Singular Perturbations Methods in Control Theory

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1 Introduction

Very often one encounters dynamical systems in which the derivatives of some of the states are multiplied by a small positive parameter; that is, the scales for the dynamics of the states are very diversified. When the small parameter ε is multipying the derivative as in system

$$\left\{ \begin{array}{ll} \dot{x} = f(t,x,z,\varepsilon), & x(t_0) = \xi(\varepsilon) \\ \varepsilon\dot{z} = g(t,x,z,\varepsilon), & z(t_0) = \zeta(\varepsilon) \end{array} \right. \tag{1}$$

the usual theory of continuous dependance of the solutions with respect to the parameters cannot be applied. The analysis of such systems is achieved with the aid of the *Singular Perturbation Theory*. The purpose of Singular Perturbation Theory is to investigate the behaviour of solutions of (1) as $\varepsilon \to 0$ for $0 \leq t \leq T$ and also for $0 \leq t < +\infty$.

This chapter is organized as follows. In Section 2 we recall Tikhonov's theorem on fast and slow systems, its extension to infinite time intervals, and Khalil's theorem on exponential stability of the origin of a fast and slow system. In Section 2.4 we define the notion of practical stability in a system depending on a parameter and we show that the extension of Tikhonov's theorem to infinite time intervals can be reformulated as a result of practical stability of the origin. In Section 3 we use the results of the preceeding section to reduce the dimension of systems in the problem of feedback stabilization. In Section 4 we discuss the peaking phenomenon in triangular systems of the form

$$\dot{x} = f(x,y), \qquad \dot{y} = G(y,\varepsilon). \tag{2}$$

Some states of the second equation may peak to very large values, before they rapidly decay to zero. Such peaking solutions can destabilize the first equation. In Section 4.2 we introduce the concept of *instantaneous stability*, to measure the fast decay to zero of the solutions of the second equation, and the concept of *uniform infinitesimal boundedness*, to measure the effects of peaking on the first equation. In Section 4.3 we motivate the case where the mapping g in the second equation depends also on the x variable. This case arises in control problems where the feedback is computed, not on the state, but on an estimation of the state given by an observer.

The singular perturbation model of a dynamical system is a state-space model (1) in which the derivatives of some of the states are multiplied by a *small* positive parameter ε. What means "small" for a scientist is usually well understood. For instance, in population dynamics, the characteristic time of reproduction is small compare to the effect of demographic pressure on natality rate. As another example we can say that the dayly activity of a predator eating its prey is fast compare to its annual reproduction [17]. What means "small" for a mathematician is somewhat different. In order to guaranty the widely spread standard of mathematical rigor, mathematics have to be written with respect to some formal langage in which proofs are unambigously written.

Classical analysis prohibits the use of a sentence like "ε is a small parameter". This is the reason why, in order to capture the idea that ε is a "small" parameter one considers the whole family of differential equations (1) where the parameter ε ranges over an interval $(0, \varepsilon_0]$, $\varepsilon_0 > 0$. The aim of *Singular Perturbation Theory* is to use the limiting behaviour of the system, when ε goes to 0, to get an idea of what the system looks like when ε is "small".

The problem with classical singular perturbation theory [8, 14, 15, 32] lies in the fact that its results are expressed in a rather abstract and sophisticated way. For this reason the use of its results is not easy for non mathematically trained people and may be subject to misinterpretations, even for mathematicians (see Lobry [18]).

Recently, Robinson [28] developped a new mathematical formalism, called Non Standard Analysis (NSA) which is proved to be equivalent to the classical one — in the sense that everything you formulate in the new formalism can be translated in the old one and vice versa — in which the sentence "ε is an infinitesimal real number strictly greater than 0" makes perfect sense. As a consequence the mathematical statement, in the new formalism: "Let us consider system (1) with ε an infinitesimal real number strictly greater than 0" have the very natural interpretation: "with ε a small parameter", where "small" has the intuitive meaning that every scientist understand.

The idea of using NSA in perturbation theory of differential equation goes back to the seventies with the Reebian scholl (cf [23, 24, 26, 27] and the references therein). It gave birth to the nonstandard perturbation theory of differential equations which has become today a well-established tool in asymptotic theory (see the five-digits classification 34E18 of the AMS 2000 Mathematics Subject Classification). To have an idea of the rich literature on the subject, the reader is referred to [1, 5, 6, 7, 23, 29, 33]. In this chapter we use the langage of NSA. For the convenience of the reader the results are formulated in both classical terms and nonstandard terms. In the Appendix we give a short description of NSA and we discuss the classical concepts in this context.

In this chapter we do not provide proofs of the presented results. For each result we send the reader to a reference in the literature. For the sake of clarity we prefer to discuss the "geometrical" meaning of the results and to illustrate them by simple examples.

2 Fast and slow systems

2.1 Tikhonov theory

System (1) is called a *fast and slow system*. The vectors $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ are its slow and fast components. The mathematical tool used to deal with this different time scales is Tikhonov's theorem which permits to reduce the complexity of the system through suitable approximations. If we go to the *fast time*, $\tau = (t - t_0)/\varepsilon$, system (1) becomes

$$\begin{cases} x' = \varepsilon f(t_0 + \varepsilon\tau, x, z, \varepsilon), & x(0) = \xi(\varepsilon) \\ z' = g(t_0 + \varepsilon\tau, x, z, \varepsilon), & z(0) = \zeta(\varepsilon) \end{cases}$$

where the prime $'$ denotes the derivative with respect to time τ. This system is a regular perturbation of system

$$\begin{cases} x' = 0, & x(0) = \xi(0) \\ z' = g(t_0, x, z, 0), & z(0) = \zeta(0) \end{cases}$$

Hence the z component of any solution of system (1) varies very quickly according to the equation

$$z' = g(t_0, \xi(0), z, 0), \qquad z(0) = \zeta(0). \tag{3}$$

This equation is called the *boundary layer equation*. It consists simply in equation

$$z' = g(t, x, z, 0), \tag{4}$$

where $t = t_0$ and $x = \xi(0)$ are frozen at their initial values. A solution of (4) may behave in one of several ways : it may be unbounded as $\tau \to \infty$, it may tend toward an equilibrium point, or it may approach a more complex attractor. Obviously, if the fast equation has multiple stable equilibria, the asymptotic behaviour of a solution is determined by its initial value. Assume that the second case occurs, that is, the solutions of (4) tend toward an equilibrium $h(t, x)$, where $z = h(t, x)$ is a root of equation

$$g(t, x, z, 0) = 0. \tag{5}$$

The manifold $\mathcal{L}$ defined by equation (5) is called the *slow manifold*. The solutions of (1) have a fast transition (boundary layer) from $(\xi(0), \zeta(0))$ to $(\xi(0), h(t_0, \xi(0)))$, namely to a point of the slow manifold $\mathcal{L}$. Then a slow motion takes place on the slow manifold, according to the equation

$$\dot{x} = f(t, x, h(t, x), 0). \tag{6}$$

This equation is called *the reduced problem*. It is obtained by taking the first equation of (1) subject to the constraint (5).

This description of system (1) was given by Tikhonov [31], under suitable hypotheses (see also [8, 14, 15, 20, 21, 32]). The crucial stability property we need for the boundary layer system (4) is the asymptotic stability of its equilibrium point $z = h(t, x)$, uniformly in the parameters t and x. The following definition states this property precisely.

