$$\frac{dx}{d\tau} = f_0(x, y_0^0), \qquad x(0) = x_0^0. \tag{8}$$

consisting of the fast equation (7) where $y = y_0^0$ together with the initial value $x(0) = x_0^0$ as the *boundary layer equation.* Let $\tilde{x}(\tau)$ be the solution of this equation. According to the Short Shadow Lemma any solution $(x(\tau), y(\tau))$ of system (5) follows $(\tilde{x}(\tau), y_0^0)$, until $\tilde{x}(\tau)$ reaches its limit set. Let us assume as in Section 3.3 that $\tilde{x}(\tau)$ tends toward an asymptotically stable equilibrium point. More precisely assume that

ASSUMPTION (C). *There exists a standard* isolated root $x = \xi(y)$ *of equation*

$$f_0(x, y) = 0, \tag{9}$$

that is, there exists an n-dimensional standard continuous vector function $\xi(y)$ *defined on a standard open subset* $Y \subset \mathbf{R}^m$ *such that for all* $y \in Y$, $(\xi(y), y)$ *lies in* D_0, $f_0(\xi(y), y) \equiv 0$ *and to every standard compact subset* $K \subset Y$ *there exists a standard number* $\delta_K > 0$ *such that the relations*

$$\|x - \xi(y)\| < \delta_K, \;\; x \neq \xi(y), \;\; y \in K$$

imply $f_0(x, y) \neq 0$.

It is not excluded that equation (9) may have other roots beside $\xi(y)$. The manifold defined by equation (9) is called the *slow manifold*. It is the set of equilibrium points of the fast equation (7).

ASSUMPTION (D). *For each* $y \in Y$, *the point* $x = \xi(y)$ *is an asymptotically stable equilibrium point of the fast equation (7) and for any standard compact subset* $K \subset Y$ *the equilibrium point* $x = \xi(y)$ *has a uniform basin of attraction over* K.

The uniformity of the basin of attraction of $x = \xi(y)$ over K means that there is a standard positive number a_K, such that $0 < a_K \leq \delta_K$, where $\delta_K$is given in Assumption (C) and for all $y \in K$ the ball $\mathcal{B} = \{x \in \mathbf{R}^n \; : \|x - \xi(y)\| \leq a_K\}$ of center $\xi(y)$ and radius a_K is included in the basin of attraction of $x = \xi(y)$.

ASSUMPTION (E). *The point* y_0^0 *lies in* Y. *The point* x_0^0 *lies in the basin of attraction of the equilibrium point* $x = \xi(y_0^0)$.

Then we have the following result (see [24]).

Proposition 4.1. *Let Assuptions (A) to (E) be satisfied. Let* $(x(\tau), y(\tau))$ *be a solution of problem (5). Let* $\tilde{x}(\tau)$ *be the solution of the boundary layer equation (8). Then there exists* $L \simeq +\infty$ *such that* $(x(\tau), y(\tau))$ *is defined at least on* $[0, L]$ *and satisfies*

$$x(\tau) \simeq \tilde{x}(\tau), \;\; y(\tau) \simeq y_0^0, \quad \text{for} \quad 0 \leq \tau \leq L.$$

Moreover, for all $\tau \geq L$, *if* $(x(\tau), y(\tau))$ *is defined and* $y(\tau)$ *is nearstandard in* Y, *then* $x(\tau) \simeq \xi(y(\tau))$.

Thus the solution $(x(\tau), y(\tau))$ *can be approximated* by a *fast transition* from (x_0^0, y_0^0) to $(\xi(y_0^0), y_0^0)$, along the boundary solution $\tilde{x}(\tau)$, followed by a *slow motion* along the slow manifold $x = \xi(y)$.

