STROBOSCOPY AND AVERAGING

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Dedicated to the memory of Jean Louis Callot and Georges Reeb

Abstract. The aim of this paper is to give a presentation of the method of stroboscopy and to apply it in several problems in the perturbation theory of differential equations and in numerical analysis. We give a proof based on stroboscopy of the KBM theorem of averaging.

Contents

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1. Introduction

The method of stroboscopy was proposed in 1977 by J. L. Callot and its writer G. Reeb (see [R,CS,S7]) for the study of the differential equation $x' = \sin(tx/\varepsilon)$ where ε is small (see Section 5.1 for the details). It was presented by Reeb at the IV International Colloquium on Differential Geometry in Santiago de Compostella (1978). The guidind principle of this method is as follows. Let $\phi : I \to \mathbb{R}^d$, $I \subset \mathbb{R}$ be a function which take limited values. Suppose there exists a sequence of points $(t_n, x_n = \phi(t_n))$ in $I \times \mathbb{R}^d$ satisfying

$$0 < t_{n+1} - t_n \simeq 0 \quad \text{and} \quad \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \simeq f(t_n, x_n)$$

where f is a standard continuous function. Then the function ϕ is infinitely close, to a solution $x(t)$ of the differential equation $x' = f(t, x)$ whenever t and $x(t)$ are both limited.

Usually the function ϕ under consideration is not wellknown (for example it is defined to be a solution of some differential equation) and it is difficult to verify the required conditions, i.e. that it takes only limited values on some interval I, or to define explicitely the sequence of points (t_n, x_n). For these reasons, I extended the method of stroboscopy (see [S2]) to functions $\phi : I \to \mathbb{R}^d$, $I \subset \mathbb{R}$ such that for each t in I with both t and $\phi(t)$ limited, there is t' in I such that

$$0 < t' - t \simeq 0 \quad \text{and} \quad \frac{\phi(t') - \phi(t)}{t' - t} \simeq f(t, \phi(t)).$$

We call t and t' *successive instants of observation of the method of stroboscopy.* Here the sequence of points (t_n, x_n) is given by some kind of external recurrence relation (see Sections 4.2. and 4.3.). Having in mide some applications, R. Lutz [Lr] extended this result and proposed a *selective stroboscopy* where the existence of the instant of observation t' of the method of stroboscopy is assumed only for those t which lie in some subset $E \subset I$. Taking in account all these ameliorations of the original idea of stroboscopy I gave [S5] a "Stroboscopy Lemma" which is presented in Section 2. In Section 3 a proof, based on stroboscopy, of the so-called "Short Shadow Lemma" is given ; we give also a standard version of this foundamental tool in Nonstandard Asymptotic Theory. Some semicontinuity properties of the orbits of a dynamical system are obtained as a consequence of the Short Shadow Lemma. In Section 4 the classical theorem of averaging of Krylov, Bogolioubov and Mitropolski (KBM theorem) is proposed also as a consequence of stroboscopy ; in this version of the KBM theorem the usually required conditions on the differential equation are weakened. In Section 5, I give some application of stroboscopy to various problems in the perturbation theory of differential equations and in numerical analysis : the adiabatic invariants, the numerical instability and ghost solutions, the error propagation in numerical schemes. In some of these problems, I prefer to use directly the stroboscopy method, rather than to deduce the answer from some general result : my hope is to convince you that the method of stroboscopy is an important tool in the study of differential equations and is valuable to solve problems. Indeed, the method of stroboscopy was appreciated by several authors in various problems (see [S7] and its bibliographical comments).

2. Stroboscopy

In this section U is a standard open subset of $\mathbb{R}^d$, $f : U \to \mathbb{R}^d$ is a standard continuous function and $\phi : I \to \mathbb{R}^d$ is a function such that $0 \in I \subset \mathbb{R}$ and $\phi(0)$ is nearstandard[1] in U.

Definition (Stroboscopic property). Let t and t' be in I. The function ϕ is said to satisfy the *stroboscopic property* $\mathcal{S}(t,t')$ if $[t,t'] \subset I$, $t' \simeq t$, $\phi(s) \simeq \phi(t)$ for all s in $[t,t']$ and

$$\frac{\phi(t') - \phi(t)}{t' - t} \simeq f(\phi(t))$$

The Stroboscopy Lemma asserts that under suitable conditions the function ϕ is approximated by a solution of the initial value problem

$$x' = f(x) \qquad x(0) = {}^o(\phi(0)). \tag{1}$$

2.1. Stroboscopy Lemma.

Theorem 1 (Stroboscopy Lemma). *Suppose that*

(i) there is $\mu > 0$ such that whenever $t \in I$ is limited and $\phi(t) \in U_{NS}$ there is $t' \in I$ such that $t' - t > \mu$ and the function ϕ satisfies the stroboscopic property $\mathcal{S}(t,t')$.

(ii) the initial value problem (1) has a unique solution $x(t)$. Let $J = [0,\omega)$, $0 < \omega \leq +\infty$, be its maximal positive interval of definition.

Then the function $\phi(t)$ is defined for any $t \in J_{NS}$ and satisfies $\phi(t) \simeq x(t)$

Proof. The proof needs the Lemma 3 below (the proof of this Lemma needs some preliminary results and is postponed to the next section). Let b be standard in $(0,\omega)$. Since $x([0,b])$ is a standard compact subset of U, there is a standard $k > 0$, such that the neighborhood $\mathcal{N}$ of $x([0,b])$ defined by $\mathcal{N} = \{x \in \mathbb{R}^d : \exists s \in [0,b] \ \ \|x - x(s)\| \leq k\}$ is included in U. Define the set

$$A = \{r \in [0,b] : [0,r] \subset I \quad \text{and} \quad \forall s \in [0,r] \ \ \|\phi(s) - x(s)\| \leq k\}$$

The set A is non empty (since $0 \in A$) and bounded above by b. Let r_0 be the lower upper bound of A. There is $r \in A$ such that $r_0 - \mu < r \leq r_0$. Thus for $s \in [0,r]$ we have $\phi(s) \in \mathcal{N}$. Hence on $[0,r]$ the function ϕ is nearstandard in U. By Lemma 3 we have $\phi(s) \simeq x(s)$ for $s \in [0,r]$. By condition (i), there is $r' > r + \mu$ such that $r' \in E$, $[r,r'] \subset I$ and $\phi(s) \simeq x(s)$ for $s \in [r,r']$. Hence $[0,r'] \subset I$ and $\|\phi(s) - x(s)\| \leq k$ for $s \in [0,r']$. Suppose $r' \leq b$, then $r' \in A$ which is a contradiction with $r' > r_0 = \sup A$. Thus $r' > b$, that is for each standard $b \in J$ we have $\phi(s) \simeq x(s)$ for all $s \in [0,b]$. □

Remarks. 1. Condition (i) is a local property of the function ϕ. From this local property, the Stroboscopy Lemma gives a global estimate on ϕ and its domain of definition I. Indeed the function ϕ is defined at least on any interval $[0,b]$ with b

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[1] A point $x \in U$ is said to be *nearstandard in U* if there is a standard $x_0 \in U$ such that $x \simeq x_0$, that is ${}^o x \in U$, where ${}^o x$ is the shadow of x. We abbreviate this as $x \in U_{NS}$, where $U_{NS} = \{x \in U : \exists^{st} x_0 \in U \ \ x \simeq x_0\}$ is the *external-set* of nearstandard points in U.

standard in J and satisfies $\phi(t) \simeq x(t)$ for all $t \in [0, b]$. By permanence there is an interval $J_0 = [0, \omega_0] \subset J$ such that $\omega_0 \simeq \omega$, $J_0 \subset I$ and $\phi(t) \simeq x(t)$ for all t in J_0.