Definition 1 *The equilibrium $z = h(t, x)$ of system (4) is*

1. stable *(in the sense of Liapunov) if for every $\mu > 0$ there exists $\eta > 0$ (depending on (t, x)), with the property that any solution $z(\tau)$ of (4) for which $\|z(0) - h(t, x)\| < \eta$, can be continued for all $\tau > 0$ and satisfies the inequality $\|z(\tau) - h(t, x)\| < \mu$.*
2. attractive *if it admits a basin of attraction, that is, a neighborhood $\mathcal{V}$ (depending on (t, x)), with the property that any solution $z(\tau)$ of (4) for which $z(0) \in \mathcal{V}$, can be continued for all $\tau > 0$ and satisfies $\lim_{\tau\to+\infty} z(\tau) = h(t, x)$.*
3. asymptotically stable *if it is stable and attractive.*
4. asymptotically stable, uniformly *in $(t, x) \in [t_0, t_1] \times X$, if for every $\mu > 0$ there exists $\eta > 0$ with the property that for any $(t, x) \in [t_0, t_1] \times X$ any solution $z(\tau)$ of (4) for which $\|z(0) - h(t, x)\| < \eta$, can be continued for all $\tau > 0$ and satisfies the inequality $\|z(\tau) - h(t, x)\| < \mu$ and $\lim_{\tau\to+\infty} z(\tau) = h(t, x)$.*

We say that the basin of attraction of the equilibrium point $z = h(t, x)$ is uniform in $(t, x) \in [t_0, t_1] \times X$, if there exists $a > 0$, such that for all $(t, x) \in [t_0, t_1] \times X$, the ball of center $h(t, x)$ and radius a

$$\mathcal{B} = \{z \in \mathbb{R}^m : \|z - h(t, x)\| < a\}$$

is a basin of attraction of $h(t, x)$.

If $z = h(t, x)$ is asymptotically stable, uniformly in $(t, x) \in [t_0, t_1] \times X$, then the basin of attraction is uniform. Conversely, if X is compact, then the asymptotic stablility of $z = h(t, x)$ together with the existence of a uniform basin of attraction imply that the asymptotic stability is uniform in $(t, x) \in [t_0, t_1] \times X$ (see [11, 20]). Hence, to formulate Tikhonov's theorem under the hypothesis that the basin of attraction is uniform as done in the present lecture or in [8, 10] is the same as formulating it under the hypothesis that the asymptotic stability is uniform as done in [15, 31, 32]. There are other versions of this theorem which use slightly different technical assumptions, for example [14].

Verification of uniform asymptotic stability of the equilibrium point may be done either by linearization or via search of a Lyapunov function. For $\mathcal{C}^1$ systems, let

$$A = \left[\frac{\partial g}{\partial z}(t, x, z, 0)|_{z=h(t,x)}\right]$$

be the Jacobian matrix evaluated at the equilibrium point $z = h(t, x)$. It can be shown that if there exists a positive constant c such that for all $(t, x) \in [t_0, t_1] \times X$, any eigenvalue λ of the Jacobian matrix A satisfies $\mathrm{Re}\lambda \leq -c$, then the equilibrium is asymptotically stable uniformly in $(t, x) \in [t_0, t_1] \times X$.

The discussion of asymptotic stability in NSA goes as follows.

Proposition 1 *Assume that g and h are standard. The equilibrium $z = h(t, x)$ of system (4) is*

1. stable *(in the sense of Liapunov) if any solution $z(\tau)$ of (4) for which $z(0) \simeq h(t, x)$, can be continued for all $\tau > 0$ and satisfies $z(\tau) \simeq h(t, x)$.*

2. attractive *if for all standard (t, x) it admits a standard basin of attraction, that is, a standard neighborhood $\mathcal{V}$ (depending on (t, x)), with the property that any solution $z(\tau)$ of (4) for which $z(0) \in \mathcal{V}$ is standard, can be continued for all $\tau > 0$ and satisfies $z(\tau) \simeq h(t, x)$ for all $\tau \simeq +\infty$.*

3. asymptotically stable *if for all standard (t, x) there exists a standard neighborhood $\mathcal{V}$ (depending on (t, x)), with the property that any solution $z(\tau)$ of (4) for which $z(0) \in \mathcal{V}$, can be continued for all $\tau > 0$ and satisfies $z(\tau) \simeq h(t, x)$ for all $\tau \simeq +\infty$.*

4. asymptotically stable, uniformly *in $(t, x) \in [t_0, t_1] \times X$, if there exists a standard $a > 0$ such that for all $(t, x) \in [t_0, t_1] \times X$ any solution $z(\tau)$ of (4) for which $z(0)$ is in the ball of center $h(t, x)$ and radius a, can be continued for all $\tau > 0$ and satisfies $z(\tau) \simeq h(t, x)$ for all $\tau \simeq +\infty$.*

Notice that if we only require that for all standard $z(0) \in \mathcal{V}$, $z(\tau) \simeq h(t, x)$ for all $\tau \simeq +\infty$, then we obtain the attractivity of the equilibrium $z = h(t, x)$. The asymptotic stability of this equilibrium is obtained if we require that for all $z(0) \in \mathcal{V}$, standard or not standard, $z(\tau) \simeq h(t, x)$ for all $\tau \simeq +\infty$. In fact a time $T(z(0))$ such that $z(\tau) \simeq h(t, x)$ for all $\tau > T(z(0))$ can grow unboundedly even if $z(0)$ ranges over a compact neighborhood of $h(t, x)$. This occurs for instance if $h(t, x)$ is attractive but not stable as in the classical example of Vinograd (see [9] Section 40).

Theorem 1 *[20] Consider the singular perturbation problem (1) and let $z = h(t, x)$ be an isolated root of (5). Assume that there exist positive constants $t_1 > t_0$, r and ε_0, and a compact domain $X \subset \mathbb{R}^n$ such that the following conditions are satisfied for all*

$$t_0 \leq t \leq t_1, \qquad x \in X, \qquad \|z - h(t, x)\| \leq r, \qquad 0 < \varepsilon \leq \varepsilon_0$$

- *The functions $f(t, x, z, \varepsilon)$, $g(t, x, z, \varepsilon)$ and $h(t, x)$ are continuous, and the initial data $\xi(\varepsilon)$ and $\zeta(\varepsilon)$ are continuous.*

- *The reduced problem (6) has a unique solution $x_0(t)$ with initial condition $x(t_0) = \xi(0)$, defined on $[t_0, t_1]$ and $x_0(t) \in X$ for all $t \in [t_0, t_1]$.*

- *The boundary layer equation (4) has the uniqueness of the solutions with prescribed initial conditions. Let $\tilde{z}(\tau)$ be the solution of equation (3).*

- *The equilibrium point $z = h(t, x)$ of this boundary layer equation is asymptotically stable uniformly in $(t, x) \in [t_0, t_1] \times X$.*
- *The initial condition $\zeta(0)$ belongs to the basin of attraction of $h(t_0, \xi(0))$.*

Then for every $\delta > 0$ there exists a positive constant ε^* such that for all $0 < \varepsilon < \varepsilon^*$, every solution $(x(t), z(t))$ of the singular perturbation problem (1) is defined at least on $[t_0, t_1]$, and satisfies

$$\|x(t) - x_0(t)\| \leq \delta$$

$$\left\|z(t) - \tilde{z}\left(\frac{t - t_0}{\varepsilon}\right) - h(t, x_0(t)) + h(t_0, \xi(0))\right\| \leq \delta$$

for all $t_0 \leq t \leq t_1$.

When the singular perturbation problem (1) has a unique solution, we denote it as $(x(t, \varepsilon), z(t, \varepsilon))$. We have

$$\lim_{\varepsilon \to 0} x(t, \varepsilon) = x_0(t), \text{ for } t_0 \leq t \leq t_1. \tag{7}$$

We have also

$$\lim_{\varepsilon \to 0} z(t, \varepsilon) = h(t, x_0(t)), \text{ for } t_0 < t \leq t_1, \tag{8}$$

but the limit holds only for $t > t_0$, since there is a boundary layer at $t = t_0$, for the z-component. Indeed, we have

$$\lim_{\varepsilon \to 0} \left(z(t, \varepsilon) - \tilde{z}\left(\frac{t - t_0}{\varepsilon}\right)\right) = h(t, x_0(t)) - h(t_0, \xi(0)), \text{ for } t_0 \leq t \leq t_1.$$

2.2 Singular perturbations on the infinite interval

Theorem 1 is valid only over compact time intervals. The estimates (7) and (8) do not hold in general for all $t \geq t_0$. This can be done if some additional conditions are added to ensure asymptotic stability of the solutions of the reduced problem (see [10, 14, 15, 20]). The following theorem extends Theorem 1 to the infinite-time interval (see also [10, 14, 15]).