4.2 Tykhonov's Theorem

The description of the slow motion near the slow manifold $x = \xi(y)$, given and the previous section, is more precise when the righthand side $h(x,y)$ of the second equation in system (5) is assumed to be of the form

$$h(x,y) = \varepsilon g(x,y),$$

where ε is an infinitesimal positive number. In that case, system 5 writes

$$\begin{cases} \dfrac{dx}{d\tau} = f(x,y) \\ \dfrac{dy}{d\tau} = \varepsilon g(x,y) \end{cases} \tag{10}$$

If we go to the *slow time*, $t = \varepsilon\tau$, system (10) becomes

$$\begin{cases} \varepsilon\dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \tag{11}$$

Since the small parameter ε is multipying the derivative, the usual theory of continuous dependance of the solutions with respect to the parameters cannot be applied. Such problems are known in the litterature as *singular perturbations.* The purpose of *Singular Perturbation Theory* is to investigate the behaviour of solutions of (11) as $\varepsilon \to 0$ for $0 \leq t \leq T$ and also for $0 \leq t < +\infty$. The investigation of such systems is given by Tikhonov's theory (see [27] and Section 39 of Wasow's book [29]). System 10 is a fast and slow system, so Proposition 4.1 applies. This result can be made more precise if we replace Assumption A in the preceeding section by the stronger

ASSUMPTION (A). *The mapping f and g are perturbations of the standard continuous mapping $f_0 : D_0 \to \mathbf{R}^n$ and $g_0 : D_0 \to \mathbf{R}^m$, $\varepsilon > 0$ is infinitesimal and the initial condition (x^0, y^0) is nearstandard in D_0. Let $(x_0^0, y_0^0) \in D_0$ be standard such that $(x^0, y^0) \simeq (x_0^0, y_0^0)$.*

According to Proposition 4.1, $(x(\tau), y(\tau))$ can be approximated by a fast transition along the boundary solution $\tilde{x}(\tau)$, followed by a slow motion along the slow manifold $x = \xi(y)$. This slow motion is a solution of equation

$$\frac{dy}{dt} = g_0(\xi(y), y). \tag{12}$$

This equation is called *the slow equation.* To obtain this approximation we need the following

ASSUMPTION (F). *The slow equation (12), defined on Y, has the unicity of the solutions with prescribed initial conditions.*

We refer to problem

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = f(x,y) & x(0) = x^0 \\ \dfrac{dy}{dt} = g(x,y) & y(0) = y^0 \end{cases} \tag{13}$$

consisting of system (11) together with the initial condition $x(0) = x^0$, $y(0) = y^0$, as the *full problem.* We refer to problem

$$\frac{dy}{dt} = g_0(\xi(y), y), \quad y(0) = y_0^0, \tag{14}$$

consisting of the slow equation (12) together with the initial condition $y(0) = y_0^0$, as the *reduced problem.* Let us formulate Tihonov's theorem concerning approximations on compact intervals of time ([24], see also [13, 28] which give applications to systems theory).

Theorem 4.2. *Let Assuptions (A) to (F) be satisfied. Let $(x(t), y(t))$ be a solution of the full problem (13). Let $\tilde{x}(\tau)$ be the solution of the boundary layer equation (8). Let $y_0(t)$ be the solution of the reduced problem (14). Let $x_0(t) = \xi(y_0(t))$. If $y_0(t)$ is defined on a standard interval $[0, T]$, then $(x(t), y(t))$ is defined on $[0, T]$ and satisfies*

$$\begin{array}{lll} x(\varepsilon\tau) \simeq \tilde{x}(\tau), \; y(\varepsilon\tau) \simeq y_0^0, & \text{for} & 0 \leq \tau \leq L, \\ x(t) \simeq x_0(t), & \text{for} & \varepsilon L \leq t \leq T, \\ y(t) \simeq y_0(t), & \text{for} & 0 \leq t \leq T. \end{array}$$

Here L is an unlimited number such that $\varepsilon L \simeq 0$.

For the formulation of the theorem concerning approximations on the infinite time interval we need one more hypothesis.

ASSUMPTION (G). *There exists a standard equilibrium point $y_\infty \in Y$ of the slow equation (12). The point $y = y_\infty$ is an asymptotically stable equilibrium point of this equation and y_0^0 lies in the basin of attraction of y_∞.*

When Assumption (G) is satisfied, the solution $y_0(t)$ of the reduced problem (14) is defined for all $t \geq 0$ and satisfies the property $y_0(t) \simeq y_\infty$ for all unlimited $t \simeq +\infty$. In this case the approximates given by Theorem 4.2 holds for all $t \geq 0$ as explained in the next result (see [24]).