2. Condition (i) may be weakened and replaced by

(i') there are $\mu > 0$ and $E \subset I$, with $0 \in E$, such that whenever $t \in E$ is limited and $\phi(t) \in U_{NS}$ there is $t' \in E$ such that $t' - t > \mu$ and the function ϕ satisfies the stroboscopic property $\mathcal{S}(t, t')$.

The proof is almost the same (see [S5] for the details). This is a slightly simplified version of the selective stroboscopy [Lr].

2.2. Preliminary lemmas.

Lemma 1. *Let $\{(t_n, x_n) : n = 0, ..., N+1\}$ be a sequence of points in $\mathbb{R} \times U$. Suppose*

$i)$ there is a standard $b > 0$ such that $0 = t_0 < t_1 < ... < t_N \leq b < t_{N+1}$, and $t_{n+1} \simeq t_n$ for all $n = 0, ..., N$,

$ii)$ for each $n \in \{0, ..., N\}$ $x_n \in U_{NS}$ and there exists $\eta_n \simeq 0$ such that $x_{n+1} \simeq x_n + (t_{n+1} - t_n)[f(x_n) + \eta_n]$.

Then the standard function $x : [0, b] \to U$ which, for t standard in $[0, b]$, is defined by $x(t) := {}^o x_n$ where n is such that $t_n \leq t < t_{n+1}$, is a solution of the initial value problem (1), and satisfies $x_n \simeq x(t_n)$ for all $n \in \{0, ..., N\}$.

Proof. Let $\eta = \max\{\eta_0, ..., \eta_N\}$ and $m = \max\{\|f(x_0)\|, ..., \|f(x_N)\|\}$. Then we have $\eta \simeq 0$ and $m = \|f(x_p)\|$ for some $p \in \{0, ..., N\}$. Since f is standard and continuous in U, and ${}^o x_p \in U$, $f({}^o x_p)$ is standard and $m \simeq \|f({}^o x_p)\|$. Hence $M = m + 1$ is limited. For $n > p$ we have

$$\|x_n - x_p\| = \|\sum_{k=p}^{n-1}(x_{k+1} - x_k)\| = \|\sum_{k=p}^{n-1}(t_{k+1} - t_k)(f(x_k) + \eta_k)\|$$
$$< \sum_{k=p}^{n-1} M(t_{k+1} - t_k) = M|t_n - t_p|.$$

Hence $t_n \simeq t_p$ implies $x_n \simeq x_p$. Thus the function $x(t)$ is continuous on $[0, b]$ and satisfies $x_n \simeq x(t_n)$ for all $n \in \{0, ..., N\}$. Since f is standard and continuous, for $n \in \{0, ..., N\}$ we have $f(x_n) = f(x(t_n)) + \alpha_n$ with $\alpha_n \simeq 0$. Let t be standard in $[a, b]$. There is $n \in \{0, ..., N\}$ such that $t_n \leq t < t_{n+1}$ and we have

$$x(t) - x(0) \simeq x_n - x_0 = \sum_{p=0}^{n-1}(t_{p+1} - t_p)[f(x_p) + \eta_p]$$
$$= \sum_{p=0}^{n-1}(t_{p+1} - t_p)[f(x(t_p)) + \alpha_p + \eta_p] \simeq \int_0^t f(x(s))ds.$$

Therefore $x(t) - x(0) = \int_0^t f(x(s))ds$ for all standard t and consequently for all $t \in [0, b]$, that is $x(t)$ is a solution of problem (1). $\square$

Lemma 2. *Suppose there is a standard $b > 0$ such that $[0, b] \subset I$ and $\phi(t) \in U_{NS}$ for all t in $[0, b]$. Suppose there is $\mu > 0$ such that for all $t \in [0, b]$ there is $t' \in I$ $t' - t > \mu$ and the function ϕ satisfies the stroboscopic property $\mathcal{S}(t, t')$. Then the function ϕ is S-continuous on $[0, b]$ and its shadow is a solution $x(t)$ of the initial value problem (1) defined on $[0, b]$, and satisfies $\phi(t) \simeq x(t)$ for all $t \in [0, b]$.*

Proof. Define the set $A = \{\lambda \in \mathbb{R} : \forall t \in [0, b] \cap E\ \exists t' \in E\ \mathcal{S}(t, t', \lambda)\}$ where $\mathcal{S}(t, t', \lambda)$ is the property :

$$[t, t'] \subset I \qquad \mu < t' - t < \lambda \qquad \forall s \in [t, t']\ \ \|\phi(s) - \phi(t)\| < \lambda$$

$$\text{and } \ \|\frac{\phi(t') - \phi(t)}{t' - t} - f(\phi(t))\| < \lambda.$$

The set A contains all the standard real numbers $\lambda > 0$. By permanence, there is $\varepsilon \simeq 0$ in A, that is there is $0 < \varepsilon \simeq 0$ such that for all $t \in [0, b]$ there is $t' \in E$ such that $P(t, t', \varepsilon$ holds. By the axiom of choice there exits a function $c : [0, b] \cap E \to E$ such that $t' = c(t)$, that is $P(t, c(t), \varepsilon)$ holds for all $t \in [0, b] \cap E$. Since $c(t) - t > \mu$, there are $N > 0$ and $\{t_n : n = 0, ..., N + 1\}$ such that $t_0 = 0$, $t_N \leq b < t_{N+1}$ and $t_{n+1} = c(t_n)$. Define $\{x_n : n = 0, ..., N + 1\}$ by $x_n = \phi(t_n)$. By Lemma 1 the shadow of the sequence (t_n, x_n) is a solution $x(t)$ of the differential equation $x' = f(x)$, defined on $[0, b]$. Since $\phi(t_n) \simeq x(t_n)$ for all $n = 0, ..., N$ and $\phi(s) \simeq \phi(t_n)$ on $[t_n, t_{n+1}]$, we have $\phi(t) \simeq x(t)$ for all $t \in [0, b]$. □