Theorem 2 *[20] Consider the singular perturbation problem (1) and let $z = h(t, x)$ be an isolated root of (5). Assume that there exist positive constants r and ε_0, and a compact domain $X \subset \mathbb{R}^n$ such that the following conditions are satisfied for all*

$$t_0 \leq t < +\infty, \qquad x \in X, \qquad \|z - h(t, x)\| \leq r, \qquad 0 < \varepsilon \leq \varepsilon_0$$

- *The functions $f(t, x, z, \varepsilon)$, $g(t, x, z, \varepsilon)$ and $h(t, x)$ are continuous, and the initial data $\xi(\varepsilon)$ and $\zeta(\varepsilon)$ are continuous.*
- *$f(t, 0, 0, 0) = 0$, $g(t, 0, 0, 0) = 0$ and $h(t, 0) = 0$*

- *the origin of the reduced problem (6) is asymptotically stable and the initial condition* $\xi(0)$ *belongs to its basin of attraction. Let* $x_0(t)$ *be the solution of the reduced system (6) with initial condition* $x_0(t_0) = \xi(0)$*. It is defined for all* $t_0 \leq t < +\infty$ *and we have* $\lim_{t\to+\infty} x_0(t) = 0$

- *The boundary layer equation (4) has the uniqueness of the solutions with prescribed initial conditions. Let* $\tilde{z}(\tau)$ *be the solution of equation (3).*

- *The equilibrium point* $z = h(t,x)$ *of this boundary layer equation is asymptotically stable uniformly in* $(t,x) \in [t_0, t_1] \times X$*.*

- *The initial condition* $\zeta(0)$ *belongs to the basin of attraction of* $h(t_0, \xi(0))$*.*

Then for every $\delta > 0$ *there exists a positive constant* ε^* *such that for all* $0 < \varepsilon < \varepsilon^*$*, every solution* $(x(t), z(t))$ *of the singular perturbation problem (1) is defined for all* $t \geq t_0$*, and satisfies*

$$\|x(t) - x_0(t)\| \leq \delta$$

$$\left\|z(t) - \tilde{z}\left(\frac{t-t_0}{\varepsilon}\right) - h(t, x_0(t)) + h(t_0, \xi(0))\right\| \leq \delta$$

for all $t \geq t_0$*.*

Assume that the singular perturbation problem (1) has a unique solution, denoted by $(x(t,\varepsilon), z(t,\varepsilon))$. Since $x_0(t)$ is defined for all $t_0 \leq t < +\infty$ and satisfies $\lim\limits_{t\to+\infty} x_0(t) = 0$ we have

$$\lim_{\varepsilon\to 0, t\to+\infty} x(t,\varepsilon) = 0, \qquad \lim_{\varepsilon\to 0, t\to+\infty} z(t,\varepsilon) = 0.$$

This last property does not mean that the origin of the full system (1) is asymptotically stable. It means only that it is *practically asymptotically stable* (see Section 2.4). If we want to use the Singular Perturbation Theory for the purpose of stabilization then we need a stronger result. This is done, in the following section, if some additional conditions are added to ensure exponential stability of the solutions of the reduced problem and the fast equation.

2.3 Stability

Definition 2 *The equilibrium* $z = h(t,x)$ *of the boundary layer system (4) is exponentially stable, uniformly in* $(t,x) \in [t_0, t_1] \times X$*, if there exist positive constants* k*,* γ*, and* r *such that for all* $(t,x) \in [t_0, t_1] \times X$*, any solution of (4) for which* $\|z(0) - h(t,x)\| \leq r$ *satisfies*

$$\|z(\tau)\| \leq k\|z(0)\|\mathrm{e}^{-\gamma\tau} \textit{ for all } \tau \geq 0.$$

We have the following result of conceptual importance, whose proof can be found in [14], Section 9.4 or [15], Section 7.5. Similar results can also be obtained using the Geometric Theory of Singular Perturbations (see [12], Section B.3).

Theorem 3 *Consider the singular perturbation problem (1). Assume that*

$$f(t,0,0,\varepsilon)=0 \text{ and } g(t,0,0,\varepsilon)=0.$$

Let $z=h(t,x)$ be an isolated root of (5) such that $h(t,0)=0$. Assume that there exist positive constants r and ε_0, and a compact domain $X\subset\mathbb{R}^n$ such that the following conditions are satisfied for all

$$t_0\le t<+\infty,\qquad x\in X,\qquad \|z-h(t,x)\|\le r,\qquad 0<\varepsilon\le\varepsilon_0$$

- *The functions $f(t,x,z,\varepsilon)$, $g(t,x,z,\varepsilon)$ and $h(t,x)$, are $\mathcal{C}^1$, and the initial data $\xi(\varepsilon)$ and $\zeta(\varepsilon)$ are continuous.*
- *The origin of the reduced problem (6) is exponentially stable.*
- *The equilibrium point $z=h(t,x)$ of the boundary layer equation is exponentially stable uniformly in $(t,x)\in[t_0,t_1]\times X$.*

Then there exists $\varepsilon^*>0$ such that for all $0<\varepsilon<\varepsilon^*$, the origin of (1) is exponentially stable.

Notice that exponential stability is necessary for this kind of result. Indeed it is well known that asymptotic stability is not robust to arbitrary small perturbations (see [13], Section 10.2). This point is illustrated by the following example.

Example 1 Consider the singularly perturbed system

$$\begin{aligned}\dot x&=-x^3+\varepsilon x\\ \varepsilon\dot z&=-z\end{aligned}$$

The reduced system is $\dot x=-x^3$. Its origin is asymptotically stable but not exponentially stable. The origin of the full system is unstable despite the fact that the origin of the boundary layer equation $z'=-z$ is exponentially stable.

It is important to notice that the origin of the full system is asymptotically stable only for small values of ε. This point is illustrated by the following example.

Example 2 Consider the singularly perturbed system

$$\left\{\begin{array}{rcl}\dot x&=&-z_2\\ \varepsilon\dot z_1&=&-z_1+x\\ \varepsilon\dot z_2&=&-z_2+z_1\end{array}\right. \tag{9}$$

The boundary layer equation is

$$z_1'=-z_1+x,\qquad z_2'=-z_2+z_1$$

It has $(z_1 = x, z_2 = x)$ as an exponentially stable equilibrium uniformly in x. The corresponding reduced problem is $\dot{x} = -x$. According to Theorem *3* the origin of system (*9*) is exponentially stable for small values of ε. Since this system is linear, elementary computations of the eigenvalues show that the origin is exponentially stable if $\varepsilon < 2$ and unstable if $\varepsilon > 2$.

It is important to notice that the origin of the full system is not in general globally asymptotically stable (GAS) even if the exponential stability of the boundary layer equation and the reduced problem are global, and ε is small enough. This point is illustrated by the following example.

Example 3 Consider the system

$$\dot{x} = -x + x^2 z, \qquad \varepsilon\dot{z} = -z. \tag{10}$$

The origin of the boundary layer equation $z' = -z$ is globally exponentially stable and the origin of the corresponding reduced problem $\dot{x} = -x$ is also globally exponentially stable. According to Theorem *3* the origin of system (*10*) is exponentially stable for small values of ε. In fact the origin is exponentially stable for all ε but the stability is not global. Since

$$\frac{d}{dt}(xz) = xz(xz - 1 - 1/\varepsilon),$$

we see that the hyperbola $xz = 1 + 1/\varepsilon$ consists of two trajectories. Thus the origin of (*10*) is not GAS. Tikhonov's theorem asserts that the x-component of the solution $x(t, \varepsilon)$ of (*10*) with initial condition (x_0, z_0) is such that, for all $t \geq 0$

$$\lim_{\varepsilon \to 0} x(t, \varepsilon) = x_0(t) := x_0 \mathrm{e}^{-t} \tag{11}$$

where $x_0(t)$ is the solution of the reduced model, with initial condition x_0. By explicit computations, it is easy to show that the basin of attraction of the origin of (*10*) is the set $\{xz < 1 + 1/\varepsilon\}$.