Theorem 4.3. *Let Assumption (A) to (F) be satisfied, then any solution $(x(t), y(t))$ of the full problem (13) is defined for all $t \geq 0$ and satisfies*

$$\begin{array}{lll} x(t) \simeq x_0(t) = \xi(y_0(t)), & \text{for all limited} & t \geq \varepsilon L, \\ y(t) \simeq y_0(t), & \text{for all limited} & t \geq 0. \end{array}$$

And $x(t) \simeq \xi(y_\infty)$, $y(t) \simeq y_\infty$ *for all unlimited* $t \simeq +\infty$.

This last result is not contained in Tihonov's paper nor Wasow's book. However, Hoppensteadt [10] studied the approximations on infinite time intervals. His conditions are more restrictive than ours, but his studies concern also non autonomous systems.

4.3 Averaging

The fundamental problem is the study of the initial value problem

$$\frac{dx}{d\tau} = \varepsilon F(\tau, x) \qquad x(0) = a \tag{15}$$

when ε is small. The aim of the method of averaging is to approximate the solutions of problem (15), for times τ of order $1/\varepsilon$, by the solutions of the averaged system

$$\frac{dx}{d\tau} = \varepsilon f(x) \qquad x(0) = a_0 \tag{16}$$

where $a \simeq a_0$ and where f is an average of F with respect to the variable τ. Such an average exists for the so-called KBM [3] vector fields. Since we look for the long time behaviour of the solutions, it is more suitable to consider systems (15) and (16) at time scale $t = \varepsilon\tau$. Then we have

$$\frac{dx}{dt} = F(\frac{t}{\varepsilon}, x) \qquad x(0) = a \tag{17}$$

$$\frac{dx}{dt} = f(x) \qquad x(0) = a_0. \tag{18}$$

Let us state complete definitions and assumptions under which solutions of problem (17) are approximated by solutions of problem (18). The reader is refeered to [22] for details and proofs Let $\mathcal{C}$ be the set of continuous functions from $\mathbf{R}^+ \times U$ into $\mathbf{R}^d$, where U is an open subset of $\mathbf{R}^d$. A nonautonomous vector field $F_0 \in \mathcal{C}$ is said to be a KBM vector field if it satisfies the following properties :

$(H.1)$ The continuity of the function F_0 in the variable $x \in U$ is uniform with respect to the variable $t \in \mathbf{R}^+$.

$(H.2)$ For all $x \in U$ there exists a limit

$$f(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F_0(t, x)dt$$

$(H.3)$ The initial value problem (18) has a unique solution $y(t)$. Let $J = [0, \omega)$, $0 < \omega \leq +\infty$ be its maximal positive interval of definition.

[3] KBM stands for Krylov Bogolioubov and Mitropolski

From conditions $(H.1)$ and $(H.2)$ we deduce that the function $f : U \to \mathbf{R}$ is continuous. So the averaged differential equation $x' = f(x)$ has a continuous righthand member. A continuous function $F : \mathbf{R}^+ \times D \to \mathbf{R}^n$ is said to be a perturbation of the standard KBM vector field $F_0 : \mathbf{R}^+ \times U \to \mathbf{R}^n$ if ${}^{NS}U \subset D$, that is D contains all the nearstandard points in U, and $F(s,x) \simeq F_0(s,x)$ for all $s \in \mathbf{R}^+$ and all $x \in {}^{NS}U$.

Theorem 4.4. (KBM Theorem of Averaging) *Let $F_0 : \mathbf{R}^+ \times U \to \mathbf{R}^d$ be a standard KBM vector field and let $a_0 \in U$ be standard. Let $F : \mathbf{R}^+ \times D \to \mathbf{R}^d$ be a perturbation of F_0. Let $\varepsilon > 0$ and $a \in D$ be such that $\varepsilon \simeq 0$ and $a \simeq a_0$. Then every maximal solution $x(t)$ of problem (17) is a perturbation of the solution $y(t)$ of the averaged equation (18), that is, for all nearstandard t in J, $x(t)$ is defined and satisfies $x(t) \simeq y(t)$.*

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