Lemma 3. *Suppose the real b in Lemma 2 is assumed to be limited (not necessarly standard). Suppose morover that the initial value problem (1) has a unique maximal solution $x(t)$. Then the function $x(t)$ is defined at least on $[0, b]$ and we have $\phi(t) \simeq x(t)$ for all $t \in [0, b]$.*

Proof. If $b \simeq 0$ there is nothing to prove. Assume b is not infinitesimal and let a be standard such that $0 < a < b$. By Lemma 2 the function $x(t)$ is defined on $[0, a]$ and $\phi(t) \simeq x(t)$ for all $t \in [0, a]$. By permanence that property holds for some $a \simeq b$. Since the standard function $x(t)$ is defined and limited in a, it is defined at least on $[0, b]$. Since $x(t) \simeq x(a)$ and $\phi(t) \simeq \phi(a)$ for all $t \in [a, b]$, we have $\phi(t) \simeq x(t)$ for all $t \in [0, b]$. □

3. Regular perturbations

Let $U \subset \mathbb{R}^d$ be the an open set. The fondamental problem of the regular perturbation theory is to compare the solutions of the initial values problems

$$x' = f_0(x) \qquad x(0) = a_0 \in U, \tag{2}$$

$$x' = f(x) \qquad x(0) = a \in U, \tag{3}$$

when $f : U \to \mathbb{R}^d$ is close to $f_0 : U \to \mathbb{R}^d$ and the initial condition a is close to a_0. The *Short Shadow Lemma* (Section 3.2.) gives an answer for this problem when $a \simeq a_0$, f_0 is standard and $f(x) \simeq f_0(x)$ for all $x \in U_{NS}$. Since the considered systems are n-dimensionnal, this approach include also the case of nonautonomous systems. We give here a proof of the Short Shadow Lemma based on the theorem of stroboscopy. There exists also another version of the Short Shadow Lemma with a direct proof (see [6] page 137). We first need a preliminary result (Section 3.1.) which compare the solutions of the differential equations $x' = f_0(t)$ and $x' = f(t,x)$ with initial condition $x(0) = x_0$ when f is close to f_0. It is not necessary that the vector field f should be limited. We require simply that it is nearly independent of the position x : such a vector field is said to be *nearly constant*.

3.1. Nearly constant vector fields.

Theorem 2. *Let D be a subset of $\mathbb{R}^d$ containing all the limited points of $\mathbb{R}^d$ and let $f_0 : \mathbb{R}^+ \to \mathbb{R}^d$ and $f : \mathbb{R}^+ \times D \to \mathbb{R}^d$ be continuous functions. Suppose that*

i) $\int_0^t f_0(s)ds$ is limited for all limited $t > 0$,

ii) $f_0(t) \simeq f(t,x)$ for all limited $t > 0$ and all limited $x \in D$.

Let x_0 be limited in D. Then any solution $x(t)$ of the initial value problem $x' = f(t,x)$, $x(0) = x_0$ is defined and limited for all limited $t > 0$ and satisfies

$$x(t) \simeq x_0 + \int_0^t f_0(s)ds.$$

Proof. By permanence there exists $\nu \simeq \infty$ such that the property $f_0(t) \simeq f(t,x)$ holds for $t \in [0,\nu]$, and $x \in B$ where $B \subset D$ and B is the ball of center 0 and radius ν. Suppose there is a limited $\bar{t} > 0$ such that $x(\bar{t}) \simeq +\infty$. Let $t > 0$ be such that $t \leq \bar{t}$, $x(t) \simeq +\infty$ and $x(s) \in B$ for all $s \in [0,t]$. Then we have

$$x(t) = x_0 + \int_0^t f(s,x(s))ds \simeq x_0 + \int_0^t f_0(s)ds.$$

Hence $x(t)$ is limited ; this is a contradiction. Therefore $x(t)$ is defined and limited for all limited $t > 0$. $\square$

3.2. Short Shadow Lemma.

Theorem 3 (Short Shadow Lemma). *Let U be a standard open subset of $\mathbb{R}^d$ and let $f_0 : U \to \mathbb{R}^d$ be standard and continuous. Let $a \in U$ be standard. Suppose the initial value problem (2) has a unique solution ϕ_0 and let $J = [0,\omega)$, $0 < \omega \leq +\infty$ be its maximal positive interval of definition. Let $f : U \to \mathbb{R}^d$ be continuous such that $f(x) \simeq f_0(x)$ for all $x \in U_{NS}$. Then every solution ϕ of the initial value problem (3) with $a \simeq a_0$, is defined for all $t \in J_{NS}$ and satisfies $\phi(t) \simeq \phi_0(t)$.*

Proof. Let $\phi : I \to U$ be a maximal solution of $x' = f(x)$ such that $\phi(0) \simeq a_0$. Let $\varepsilon > 0$ be infinitesimal, and let $t_0 \in I$ be limited such that $x_0 = \phi(t_0) \in U_{NS}$. The successive instant of observation of the method of stroboscopy is choosen as follows. Under the change of variables

$$T = \frac{t - t_0}{\varepsilon} \qquad X(T) = \frac{x(t_0 + \varepsilon T) - x_0}{\varepsilon}$$

the system (3) becomes

$$\frac{dX}{dT} = f(x_0 + \varepsilon X). \tag{4}$$

Since $x_0 \in U_{NS}$, the function $X \to f(x_0 + \varepsilon X)$ is defined for all limited points $X \in \mathbb{R}^d$. Since f_0 is continuous on U and $f(x) \simeq f_0(x)$ for all $x \in U_{NS}$, we have $f(x_0 + \varepsilon X) \simeq f(x_0 + \varepsilon X) \simeq f_0(x_0)$ for all limited X. By Theorem 2 we have $X(T) \simeq T f_0(x_0)$ for all limited T. Define t_1 by $t_1 = t_0 + \varepsilon$, then we have

$$\frac{\phi(t_1) - \phi(t_0)}{t_1 - t_0} = X(1) \simeq f_0(x_0)$$

and $\phi(t) - \phi(t_0) = \varepsilon X(T) \simeq 0$ for $t \in [t_0, t_1]$. So there is $\mu = \varepsilon/2$ such that whenever $t_0 \in I$ is limited and $\phi(t_0) \in U_{NS}$ there is $t_1 \in I$ such that ϕ satisfies the stroboscopic property $\mathcal{S}(t_0, t_1)$. By the Stroboscopy Lemma ϕ is defined for all $t \in J_{NS}$ and satisfies $\phi(t) \simeq \phi_0(t)$. □

Remarks. 1. In practice the function f is not needed to be defined on U. The same proof works if $f : H \to \mathbb{R}^d$ where H is a subset of $\mathbb{R}^d$ containing all the nearstandard points in U. The proof is the same. The only point we have to verify is that the system (4) is defined for all limted $X \in \mathbb{R}^d$. This is true since $x_0 \in U_{NS} \subset H$.