2.4 Practical stability

In this section we do not assume that $f(t, 0, 0, \varepsilon) = 0$ and $g(t, 0, 0, \varepsilon) = 0$, for all t, so that the origin is not an equilibrium of system (1). We begin with the definition of practical stability in a system

$$\dot{x} = f(x, \varepsilon) \tag{12}$$

depending on a parameter ε, without explicitly asking for the separation of variables into fast and slow variables as in system (1).

Definition 3 *The origin $x = 0$ of system (12) is practically asymptotically stable when $\varepsilon \to 0$ if there exists $A > 0$ such that for all $r > 0$ there exist $\varepsilon_0 > 0$ and $T > 0$*

satisfying that for all $0 < \varepsilon < \varepsilon_0$, any solution of (12) starting in the ball of radius A is at time T in the ball of radius r and never leave this ball. If this property holds for all $A > 0$ then the origin is said to be practically semiglobally asymptotically stable (PSGAS) when $\varepsilon \to 0$.

Example 4 *In the following systems, where x is real, the origin is PSGAS when $\varepsilon \to 0$.*

- $\dot{x} = x(\varepsilon x - 1)$. *The origin is stable but not GAS.*
- $\dot{x} = \varepsilon - x$. *The origin is not an equilibrium.*
- $\dot{x} = x^2(\varepsilon - x)$. *The origin is unstable.*

The notion of practical stability has a long history in the theory of differential equations [16]. Our notion of practical stability in a system depending on a parameter is strongly related to the notion of practical stabilizability introduced by Byrnes and Isidori [2]. There is a beautiful nonstandard characterization of practical stability for standard systems:

Proposition 2 *If f is standard then the origin of (12) is PSGAS when $\varepsilon \to 0$ if and only if for any infinitesimal ε, any solution of (12) with limited initial condition is infinitely close to the origin for all unlimited time.*

Let us formulate a new practical stability result derived from Tikhonov theory.

Theorem 4 *[20, 21] Consider the singularly perturbed system (1). Assume that*

$$f(t, 0, 0, 0) = 0 \text{ and } g(t, 0, 0, 0) = 0.$$

Let $z = h(t, x)$ be an isolated root of (5) such that $h(t, 0) = 0$. We assume that

1. *the equilibrium $z = h(t, x)$ of the boundary layer equation (4) is asymptotically stable uniformly in (t, x).*
2. *the origin of the corresponding reduced model (6) is asymptotically stable*

The the origin of system (1) is practically asymptotically stable when $\varepsilon \to 0$. If in addition the equilibrium $z = h(t, x)$ of the boundary layer equation is GAS and the origin of the corresponding reduced model is GAS then the origin of system (1) is PSGAS.

3 Feedback Stabilization

3.1 Stabilization

Consider the feedback stabilization of the system on Figure 1

$$\begin{array}{rcl} \dot{x} & = & f(x, z), \\ \varepsilon \dot{z} & = & g(z, u). \end{array} \tag{13}$$

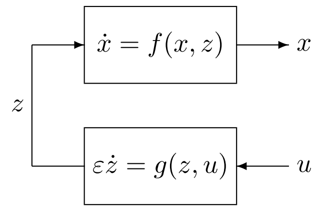

Figure 1: The system has an open-loop equilibrium point at the origin. The control task is to design a state feedback control law to stabilize the origin. The z dynamics are much faster than the x dynamics.

where $f(0, 0) = 0$ and $g(0, 0) = 0$. The z dynamics is much faster than the x dynamics. Suppose that equation $g(z, u) = 0$ has an isolated root $z = h(u)$, such that $h(0) = 0$, which is an exponentially stable equilibrium of the boundary-layer equation

$$z' = g(z, u), \tag{14}$$

uniformly in u. A procedure to design a state feedback control law to stabilize the origin is given below. We can simplify the design problem, by neglecting the z dynamics and substituting $z = h(u)$ in the x equation. The reduced-order model is

$$\dot{x} = f(x, h(u))$$

We use this model to design a state feedback control law $u = \gamma(y)$ such that the origin of the closed-loop model

$$\dot{x} = f(x, h(\gamma(x))) \tag{15}$$

is exponentially stable (see Figure 2). We shall refer to this model as the reduced closed-loop system. Suppose we have designed such a control law. Will this control stabilize the actual system with the z dynamics included ? When the control is applied to the actual system, the closed-loop system is

$$\begin{aligned} \dot{x} &= f(x, z), \\ \varepsilon\dot{z} &= g(z, \gamma(x)) \end{aligned} \tag{16}$$

We have the singular perturbation problem of Figure 2, where the full singularly perturbed model is the actual closed-loop system and the reduced model is the reduced closed-loop system. By design, the origin of the reduced model is exponentially stable. Since the equilibrium $z = h(\gamma(x))$ is an exponentially stable equilibrium of the boundary-layer equation

$$z' = g(z, \gamma(x)),$$

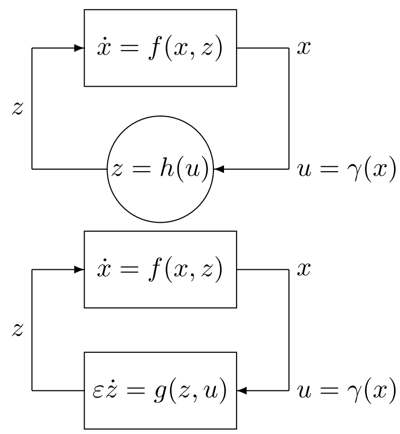

Figure 2: The control $u = \gamma(x)$ which stabilizes the reduced problem will stabilize the full problem for small ε

uniformly in x, by Theorem 3, the origin of the actual closed-loop system is exponentially stable for sufficiently small ε. This result legitimizes the procedure of simplification which consists in neglecting the fast dynamics. This result is summarized in the following theorem.

Theorem 5 *Assume that $f(0,0) = 0$ and $g(0,0) = 0$. Let $z = h(u)$ be an isolated root of equation $g(z,u) = 0$ such that $h(0) = 0$. Assume that $z = h(u)$ is an exponentially stable equilibrium of the boundary-layer equation (14) uniformly in u. Assume that we can design a control law $u = \gamma(y)$ such that the origin of the reduced model (15) is exponentially stable. Then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the origin of the dynamical feedback system (16) is exponentially stable.*

Now, we do not assume that $z = h(u)$ is an exponentially stable equilibrium uniformly in u of the boundary layer equation (14). Since we have a control u to our disposal, we can design it so that this root becomes exponentially stable. A procedure to design a composite state feedback control law to stabilize the origin is given below.

Step 1 Design a control $u = \gamma(y)$ such that the origin of the reduced system (15) is exponentially stable uniformly in x.

Step 2 With the knowledge of γ design a control law $u = \Gamma(x,z)$, such that $\Gamma(x, h(\gamma(x))) = 0$, which stabilizises the fast equation

$$z' = g(z, \gamma(x) + u)$$

at $z = h(\gamma(x))$, that is to say the equilibrium point $z = h(\gamma(x))$ of the closed-loop system

$$z' = g(z, \gamma(x) + \Gamma(x, z)) \tag{17}$$

is exponentially stable uniformly in x (see Figure 3).

Step 3 Then the composite feedback control

$$u = \gamma(x) + \Gamma(x, z))$$

will stabilizise (13), that is, for small ε, the origin is an exponentially stable equilibrium of the closed-loop system

$$\begin{array}{rcl} \dot{x} & = & f(x, z), \\ \varepsilon \dot{z} & = & g(z, \gamma(x) + \Gamma(x, z)) \end{array} \tag{18}$$

This result is summarized in the following theorem.