2. By permanence we find an interval $J_0 = [0, \omega_0] \subset J$ such that $\omega_0 \simeq \omega$, ϕ is defined on J_0 and $\phi(t) \simeq \phi_0(t)$ for all $t \in J_0$.

3. In some applications it is useful to compare the solutions of two differential equations $x' = f_1(x)$ and $x' = f_2(x)$ where $f_1 : H_1 \to \mathbb{R}^d$ and $f_2 : H_2 \to \mathbb{R}^d$ are continuous functions, none being standard. The Short Shadow Lemma permits this comparison whenever there is a standard continuous function $f_0 : U \to \mathbb{R}^d$ having the unicity of the solutions of problem (2) and such that H_1 and H_2 contain all the nearstandard points in U and $f_1(x) \simeq f_0(x)$ and $f_2(x) \simeq f_0(x)$ for all $x \in U_{NS}$. Indeed let $a \in U$ be standard and let ϕ_0 be the maximal solution of the initial value problem (2). Let $J = [0, \omega)$, $0 < \omega \leq +\infty$ be its positive interval of definition. Then for all solutions ϕ_1, ϕ_2 of the differential equations $x' = f_1(x)$ and $x' = f_2(x)$ with initial conditions $\phi_1(0) \simeq a_0$ and $\phi_2(0) \simeq a_0$ there is an interval $J_0 = [0, \omega_0] \subset J$ such that $\omega_0 \simeq \omega$, and such that ϕ_1 and ϕ_2 are defined on J_0 and satisfy $\phi_1(t) \simeq \phi_2(t) \simeq \phi_0(t)$ for all $t \in J_0$.

3.3. Lower semicontinuity properties of orbits. We recall some facts on general topology and lower semicontinuity [S8]. Let $U \subset \mathbb{R}^d$ be an open set. Let $\mathcal{C} = \mathcal{C}(U, \mathbb{R}^d)$ be the set of continuous functions $f : U \to \mathbb{R}^d$. Let $\mathcal{K}$ be the set of compact subsets of U. The topology on $\mathcal{C}$ defined by the formula

$$\forall^{st} f_0 \in \mathcal{C} \;\; \forall f \in \mathcal{C} \;\; (f \simeq f_0 \Leftrightarrow \forall^{st} K \in \mathcal{K} \;\; \forall x \in K \;\; f(x) \simeq f_0(x))$$

is nothing than the *topology of uniform convergence on compacta*. Since $\mathbb{R}^d$ is a locally compact space, to say $f \simeq f_0$ is the same as saying $\forall x \in U_{NS} \;\; f(x) \simeq f_0(x)$.

Let $\Gamma : X \to \mathcal{P}(Y)$ be a standard set-valued mapping of the standard topological space X into the power set $\mathcal{P}(Y)$ of the standard topological space Y. Let x be standard in X.The mapping Γ is said to be *lower semicontinuous* at x if :

$$\forall y \in X (y \simeq x \Rightarrow \Gamma(x) \subset {}^o\Gamma(y))$$

where ${}^o\Gamma(y) = {}^S\{u \in Y : \exists v \in \Gamma(y) \;\; v \simeq u\}$ is the shadow of the subset $\Gamma(y) \subset Y$. This is the same as the usual definition.

Let $\mathcal{X}(U)$ be the set of continuous vector fields $f : U \to \mathbb{R}^d$ on the open set $U \subset \mathbb{R}^d$, such that, for any $a \in U$ the solution $\pi_f(t, a)$ of the differential equation $x'(t) = f(x(t))$, with the initial condition $\pi_f(0, a) = a$ is unique. Let $I_f(a) = (\alpha_f(a), \omega_f(a))$, $-\infty \leq \alpha_f(a) < \omega_f(a) \leq \infty$, be its maximal interval of definition. Let $\gamma_f(a) = \{\pi_f(t, a) \in U : t \in I_f(a)\}$ be the corresponding orbit through a. We have the following classical semicontinuity properties ([L] page 28) : the mappings $I_f : U \to \mathcal{P}(\mathbb{R})$ and $\gamma_f : U \to \mathcal{P}(U)$ are lower semicontinuous. This results are immediate consequences of the Short Shadow Lemma. In fact this lemma has a more general consequence : let I (resp. γ) be the mapping of $\mathcal{X}(U) \times U$ into $\mathcal{P}(\mathbb{R})$ (resp. $\mathcal{P}(U)$) which carries (f, a) into $I_f(a)$ (resp. $\gamma_f(a)$).

Theorem 4. *The mappings $I : \mathcal{X}(u) \times U \to \mathcal{P}(\mathbb{R})$ and $\gamma : \mathcal{X}(U) \times U \to \mathcal{P}(U)$ are lowercontinuous, when $\mathcal{X}(U)$ is endowed with the topology of uniform convergence on compacta.*

Proof. Assume that f and a are standard then, by the Short Shadow Lemma for every $b \simeq a$, every $g \simeq f$ and every standard $t \in I_f(a)$, we have $t \in I_g(b)$ and $\pi_g(t, b) \simeq \pi_f(t, a)$. Hence $I_f(a) \subset {}^S I_g(b) \subset {}^o I_g(b)$ and $\gamma_f(a) \subset {}^o\gamma_g(b)$. □

3.4. A standard version of the Short Shadow Lemma. Let us give a standard translate of Theorem 3. The set $\mathcal{C} = \mathcal{C}(U, \mathbb{R}^d)$ of continuous vector fields $f : U \to \mathbb{R}^d$ is endowed with the topology of uniform convergence on compacta.

Theorem 3 bis. *Let U be an open subset of $\mathbb{R}^d$. Let $a_0 \in U$ and let $f_0 \in \mathcal{C}$ having the unicity of the solution of problem (2). Let $\phi_0 : J \to U$ be its maximal solution. Then for all $\delta > 0$ and all $l \in J$, there are $\eta > 0$ and a neighborhood $\mathcal{V}$ of f_0 such that for all $a \in U$ and all $f \in \mathcal{C}$, any maximal solution $\phi(t)$ of problem (3) is defined at least on $[0, l]$ and satisfies $\|\phi(t) - \phi_0(t)\| < \delta$ for all $t \in [0, l]$, whenever $\|a - a_0\| < \eta$ and $f \in \mathcal{V}$.*

The proof follows the usual way to translate external notions and is left to the reader. Notice that the results presented in the Remarks of Section 3.2, in particular in the case where the perturbed vector field f is not assumed to be defined on the same set U as the unperturbed vector field f_0, would have very cumbersome translates.

4. Averaging

The fundamental problem is the study of the initial value problem

$$\frac{dx}{d\tau} = \varepsilon F(\tau, x) \qquad x(0) = a \tag{5}$$

when ε is small. The aim of the method of averaging is to approximate the solutions of problem (5), for times τ of order $1/\varepsilon$, by the solutions of the averaged system

$$\frac{dx}{d\tau} = \varepsilon f(x) \qquad x(0) = a_0 \tag{6}$$

where $a \simeq a_0$ and where f is an average of F with respect to the variable τ. Such an average exists for periodic vector fields (Section 4.1.), for almost periodic vector fields (Section 4.2.) and for the so-called KBM[2] vector fields (Section 4.3.). Since we look for the long time behaviour of the solutions, it is more suitable to consider the systems (5) and (6) at the sime scale $t = \varepsilon\tau$. Then we have

$$\frac{dx}{dt} = F(\frac{t}{\varepsilon}, x) \qquad x(0) = a \tag{7}$$

$$\frac{dx}{dt} = f(x) \qquad x(0) = a_0. \tag{8}$$

Suppose problem (8) has a unique solution $y(t)$. Let $J = [0, \omega)$, $0 < \omega \leq +\infty$ be its maximal positive interval of definition.

4.1. Periodic vector fields. Let $U \subset \mathbb{R}^d$ be an open subset. Let $F_0 : \mathbb{R}\times U \to \mathbb{R}^d$ be a standard continuous function which is 2π-periodic in the time variable. The function F_0 is continuous in the second variable uniformly with respect to the first variable. Indeed let $s \in \mathbb{R}$ and $k \in \mathbb{Z}$ be such that $s - k2\pi$ is limited, then for any standard $x_0 \in U$ and any $x \simeq x_0$ we have :

$$F_0(s, x) = F_0(s - 2k\pi, x) \simeq F_0(s - 2k\pi, x_0) = F_0(s, x_0).$$

Let $f : U \to \mathbb{R}^d$ be the average of F_0 over one period :

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} F_0(\tau, x)d\tau.$$

Theorem 5. *Let $F_0 : \mathbb{R} \times U \to \mathbb{R}^d$ be a standard 2π-periodic continuous function. Let a_0 be standard. Let H be a subset of $\mathbb{R}^d$ containing all the nearstandard points in U and let $F : \mathbb{R}^+ \times H \to \mathbb{R}^d$ be continuous and such that $F(s, x) \simeq F_0(s, x)$ for all $s > 0$ and all $x \in U_{NS}$. Let $\varepsilon > 0$ and $a \in U$ be such that $\varepsilon \simeq 0$ and $a \simeq a_0$. Then for every maximal solution x of problem (7) there is an interval $J_0 = [0, \omega_0] \subset J$ such that $\omega_0 \simeq \omega$, x is defined at least on J_0 and $x(t) \simeq y(t)$ for all $t \in J_0$.*

Proof. Let $x : I \to U$ be a maximal solution of problem (7). Let $t_0 \in I$ be limited, such that $x_0 = x(t_0) \in U_{NS}$. The successive instant of observation of the method of stroboscopy is choosen as follows. Under the change of variables :

$$T = \frac{t - t_0}{\varepsilon} \qquad X = \frac{x - x_0}{\varepsilon}$$

the system (7) becomes

$$\frac{dX}{dt} = F(\frac{t_0}{\varepsilon} + T, x_0 + \varepsilon X).$$

Let us denote by $s = t_0/\varepsilon$. Since $x_0 \in U_{NS}$, $x_0 + \varepsilon X \in U_{NS}$ for all limited X in $\mathbb{R}^d$. Then we have

$$F(s + T, x_0 + \varepsilon X) \simeq F_0(s + T, x_0 + \varepsilon X) \simeq F_0(s + T, x_0)$$

[2]KBM stands for Krylov Bogolioubov and Mitropolski

for all $T > 0$ and all limited X in $\mathbb{R}^d$. By Theorem 2, for all limited $T > 0$ we have

$$X(T) \simeq \int_0^T F(s+r, x_0)dr. \tag{9}$$

Define $t_1 = t_0 + \varepsilon 2\pi$. Then we have

$$\frac{x_1 - x_0}{t_1 - t_0} = \frac{X(2\pi)}{2\pi} \simeq \frac{1}{2\pi}\int_0^{2\pi} F_0(s+r, x_0)dr = f(x_0).$$

Since $x(t) - x(t_0) = \varepsilon X(T) \simeq 0$ for all $t \in [t_0, t_1]$, we have proved that there is $\mu = \varepsilon$ such that whenever $t_0 \in I$ is limited and $x(t_0) \in U_{NS}$ there is $t_1 \in I$ such that $t_1 - t_0 > \mu$ and the function x satisfies the stroboscopic property $\mathcal{S}(t_0, t_1)$. By the Stroboscopy Lemma there is an interval $J_0 = [0, \omega_0] \subset J$ such that $\omega_0 \simeq \omega$, x is defined at least on J_0 and $x(t) \simeq y(t)$ for all $t \in J_0$. □

4.2. Almost periodic vector fields. Let $U \subset \mathbb{R}^d$ be an open subset. Let $F_0 : \mathbb{R} \times U \to \mathbb{R}^d$ be a standard continuous function which is continuous in the second variable uniformly with respect to the first variable. Suppose F_0 has an average, that is there exists a limit

$$f(x) = \lim_{T\to\infty} \frac{1}{T}\int_s^{s+T} F_0(\tau, x)d\tau$$

which is uniform with respect to $s \in \mathbb{R}$. The almost periodic functions have such properties. The function f is standard and continuous and satisfies

$$f(x) \simeq \frac{1}{T}\int_s^{s+T} F_0(\tau, x)d\tau$$

for all $s \in \mathbb{R}$, all $T \simeq \infty$ and all $x \in U_{NS}$. Theorem 5 is true if F_0 is assumed to have the above properties. The proof is the same, the only difference being the choice of the instant t_1 of the method of stroboscopy. In the present case we proceed as follows. The property (9) holds for all limited T. By permanence it is still true for some unlimited T which can be choosen such that $\varepsilon T \simeq 0$. Define $t_1 = t_0 + \varepsilon T$. Then we have

$$\frac{x_1 - x_0}{t_1 - t_0} = \frac{X(T)}{T} \simeq \frac{1}{T}\int_0^T F_0(s+r, x_0)dr = \frac{1}{T}\int_s^{s+T} F_0(r, x_0)dr \simeq f(x_0).$$

Notice that this is the first time where the successive instant t_1 of the method of stroboscopy is obtained from the time t_0 by an external construction. But this is allowed by the Stroboscopy Lemma. In the next section the choice of t_1 will be more subtle[3].