Theorem 6 *Assume that $f(0,0) = 0$ and $g(0,0) = 0$. Let $z = h(u)$ be an isolated root of equation $g(z,u) = 0$ such that $h(0) = 0$. Assume that we can design a control law $u = \gamma(y)$ such that the origin of the reduced model (15) is exponentially stable. Assume that we can design a control law such that the equilibrium $z = h(\gamma(x))$ of system (17) is exponentially stable uniformly in x. Then there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the origin of the dynamical feedback system (18) is exponentially stable.*

3.2 Practical Stabilization

Asymptotic stability alone, instead of exponential stability, may not be enough and stronger technical conditions are needed (see [15], Section 7.6). Asymptotic stability is enough to guarantee practical stability. Consider again the system (13). Suppose that $z = h(u)$ is an asymptotically stable equilibrium of the boundary-layer equation (14) uniformly in u. Suppose that we can design a control $u = \gamma(y)$ such that the origin of the closed-loop model (15) is asymptotically stable. By Theorem 4, the origin of the actual closed-loop system (16) is practically asymptotically stable when ε goes to 0. This result is summarized in the following theorem.

Theorem 7 *Assume that $f(0,0) = 0$ and $g(0,0) = 0$. Let $z = h(u)$ be an isolated root of equation $g(z,u) = 0$ such that $h(0) = 0$. Assume that $z = h(u)$ is an asymptotically stable equilibrium of the boundary-layer equation (14) uniformly in u. Assume that we can design a control law $u = \gamma(y)$ such that the origin of the reduced model (15) is asymptotically stable. Then the origin of the dynamical feedback system (16) is practically asymptotically stable when ε goes to 0.*

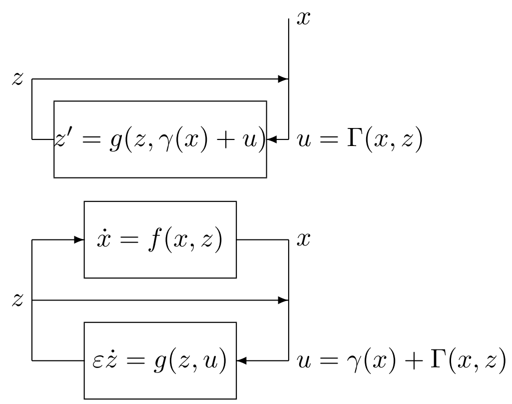

Figure 3: The composite control $u = \gamma(x) + \Gamma(x, z)$ will stabilize the full problem for small ε

Now, we do not assume that $z = h(u)$ is an asymptotically stable equilibrium uniformly in u of the boundary layer equation (14). Design a control $u = \gamma(y)$ such that the origin of the reduced system (15) is asymptotically stable. With the knowledge of γ design a control law $u = \Gamma(x, z)$ such that the equilibrium point $z = h(\gamma(x))$ of the closed-loop system (17) is asymptotically stable uniformly in x. Then the composite feedback control

$$u = \gamma(x) + \Gamma(x, z))$$

will stabilizise practically (13) for small ε. This result is summarized in the following theorem.

Theorem 8 *Assume that $f(0,0) = 0$ and $g(0,0) = 0$. Let $z = h(u)$ be an isolated root of equation $g(z, u) = 0$ such that $h(0) = 0$. Assume that we can design a control law $u = \gamma(y)$ such that the origin of the reduced model (15) is asymptotically stable. Assume that we can design a control law such that the equilibrium $z = h(\gamma(x))$ of system (17) is asymptotically stable uniformly in x. Then the origin of the dynamical feedback system (18) is practically asymptotically stable when ε goes to 0.*

4 The Peaking Phenomenon

4.1 The limitations of Tikhonov's theory

In this section, we are concerned with the asymptotic behavior, when the parameter $\varepsilon \to 0$, of the nonlinear triangular system (2), where $\dot{} = d/dt$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$,

$\varepsilon \in (0, \varepsilon_0]$ and the mappings

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \text{ and } G : \mathbb{R}^m \times (0, \varepsilon_0] \to \mathbb{R}^m$$

are of class $\mathcal{C}^1$. Notice that the limit of G is not assumed to be defined when $\varepsilon \to 0$, so that system (2) is a singular perturbation. We can think of the first equation in system (2) as a controlled system taking its inputs from the second equation in system (2). The *zero-input system* is the system

$$\dot{x} = f(x, 0). \tag{19}$$

We assume that

H1 : system (19) has 0 as a GAS equilibrium.

The condition H1 implies that the solutions of (19) tend to 0 as $t \to +\infty$. Our aim is to state conditions on f and G such that the solutions of system (2) tend to zero when $\varepsilon \to 0$ and $t \to +\infty$.

Intuitively, if we require that the solutions of $\dot{y} = G(y, \varepsilon)$ tend to 0 arbitrarily fast in t when $\varepsilon \to 0$, then, the idea that the solutions of system (2) ought to tend to zero appears plausible for the following heuristics. The second equation in system (2) drives any initial condition very fast in t near the manifold $y = 0$, where the zero-input system takes over and drives x to zero. Due to the peaking phenomenon, this idea fails. Of course, some solutions of the second equation in system (2) may peak to very large values before they decay to zero. The interaction of this peaking with the nonlinear growth in the first equation in system (2) could destabilize system (2). In general the origin of system (2) is not GAS, even if there is no peaking. The best result one can expect for system (2) is that its origin is PSGAS when $\varepsilon \to 0$.

Our first objective is to give a precise meaning to the hypothesis that the solution of the second equation in (2) tend to 0 arbitrarily fast in t when $\varepsilon \to +\infty$. For this purpose we shall use *Internal Set Theory* (IST).

In NSA, instead of considering a family of systems (2) depending on the parameter ε and dealing with its asymptotic propreties when $\varepsilon \to 0$, we consider just one (nonstandard) system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(y), \tag{20}$$

where $g(y)$ plays the role of $G(y, \varepsilon)$ with $\varepsilon > 0$ a *fixed infinitesimal real number* and we look to the (external) properties of system (20).

We introduce the concepts of *instantaneous stability*, which quantify the fast decay to zero of the solutions of the second equation in system (2) or (20) as well as the concepts of *uniform infinitesimal boundedness*, which measure the effects of the solutions of the second equation in system (2) or (20) on the first equation in this system. All these concepts are defined in both standard and nonstandard terms. The notion of *instantaneous stabilizabilty* was previously defined (see [4]) for linear systems within the IST framework. Our concept of instantaneous stability for a general system is a straightforward extension of their definition.

Let us consider the particular case where system (2) is of the form

$$\dot{x} = f(x, y), \qquad \dot{y} = G(y)/\varepsilon, \tag{21}$$

where $G : \mathbb{R}^m \to \mathbb{R}^m$ is of class $\mathcal{C}^1$. Assume that:

H2 : the vector field $y \to G(y)$ has 0 as a GAS equilibrium.

System (21) is very well understood in singular perturbation theory. The main tool is Tikhonov's theorem (cf. Section 2.2 and [10, 14, 20]) on the infinite interval of time.

Theorem 9 (Tikhonov) *Assume that f and G are standard, $\varepsilon > 0$ is infinitesimal and H2 holds. Then for all limited x_0 and y_0, the x-component $x(t)$ of the solution of system (21) with initial condition (x_0, y_0) satisfies $x(t) \simeq x_0(t)$ as long as t and $x_0(t)$ are limited, where $x_0(t)$ is the solution of the zero-input system (19), with initial condition x_0. Moreover, if H1 also holds then $x(t) \simeq x_0(t)$ for all $t \geq 0$.*

Theorem 10 *Assume that f and G are standard, $\varepsilon > 0$ is infinitesimal, and H1 and H2 hold. Then for all limited x_0 and y_0, the solution $(x(t), y(t))$ of (21) with initial condition (x_0, y_0) satisfies $x(t) \simeq 0$ and $y(t) \simeq 0$ for all positive unlimited t, that is to say, it is PSGAS when $\varepsilon \to 0$.*

Theorem 11 *Assume that H1 and H2 hold, then for all (x_0, y_0), the solution $(x(t, \varepsilon), y(t, \varepsilon))$ of (21) with initial condition (x_0, y_0) tends to 0 as $t \to +\infty$ and $\varepsilon \to 0$. For every $A > 0$, the convergence is uniform with respect to x_0 and y_0 for $\|x_0\| \leq A$ and $\|y_0\| \leq A$.*

Let us consider now the particular case where system (2) is of the form

$$\dot{x} = f(x, y), \qquad \dot{y} = G(\varepsilon)y, \tag{22}$$

where $G(\varepsilon)$ is a square matrix of order m. Assume that

H3 : the real parts of the eigenvalues of $G(\varepsilon)$ tend to $-\infty$ when $\varepsilon \to 0$.