[3]The KBM theorem of averaging in the almost periodic case and in the general case was my motivation to extend the method of stroboscopy to the situation where the sequence of instants t_n of observation is not given *a priori*. Indeed in this case this sequence is given by some kind of external recurrence relation

4.3. KBM vector fields. Let $\mathcal{C}$ be the set of continuous functions from $\mathbb{R}^+ \times U$ into $\mathbb{R}^d$, where U is an open subset of $\mathbb{R}^d$. A nonautonomous vector field $F_0 \in \mathcal{C}$ is said to be a KBM vector field if it satisfies the following properties :

$(H.1)$ The continuity of the function F_0 in the variable $x \in U$ is uniform with respect to the variable $t \in \mathbb{R}^+$, that is :

$$\forall x \in U \;\; \forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall t \in \mathbb{R} \;\; \forall x' \in U \;\; (\|x' - x\| < \delta \Rightarrow \|F_0(t, x') - F_0(t, x)\| < \varepsilon)$$

$(H.2)$ For all $x \in U$ there exists a limit

$$f(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T F_0(t, x) dt$$

$(H.3)$ The initial value problem (8) has a unique solution $y(t)$. Let $J = [0, \omega)$, $0 < \omega \leq +\infty$ be its maximal positive interval of definition.

From conditions $(H.1)$ and $(H.2)$ we deduce that the function $f : U \to \mathbb{R}$ is continuous (see Lemma 4). So the averaged differential equation $y' = f(y)$ has a continuous righthand.

The fundamental problem in the theory of averaging of Krylov, Bogolioubov and Mitropolski is to approximate the solutions of the initial value problem (7) by the function y on some subinterval J_0 of J, whenever F is close to F_0, a is close to a_0 and $\varepsilon > 0$ is sufficiently small.

First we give a nonstandard formulation of the result (Theorem 6). Then we translate a simplified version of the result into standard terms (Theorem 7) and present a generalization (Theorem 8).

Theorem 6 (KBM Theorem of Averaging). *Let $F_0 : \mathbb{R}^+ \times U \to \mathbb{R}^d$ be a standard KBM vector field and let $a_0 \in U$ be standard. Let H be a subset of $\mathbb{R}^d$ containing all the nearstandard points in U and let $F : \mathbb{R}^+ \times H \to \mathbb{R}^d$ be continuous and such that $F(s, x) \simeq F_0(s, x)$ for all $s > 0$ and all $x \in U_{NS}$. Let $\varepsilon > 0$ and $a \in U$ be such that $\varepsilon \simeq 0$ and $a \simeq a_0$. Then for every maximal solution x of problem (7) there is an interval $J_0 = [0, \omega_0] \subset J$ such that $\omega_0 \simeq \omega$, x is defined at least on J_0 and $x(t) \simeq y(t)$ for all $t \in J_0$.*

Proof. The proof needs the Lemma 5 below (the proof of this Lemma needs some preliminary result and is postponed to the next section). Let $x : I \to U$ be a maximal solution of problem (7). Let $t_0 \in I$ be limited, such that $x_0 = x(t_0) \in U_{NS}$. By Lemma 5, there is $\alpha > 0$, $\alpha \simeq 0$ such that for all limited $T \geq 0$ we have

$$\frac{1}{S} \int_s^{s+TS} F_0(r, x_0) dr \simeq T f(x_0)$$

where $s = t_0/\varepsilon$ and $S = \alpha/\varepsilon$. The successive instant of observation of the method of stroboscopy is choosen as follows. Under the change of variables :

$$T = \frac{t - t_0}{\alpha} \qquad X(T) = \frac{x(t_0 + \varepsilon T) - x_0}{\alpha}$$

the system (7) becomes

$$\frac{dX}{dT} = F(s + ST, x_0 + \alpha X).$$

Since $x_0 \in U_{NS}$, $x_0 + \alpha X \in U_{NS}$ for all limited X in $\mathbb{R}^d$. Then we have

$$F(s+ST, x_0+\alpha X) \simeq F_0(s+ST, x_0+\alpha X) \simeq F_0(s+ST, x_0)$$

for all $T > 0$ and all limited X in $\mathbb{R}^d$. Moreover we have

$$\int_0^T F_0(s+Sr, x_0)dr = \frac{1}{S}\int_s^{s+TS} F_0(r, x_0)dr \simeq Tf(x_0),$$

which is limited for all limited $T > 0$. By Theorem 2 we have $X(T) \simeq Tf(x_0)$ for all limited $T > 0$. Define t_1 by $t_1 = t_0 + \alpha$. Then we have :

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(1) \simeq f(x_0)$$

Since $t_1 - t_0 = \alpha > \varepsilon$ and $x(t) - x(t_0) = \alpha X(T) \simeq 0$ for all $t \in [t_0, t_1]$, we have proved that there is $\mu = \varepsilon$ such that whenever $t_0 \in I$ is limited and $x(t_0) \in U_{NS}$ there is $t_1 \in I$ such that the function x satisfies the stroboscopic property $\mathcal{S}(t_0, t_1)$. By the Stroboscopy Lemma there is an interval $J_0 = [0, \omega_0] \subset J$ such that $\omega_0 \simeq \omega$, x is defined at least on J_0 and $x(t) \simeq y(t)$ for all $t \in J_0$. □

4.4. Preliminary lemmas. In this section U is a standard open subset of $\mathbb{R}^d$ and $F_0 \in \mathcal{C}$ is standard and satisfies conditions $(H.1)$ and $(H.2)$. The external translations of these conditions are

$(H.1')$ $\forall^{st} x_0 \in U \;\; \forall t > 0 \;\; \forall x \in U \;\; (x \simeq x_0 \Rightarrow F_0(t,x) \simeq F_0(t, x_0))$

$(H.2')$ There is a standard function $f : U \to \mathbb{R}^d$ such that

$$\forall^{st} x_0 \in U \;\; \forall T \simeq +\infty \;\; f(x_0) \simeq \frac{1}{T}\int_0^T F_0(t, x_0)dt$$

Lemma 4. *The function f is continuous and we have*

$$f(x) \simeq \frac{1}{T}\int_0^T F_0(t,x)dt$$

for each $x \in U_{NS}$ and each $T \simeq +\infty$.

Proof. Let $x_0 \in U$ and $x \in U$ be such that x_0 is standard and $x \simeq x_0$. Let $\delta > 0$ be infinitesimal. By condition $(H.2)$, there is $T_0 > 0$ such that

$$|f(x) - \frac{1}{T}\int_0^T F_0(t,x)dt| < \delta$$

for all $T > T_0$. Hence for some $T \simeq +\infty$ we have

$$f(x) \simeq \frac{1}{T}\int_0^T F_0(t,x)dt.$$

By condition $(H.1')$ we have $F_0(t,x) \simeq F_0(t,x_0)$ for all $t > 0$. Therefore

$$f(x) \simeq \frac{1}{T}\int_0^T F_0(t,x_0)dt.$$

By condition $(H.2')$ we deduce that $f(x) \simeq f(x_0)$. Thus f is continuous. Moreover for all $T \simeq +\infty$ we have

$$f(x) \simeq f(x0) \simeq \frac{1}{T}\int_0^T F_0(t,x_0)dt \simeq \frac{1}{T}\int_0^T F_0(t,x)dt.$$

□

Lemma 5. *For each limited $t > 0$ and each $x \in U_{NS}$, there is $\alpha > 0$, $\alpha \simeq 0$ such that for all limited $T \geq 0$ we have*

$$\frac{1}{S}\int_s^{s+TS} F_0(r,x)dr \simeq Tf(x) \quad \text{where} \quad s = \frac{t}{\varepsilon} \quad \text{and} \quad S = \frac{\alpha}{\varepsilon}.$$

Proof. Let $t > 0$ be limited and let $x \in U_{NS}$. We use the notations $s = t/\varepsilon$ and $g(r) = F_0(r,x)$. Let $T > 0$ and $S > 0$ be real numbers.

i) Suppose s is limited. Let S be unlimited. If $s + TS$ is limited then we have $T \simeq 0$ and

$$\frac{1}{S}\int_s^{s+TS} g(r)dr \simeq 0 \simeq Tf(x).$$

If $s + TS \simeq +\infty$ we write

$$\frac{1}{S}\int_s^{s+TS} g(r)dr = \left(T + \frac{s}{S}\right)\frac{1}{s+TS}\int_0^{s+TS} g(r)dr - \frac{1}{S}\int_0^s g(r)dr.$$

By Lemma 4 we have

$$\frac{1}{s+TS}\int_0^{s+TS} g(r)dr \simeq f(x).$$

Since $\frac{1}{S}\int_0^s g(r)dr \simeq 0$ and $\frac{s}{S} \simeq 0$, we have

$$\frac{1}{S}\int_s^{s+TS} g(r)dr \simeq Tf(x).$$

This property is satisfied for all $S \simeq +\infty$. It suffices to take $S = 1/\sqrt{\varepsilon}$ for which $\alpha = \varepsilon S \simeq 0$.

ii) Suppose s is unlimited. We write

$$\frac{1}{S}\int_s^{s+TS} g(r)dr = T\frac{1}{s+TS}\int_0^{s+TS} g(r)dr + \frac{s}{S}\left(\frac{1}{s+TS}\int_0^{s+TS} g(r)dr - \frac{1}{s}\int_0^s g(r)dr\right).$$

Let us denote by

$$\eta(u) = \frac{1}{s+u}\int_s^{s+u} g(r)dr - f(x).$$

By Lemma 4 we have $\eta(u) \simeq 0$ for all $u \geq 0$. Then we have

$$\frac{1}{S}\int_s^{s+TS} g(r)dr = T\big[f(x) + \eta(TS)\big] + \frac{s}{S}\big[\eta(TS) - \eta(0)\big].$$

The quantity $T\eta(TS) + \frac{s}{S}\big[\eta(TS) - \eta(0)\big]$ is infinitesimal for all T limited and all S such that $\frac{s}{S}$ is limited. By permanence this property holds for some S for which $\frac{S}{s} \simeq 0$. Since $t = \varepsilon s$ is limited and $\frac{S}{s} \simeq 0$ we have $\alpha = \varepsilon S \simeq 0$. $\square$

4.5. A standard version of the KBM Theorem. Let us give now a standard translation of some simplified version of Theorem 6. It still is possible to translate the actual version of Theorem 6 but at the price of a somewhat complicated formulation. This translation is left to the reader. The set $\mathcal{C} = \mathcal{C}(\mathbb{R}^+ \times U, \mathbb{R}^d)$ of continuous vector fields is endowed with the topology of uniform convergence on the product of $\mathbb{R}^+$ by compact subsets of U.

Theorem 7. *Let $F_0 : \mathbb{R}^+ \times U \to \mathbb{R}^d$ be a KBM vector field. Then for all $\delta > 0$ and all $l \in J$, there are $\eta > 0$ and a neighborhood $\mathcal{V}$ of F_0 in $\mathcal{C}$ such that for all $\varepsilon > 0$, for all $b \in U$ and all $F \in \mathcal{C}$, any maximal solution $x(t)$ of problem (7) is defined at least on $[0, l]$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \in [0, l]$, whenever $\varepsilon < \eta$, $\|b - a\| < \eta$ and $F \in \mathcal{V}$.*

Remark. In the usual formulations of the KMB theorem it is required that the function $F(t, x)$ is Lipschitz continuous in x (see [E,SV]). In our formulation this condition has been weakened : only the continuity in x uniformly with respect to t is assumed. Therefore averaging is valid in situations which are not covered by the classical results. For example, by Theorem 6 the solutions of the differential equation

$$\frac{dx}{dt} = x^{2/3} \cos \frac{t}{\varepsilon}$$

are nearly constant for all limited $t > 0$. This result may also be verified by solving the equation. We may obtain also second order approximations of the solutions (see [S1]).

4.6. A generalization of the averaging theorem. In order to motivate the generalization we propose, let us first give an example. Consider the function $F_0(t, x) = \sin(tx)$. The properties $(H.2)$ and $(H.3)$ are satisfied, with $f(x) = 0$, but the function F_0 is not continuous in x uniformly with respect to t. We cannot apply Theorem 6. In fact the conclusion of this theorem would be false since the solutions of the differential equation $x' = \sin \frac{tx}{\varepsilon}$ are not nearly constant. However the change of variable $\theta = tx$ transforms this equation into the system

$$\begin{aligned} \frac{dx}{dt} &= \sin \frac{\theta}{\varepsilon} \\ \frac{d\theta}{dt} &= x + t \sin \frac{\theta}{\varepsilon}. \end{aligned}$$

When $(x, t) \in U := \{(x, t) \in \mathbb{R}^2 : x^2 > t^2\}$ this system is equivalent to

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{\sin(\theta/\varepsilon)}{x + t \sin(\theta/\varepsilon)} \\ \frac{dt}{d\theta} &= \frac{1}{x + t \sin(\theta/\varepsilon)}. \end{aligned}$$

This system satisfies the conditions of Theorem 6. The averaged system is (for $x > t > 0$)

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{\sqrt{x^2 - t^2} - x}{t\sqrt{x^2 - t^2}} \\ \frac{dt}{d\theta} &= \frac{1}{\sqrt{x^2 - t^2}}. \end{aligned}$$

Therefore the solutions of the differential equation $\frac{dx}{dt} = \sin \frac{tx}{\varepsilon}$ are infinitely close (in the subset $x > t > 0$) to the solutions of the standard differential equation

$$\frac{dx}{dt} = \frac{\sqrt{x^2 - t^2} - x}{t}.$$

This result was first proved by J.L. Callot [R] using the stroboscopic method (see also Section 5.1) and was at the origin of the method of stroboscopy.