System (22) was considered in automatic control literature. In this context the second equation in system (22) is understood as a "high gain" dynamic feedback [30]. More precisely, we start with a state feedback-partially linear system of the form

$$\dot{x} = f(x, y), \qquad \dot{y} = Ay + Bu,$$

where the pair (A, B) is controllable. The second equation can be easily stabilized by $u = Ky$, where K is designed such that the matrix $G := A + BK$ is Hurwitz. Asymptotic stability of the origin of the full closed-loop system

$$\dot{x} = f(x, y) \qquad \dot{y} = Gy \tag{23}$$

will now follow from assumption H1 and the concept of *input to state stability* (see [14], p. 548). What about global stabilization ? If the linear feedback control $u = Ky$ is designed to assign the eigenvalues of G far to the left in the complex plane so that the solutions of $\dot{y} = Gy$ decay to zero arbitrarily fast, one might think that the origin of the triangular system (23) can be GAS. It turns out that such a strategy may fail and (22) is not reductible to Tikhonov's theorem because of the so called "peaking phenomenon". The following example of Sussmann and Kokotovic (cf. [30] Example 1.1) shows a system of type (22) such that the solution $y(t, \varepsilon)$ becomes unbounded when $\varepsilon \to 0$, even if its decay to zero is arbitrarily fast. In that case, the origin of (22) is not GAS. Even worse, for some initial conditions, the solution escapes to infinity in finite time.

Example 5 *Consider the three dimensional system*

$$\dot{x} = (1 + y_2)\varphi(x), \qquad \dot{y} = G(\varepsilon)y, \tag{24}$$

where $\varphi(x) = -x^3/2$ *and*

$$G(\varepsilon) = \begin{pmatrix} 0 & 1 \\ -1/\varepsilon^2 & -2/\varepsilon \end{pmatrix}.$$

The condition H1 holds. Since both eigenvalues of $G(\varepsilon)$ *are equal to* $-1/\varepsilon$*, condition H3 is also satisfied. The exponential matrix*

$$\mathrm{e}^{tG(\varepsilon)} = \begin{pmatrix} 1 + t/\varepsilon & t \\ -t/\varepsilon^2 & 1 - t/\varepsilon \end{pmatrix} \mathrm{e}^{-t/\varepsilon} \tag{25}$$

shows that as $\varepsilon \to 0$*, the solution* $y(t, \varepsilon) = \mathrm{e}^{tG(\varepsilon)}y_0$ *will decay to zero arbitrarily fast. The component* $(-t/\varepsilon^2)\mathrm{e}^{-t/\varepsilon}$ *of (25) reaches the value* $-1/(\varepsilon \mathrm{e})$ *at* $t = \varepsilon$*. Then some solutions have a transient behaviour with a peak of order* $1/\varepsilon$ *before they rapidly decay to 0. This phenomenon is known as the* peaking phenomenon*. The interaction of this peaking with the nonlinear growth in the first equation in system (24) could destabilize the system. Let* $y_0 = (y_{10}, y_{20})$*, then we have*

$$x(t, \varepsilon) = \frac{x_0}{\sqrt{1 + x_0^2 \left[t - y_{10} + (y_{10}(t/\varepsilon + 1) + y_{20}t)\mathrm{e}^{-t/\varepsilon} \right]}}.$$

If $x_0^2 y_{10} > 1$ *the solution will have a finite escape time* $t_e > 0$ *and* t_e *tends to zero when* $\varepsilon \to 0$*. For* ε *large enough, the solutions are attracted to 0 as soon as* $x_0^2 y_{10} < 1$ *and for all* $t > 0$ *we have*

$$\lim_{\varepsilon \to 0} x(t, \varepsilon) = x_0(t) := \frac{x_0^*}{\sqrt{1 + x_0^{*2} t}}. \tag{26}$$

Here, $x_0(t)$ *is the solution of the zero-input system* $\dot{x} = -x^3/2$*, with initial condition* $x_0^* = x_0/\sqrt{1 - x_0^2 y_{10}}$*. Since the limit (26) does not hold for* $t = 0$*, there is a boundary layer at* $t = 0$ *which quickly drives the state* $x(t)$ *from* x_0 *to* x_0^**. The peaking* *phenomenon explains both this boundary layer for the solutions attracted to 0 and the destabilizing effect of the second equation in system (24) on the first equation in this system.*

Example 6 *If $\varphi(x) = -x$ in (24) then the origin is GAS in spite of peaking. However, the solutions still have a boundary layer at $t = 0$. Since*

$$x(t,\varepsilon) = x_0 \mathrm{e}^{y_{10} - t - (y_{10}(t/\varepsilon+1) + y_{20}t)\mathrm{e}^{-t/\varepsilon}}$$

we see that, all the solutions are attracted to 0 and for all $t > 0$ we have

$$\lim_{\varepsilon \to 0} x(t,\varepsilon) = x_0(t) := x_0^* \mathrm{e}^{-t}. \tag{27}$$

Here, $x_0(t)$ is the solution of the zero-input system $\dot{x} = -x$, with initial condition $x_0^* = x_0 \mathrm{e}^{y_{10}}$. The boundary layer at $t = 0$ quickly drives the state $x(t)$ from x_0 to x_0^*.

The limit (11) was obtained from Tikhonov's theorem. The limits (26) and (27) were obtained by direct computation, because Tikhonov's theorem does not apply to Examples 5 and 6.

4.2 Instantaneous Stability and Uniform Infinitesimal Boundedness

In the previous particular cases (21) and (22) of system (20), the fast attractivity of the origin was guaranteed by the special form of the second equation in the system and by conditions H2 or H3. In the general case, there is no such criteria and we must introduce fast attractivity as a hypothesis. For this purpose, we need the following definitions.

Definition 4 *A function $t \mapsto u(t)$ is said to be an impulse if $u(t) \simeq 0$, for all positive non infinitesimal t.*

Definition 5 *The origin of system $\dot{y} = g(y)$ is instantaneously stable (IS) if for every limited y_0, the function $t \mapsto y^g(t, y_0)$ is an impulse.*

Example 7 *If $\varepsilon > 0$ is infinitesimal, then the origin of the second equation in system (24), considered in Example 5, is IS.*

The effects of any eventual peaking of the solutions of the second equation in system (20) on the first equation in this system must be controlled. Thus, we need a hypothesis on the behaviour of the x-component of the solutions of (20) during the very short time where the peaking can destabilize the system. For this purpose, we need the following definition.

Definition 6 *The system (20) is uniformly infinitesimally bounded (UIB) if for all limited x_0 and y_0 and for all positive infinitesimal t, the x-component*

$$x(t) = x^{(f,g)}(t, (x_0, y_0))$$

of the solution of (20) with initial condition (x_0, y_0) is limited.

We consider the condition below :

H4 : system (20) is UIB and the origin of system $\dot{y} = g(y)$ is IS.

Theorem 12 *[19] Assume that f is standard and H4 holds. Then for all limited x_0 and y_0, there exists a limited x_0^* such that the x-component $x(t)$ of the solution of (20) with initial condition (x_0, y_0) satisfies $x(t) \simeq x_0(t)$, as long as $x_0(t)$ is limited and t is appreciable positive, where $x_0(t)$ is the solution of the zero-input system (19), with initial condition x_0^*. Moreover, if H1 also holds then $x(t) \simeq x_0(t)$ for all non infinitesimal positive t.*

This result shows that a solution of system (20) starting from a limited point (x_0, y_0) is approximated by a solution of the zero-input system (19) starting from a limited point x_0^*. It is a Tikhonov-like result (compare with Theorem 9). In Tikhonov's case we had $x_0^* \simeq x_0$. However in the general case, due to the eventual peaking of the solutions of the second equation in system (20), $x_0^* \not\simeq x_0$ and there is a boundary layer at $t = 0$.

Theorem 13 *[19] Assume that f is standard, and that H1 and H4 hold. Then the origin of system (20) is PSGAS when $\varepsilon \to 0$.*

We introduced the nonstandard concepts of IS and UIB for system (20). Let us give their standard formulation for the system (2) depending on the parameter ε.

Definition 7 *The origin of system $\dot{y} = G(y, \varepsilon)$ is IS when $\varepsilon \to 0$, if for every $\delta > 0$, $A > 0$ and $t_0 > 0$, there exists $\varepsilon_0 > 0$ such that whenever an initial condition y_0 satisfies $\|y_0\| \leq A$, it follows that $\|y^G(t, y_0, \varepsilon)\| < \delta$ for all $t \geq t_0$ and all $\varepsilon < \varepsilon_0$.*

Definition 8 *The system (2) is UIB when $\varepsilon \to 0$, if for every $A > 0$, there exist $B > 0$, $t_0 > 0$ and $\varepsilon_0 > 0$ such that whenever an initial condition (x_0, y_0) satisfies $\|x_0\| \leq A$ and $\|y_0\| \leq A$, it follows that $\|x^{(f,G)}(t, (x_0, y_0), \varepsilon)\| \leq B$ for all $t \in [0, t_0]$ and all $\varepsilon < \varepsilon_0$.*

We consider the condition below

H5 : the origin of system $\dot{y} = G(y, \varepsilon)$ is IS when $\varepsilon \to 0$, and system (2) is UIB when $\varepsilon \to 0$.

Theorem 14 *[19] Assume that H1 and H5 hold. Then the origin of system (2) PSGAS when $\varepsilon \to 0$.*

4.3 Further developments

Thus far, we considered only triangular systems, in order to focus on the new concepts of IS and UIB without being burdened by technicalities. The second equation does not contain the first variable and the behavior of y does not depend on x. There are many applications for this particular case. There are also more realistic problems where the second equation in system (20) depends on x also

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y), \tag{28}$$

and the stability of system (28) is investigated under condition H1 and the assumption that the origin of the second equation in (28) is IS in some sense to be precised.

For instance, consider the "high gain" observer problem which is well known in automatic control. Let us begin with the linear case which is well understood. We consider the system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{29}$$

with the usual assumptions of controllability of the pair (A, B) and observability of the pair (A, C). System (29) can be stabilized by $u = Rx$ where R is designed such that the matrix $A + BR$ is Hurwitz. Consider now the Luenberger observer

$$\dot{\hat{x}} = A\hat{x} + Bu + KC(\hat{x} - x).$$

The error between the state $x(t)$ and its observation $\hat{x}(t)$ is $y = \hat{x} - x$ and y is solution of the differential equation

$$\dot{y} = (A + KC)y. \tag{30}$$

If K is taken such that the matrix $G := A + KC$ is Hurwitz, then the error tends to 0. Moreover, if we assign the eigenvalues of G far to the left in the complex plane, then the origin of system (30) is IS.

Consider now the case where the feedback is based on the estimation given by the Luenberger observer. The full system is

$$\dot{x} = Ax + BR\hat{x}, \qquad \dot{\hat{x}} = A\hat{x} + BR\hat{x} + KC(\hat{x} - x),$$

which can be rewritten using the variables x and y as

$$\dot{x} = (A + BR)x + BRy, \qquad \dot{y} = (A + KC)y. \tag{31}$$

This system turns out to be GAS. This can be seen by elementary algebraic considerations, but also through the previous theory, in the case where we choose K such that the origin of the second equation in (31) is IS.

For the nonlinear case, there are circumstances where one can built Luenberger like observers. Let us consider the problem of stabilization of the control system

$$\dot{x} = \phi(x, u)$$

under some feedback law $u = R(x)$. Assume that we have designed some R such that system

$$\dot{x} = \phi(x, R(x))$$

has the origin as a GAS equilibrium. Assume now that the state vector x is not accessible to measurement, which means that only a certain function $\xi = \varphi(x)$ of the state is available. Is it possible to recover the state $x(t)$ from $\xi(t)$? The answer is yes, to some extent. Namely, under some assumptions that we do not detail here, there is a system of the form

$$\dot{\hat{x}} = \vartheta(\hat{x}, \varphi(x))$$

such that the error $y = \hat{x} - x$ tends to zero as $t \to +\infty$. Unlike in the linear case (31), the differential equation of the error does contain the variable x. Actually one has

$$\dot{x} = \phi(x, R(\hat{x})), \qquad \dot{\hat{x}} = \vartheta(\hat{x}, \varphi(x)),$$

which can be rewritten in the variables x and y as

$$\begin{aligned} \dot{x} &= \phi(x, R(x+y)) = f(x,y) \\ \dot{y} &= \vartheta(x+y, \varphi(x)) - \phi(x, R(x+y)) = g(x,y) \end{aligned} \tag{32}$$

Condition H1 is satisfied. If the origin of the second equation in (32) is IS in some sense, we may hope that the origin of (32) is PSGAS. This case is not covered by the results of this paper and thus deserves further attention.

A A short tutorial on Nonstandard Analysis

Here, we begin with an elementary approach to NSA. We follow Callot [3] and Lutz [22]. The first object to look at it is the set of natural numbers 0, 1, 2, $\cdots$, n, $n+1$, $\cdots$ and to decide that some of them are *infinitely large* (or simply *large*, or *unlimited*) with the following rules:

1. 0 is not large (it is limited),
2. if p is not large so is $p+1$,
3. if p is not large and $q \leq p$ then q is not large,
4. there exists n such that n is large.

An easy consequence of Rule 2 is the following: if n is large, so is $n-1$. The first objection to this set of rules is the induction principle which states that a property of integers which satisfies rules 1 and 2 must be true for all integers. Thus from now we must accept to work in a mathematical world where the induction principle is not true for all the properties. Namely the induction principle is not true for all the properties related to orders of magnitude. In other words *the collection of not large integers is* *not a subset of the set of integers*. Actually, the reader is well trained to this way of thinking, where the induction principle does not hold: "There should be a finite chain linking some monkey to Darwin, repecting the rules: a monkey's son is a monkey, the father of a man is a man". The induction principle does not hold for the properties of "being a monkey's son" or "being the father of a man" and actually there is some fuzzy border between the two properties.

The question is whether there exists a mathematical formalism that recognizes this fuzzy aspect of informal langage. At the first glance the answer seems to be no, so much we have in mind that everything in mathematics has to be precise. It turns out that Robinson [28] proved the following

- There exists a mathematical formalism called Nonstandard Analysis (NSA), in which the adjective "large" for a natural number with the rules 1, 2, 3 and 4 makes perfect sense.

- The new formalism contains no more contradiction than the old one and everything which was true for the old formalism is still true when expressed in the new one.

The consequences of this are very important. It says that we can use freely all the previous mathematics and give statements in the new langage of classical statements, give new proofs, maybe prove new theorems in the new langage. We hope that NSA statements of classical results will appear more illuminating.

Let us describe briefly *Internal Set Theory* (IST) which is an axiomatic approach, given by Nelson [25], of NSA. In IST we adjoin to ordinary mathematics (say ZFC) a new undefined unary predicate *standard* (st). The axioms of IST are the usual axioms of ZFC plus three others which govern the use of the new predicate. Hence, *all theorems of* ZFC *remain valid*. What is new in IST is an addition, not a change. We call a formula of IST *external* in the case where it involves the *new* predicate "st"; otherwise, we call it *internal*. Thus, internal formulas are the formulas of ZFC. The theory IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC.

Roughly speaking, standard objects of IST are those objects of classical mathematics as the integers 0, 1, 100 ..., the real numbers π, e ..., the functions $\sin x$, $\ln x$ ... The limited (resp. large) integers of the elementary approach to NSA in the beginning of this section are simply the standard (resp. nonstandard) integers of IST.

Definition 9 *A real number ω is said to be infinitely large (or simply large, or unlimited) if $|\omega|$ is greater than some large integer. A real number ε is said to be infinitesimal (or simply small) if $\varepsilon = 0$ or $1/\varepsilon$ is large. A real number which is not large is said to be limited. A limited number which is not infinitesimal is said to be appreciable. A real number y is said to be infinitely close (or simply close) to x, which is denoted by $y \simeq x$, if $|y - x|$ is infinitesimal.*

Thus a real number x is *infinitesimal* when $|x| < a$ for all standard $a > 0$, *limited* when $|x| \leq a$ for some standard a, and *unlimited*, when it is not limited. We use the following notations : $x \simeq 0$ for x infinitesimal, $x \simeq +\infty$ for x unlimited positive, $x \gg 0$ for x non infinitesimal positive. Thus we have

$$\begin{array}{rcl} x \simeq 0 & \Leftrightarrow & \forall^{\mathrm{st}} a > 0 \; |x| < a \\ x \gg 0 & \Leftrightarrow & \exists^{\mathrm{st}} a > 0 \; x \geq a \\ x \text{ limited} & \Leftrightarrow & \exists^{\mathrm{st}} a \; |x| \leq a \\ x \simeq +\infty & \Leftrightarrow & \forall^{\mathrm{st}} a \; x > a \end{array} \tag{33}$$

The orders of magnitude and proximity introduced in Defintion 9 extend to vectors as follows

Definition 10 *An element $y \in \mathbb{R}^d$, where d is limited, is said to be limited if $\|y\|$ is limited. It is said to be infinitely close (or simply close) to $x \in \mathbb{R}^d$, which is denoted by $y \simeq x$, if $\|y - x\|$ is infinitesimal.*

In this definition one takes for $\|x\|$ the Euclidean norm or

$$\|x\| = \max_{1 \leq i \leq d} |x_i|, \quad \text{or} \;\; \|x\| = \sum_{i=1}^{d} |x_i|.$$

Since the dimension d is assumed to be limited, the notions of *limited* or *close* are the same for the three norms.

We may not use external formulas to define subsets. The notations $\{x \in \mathbb{R} : x \text{ is limited}\}$ or $\{x \in \mathbb{R} : x \simeq 0\}$ are not allowed. Moreover, we can prove that there do not exist subsets L and I of $\mathbb{R}$ such that, for all x in $\mathbb{R}$, x is in L if and only if x is limited, or x is in I if and only if x is infinitesimal.

B Some concepts of calculus revisited in NSA

Let us look to few concepts of calculus revisited in NSA.

a) The sequence u_n tends to l when n goes to infinity:

Classical	$\forall \varepsilon > 0 \; \exists N$	$\forall n \; (n \geq N \Rightarrow |u_n - l| \leq \varepsilon)$.
NSA		$\forall n \; (n \text{ large} \Rightarrow u_n \simeq l)$.

The NSA characterization is read: if n is large then u_n is close to l, which is very close to the intuitive idea of limit we want to formalize.

These concepts are equivalent only for standard sequences. Notice that the sequence $u_n = (-1)^n \varepsilon$ with ε infinitesimal, tends to 0 in NSA but not in the classical sense. Conversely the sequence $u_n = 1/(n\varepsilon)$, with $\varepsilon \neq 0$ infinitesimal, tends to 0 in the classical sense but not in NSA. Of course, these examples for which the classical and NSA concepts are not equivalent, are not standard, since their definition make use of the new concept *infinitesimal*.

b) The function f is continuous at x_0:

Classical $\quad \forall\varepsilon > 0\ \exists\eta > 0 \quad \forall x\ (|x - x_0| \leq \eta \Rightarrow |f(x) - f(x_0)| \leq \varepsilon).$

NSA $\quad \forall x\ (x \simeq x_0 \Rightarrow f(x) \simeq f(x_0)).$

The NSA characterization is read: if x is close to x_0 then $f(x)$ is close to $f(x_0)$. Again we must emphasize on the fact that the classical and NSA concepts are equivalent only for standard functions. Notice that the function $f(x) = O$ if $x \leq 0$, $f(x) = \varepsilon$ with $\varepsilon \neq 0$ infinitesimal if $x > 0$, is continuous at 0 in NSA but not in the classical sense. Conversely the function $f(x) = x/\varepsilon$, with $\varepsilon \neq 0$ infinitesimal, is continuous at 0 in the classical sense but not in NSA. With the aid of the concepts and notations introduced in Definition 10 the discussion of continuity for functions of several variable, within NSA, is exactly the same as above.

c) Let $\dot{x} = f(x)$ be a system of differential equations on $\mathbb{R}^d$, with d standard. Let $x(t, x_0)$ be the solution of initial condition $x(0) = x_0$. Assume that $f(\xi) = 0$, that is $x = \xi$ is an equilibrium. This equilibrium is stable (in the sense of Lyapunov) if:

Classical $\quad \forall\varepsilon > 0\ \exists\eta > 0 \quad \forall x_0\ (\|x_0 - \xi\| \leq \eta \Rightarrow \forall t \geq 0\ \|x(t, x_0) - \xi\| \leq \varepsilon).$

NSA $\quad \forall x_0\ (x_0 \simeq \xi \Rightarrow \forall t \geq 0\ x(t, x_0) \simeq \xi)$

The NSA characterization is read: if x_0 is close to ξ, so is $x(t, x_0)$, for every $t \geq 0$.

Notice that the equilibrium $\xi = (0, 0)$ of the differential system

$$\left\{ \begin{array}{rcl} \dot{x} & = & y + (\varepsilon - x^2 - y^2)x \\ \dot{y} & = & -x + (\varepsilon - x^2 - y^2)y \end{array} \right.$$

with ε infinitesimal positive, is stable in the sense of NSA but not in the classical sense. Indeed, the eigenvalues $\lambda = \varepsilon \pm \mathrm{i}$ of the linear part are of positive real part, so the equilibrium is unstable. However, the circle of infinitesimal radius $x^2 + y^2 = \varepsilon$ is an attracting limit cycle, so any solution starting close to ξ remains close to it for every $t \geq 0$. Conversely the equilibrium $\xi = (0, 0)$ for the differential system

$$\left\{ \begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -\varepsilon^2 x \end{array} \right.$$

with ε infinitesimal, is stable in the classical sense but not in NSA. Indeed, since $y^2 + \varepsilon^2 x^2$ is constant, this equilibrium is a center, so is stable in the classical sense. But the solution $x(t) = \sin(\varepsilon t)$, $y(t) = \varepsilon\cos(\varepsilon t)$, with initial condition $(0, \varepsilon) \simeq \xi$, is not close to ξ for $t = \pi/(2\varepsilon)$.

d) Let us discuss now *global asymptotic stability* of equilibria. The equilibrium ξ is globally asymptotically stable (GAS) if:

Classical	It is stable and $\forall x_0\ \lim_{t\to+\infty} x(t,x_0)=\xi$.
NSA	If x_0 is limited and $t\simeq+\infty$ then $x(t,x_0)\simeq\xi$.

Notice that the equilibrium $\xi = 0$ of the system differential $\dot{x} = x(\varepsilon - x^2)$ with ε infinitesimal positive, is GAS in the sense of NSA but not in the classical sense. Conversely the equilibrium $\xi = 0$ for the differential system $\dot{x} = -\varepsilon x$ with $\varepsilon > 0$ infinitesimal, is GAS in the classical sense but not in NSA. Indeed, $x(t,x_0) = x_0\exp(-\varepsilon t)$ is the solution. Thus $x(1/\sqrt{\varepsilon}, x_0) = x_0\exp(-\sqrt{\varepsilon})$ is not infinitesimal if x_0 is appreciable.

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