This example shows that it may be interesting to study systems of more general type (see [S2]) :

$$\begin{aligned} &\frac{dx}{dt} = F(\frac{\theta}{\varepsilon}, x) \qquad x(0) = x_0 \\ &\frac{d\theta}{dt} = G(\frac{\theta}{\varepsilon}, x) \qquad \theta(0) = \theta_0 \end{aligned} \tag{10}$$

where $\theta \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$ (the problem (7) is obtained by setting $G = 1$). More precisely let U be an open subset of $\mathbb{R}^d$ and let $F_0 : \mathbb{R}^+ \times U \to \mathbb{R}^d$ and $G_0 : \mathbb{R}^+ \times U \to \mathbb{R}$ be continuous functions. Suppose

$(H.1*)$ The continuity of F_0 and G_0 in x is uniform with respect to θ and $G_0(x, \theta) \neq 0$ for all $(x, \theta) \in \mathbb{R}^+ \times U$.

$(H.2*)$ For all $x \in U$ there exist the limits

$$f(x) = \lim_{T\to\infty} \int_0^T \frac{F_0(\theta, x)}{G_0(\theta, x)} d\theta \qquad g(x) = \lim_{T\to\infty} \int_0^T \frac{1}{G_0(\theta, x)} d\theta.$$

$(H.3*)$ The initial value problem

$$\frac{dy}{dt} = \frac{f(y)}{g(y)} \qquad y(0) = a_0 \in U$$

has a unique solution $y(t)$. Let $J = \{0, \omega)$, $0 < \omega \leq +\infty$ be its maximal positive interval of definition.

From conditions $(H.1*)$ and $(H.2*)$ we deduce that the functions $f : U \to \mathbb{R}^d$ and $g : U \to \mathbb{R}$ are continuous and $g(x) \neq 0$ on U, so the averaged differential equation in $(H.3*)$ is well defined.

Theorem 8. *Let U, F_0 and G_0 be standard and satisfy $(H.1*)$, $(H.2*)$ and $(H.3*)$. Let $a_0 \in U$ be standard. Let H be a subset of $\mathbb{R}^d$ containing all the nearstandard points in U. Let $F : \mathbb{R}^+ \times H \to \mathbb{R}^d$ and $G : \mathbb{R}^+ \times H \to \mathbb{R}$ be continuous functions such that $F(\theta, x) \simeq F_0(\theta, x)$ and $G(\theta, x) \simeq G_0(\theta, x)$ for all $\theta > 0$ and all $x \in U_{NS}$. Let $\varepsilon > 0$, $x_0 \in H$ and $\theta > 0$ be such that $\varepsilon \simeq 0$ and $x_0 \simeq a_0$. Then for every maximal solution $(x(t), \theta(t))$ of the initial value problem (10) there is an interval $J_0 = [0, \omega_0] \subset J$ such that $\omega_0 \simeq \omega$, the functions $x(t)$ and $\theta(t)$ are defined for all $t \in J_0$ and satisfy :*

$$x(t) \simeq y(t) \qquad \theta(t) \simeq \theta_0 + \int_0^t \frac{ds}{g(y(s))}.$$

Proof. The subset $K = \{x \in H : \forall \theta > 0 \ \ G(\theta, x) \neq 0$ contains all the nearstandard points in U. On K the system (10) may be rewritten as follows :

$$\frac{dx}{d\theta} = \frac{F(\theta/\varepsilon, x)}{G(\theta/\varepsilon)} \qquad \frac{dt}{d\theta} = \frac{1}{G(\theta/\varepsilon, x)}.$$

This system satisfies all the conditions in Theorem 6. The associated averaged system defined on U is

$$\frac{dx}{d\theta} = f(x) \qquad \frac{dt}{d\theta} = g(x).$$

Since $g \neq 0$ on U, this averaged system is equivalent to

$$\frac{dx}{dt} = \frac{f(x)}{g(x)} \qquad \frac{d\theta}{dt} = \frac{1}{g(x)}.$$

The solution of the first equation with initial condition a_0 is $y(t)$, it gives an approximate of $x(t)$. The solution of the second equation with initial condition θ_0 is $\theta_0 + \int_0^t \frac{ds}{g(y(s))}$. It gives an approximate of $\theta(t)$. □

5. Applications

The aim of this paragraph is to give the reader a better understanding of the practice of stroboscopy. Let ϕ be the function we study : the technique of stroboscopy consists in selecting some instants of observation t_n, eventually very irregularely distribued, in such a way that the ratio $\frac{x_{n+1}-x_n}{t_{n+1}-t_n}$ is always close to the value $g(t_n, x_n)$ of a standard function g, where $x_n = \phi(t_n)$. The Stroboscopy Lemma dispense us from selecting all the instants t_n. We have only to indicate how to choose the successive instant of observation. This choice is often obtained as follows.

(i) Let t_0 be an instant of observation of the method of stroboscopy. Observe the function ϕ near (t_0, x_0) under some suitable microscope (where $\alpha \simeq 0$) :

$$T = \frac{t - t_0}{\alpha} \qquad X = \frac{x - x_0}{\alpha}.$$

(ii) Use Theorem 2 (or Theorem 3) to solve approximately the differential equation satisfied by ϕ under this microscope. Choose a time T (depending on t_0 in general) such that $\frac{X(T)}{T} \simeq g(t_0, x_0)$ where g is some standard function.

(iii) Choose $t_1 = t_0 + \alpha T$ as the successive instant of observation of the method of stroboscopy. Then you have

$$\frac{x_1 - x_0}{t_1 - t_0} = \frac{X(T)}{T} \simeq g(t_0, x_0).$$

(iv) Conclude by the theorem of stroboscopy.

The nontrivial part of the method is the choice of the focusing factor α characterizing the microscope. In the proof of Theorem 3 the choice of α was arbitrary. In the proof of Theorem 5 the focusing factor α was taken equal to the infinitesimal ε which appears naturally in problem (7). In the proof of Theorem 6 this choice was more subtle and depended on t_0 (see Lemma 5). Morover, in this last case, α is not directly obtained from the infinitesimal ε which appears in the formulation of the problem. In many other applications the choice of α is more constructive. Often we may take $\alpha = \varepsilon$. Let us illustrate the method in some problems in the perturbation theory of differential equations and in numerical analysis.

5.1. The differential equation $x' = \sin\frac{tx}{\varepsilon}$. Due to the symmetries of the associated vector fields we may restrict our study to the subset defined by $x > 0$ and $t \geq 0$. The vector field is horizontal on the isoline I_k defined by $tx = 2k\pi\varepsilon$, its slope is -1 on the isoline I'_k defined by $tx = \left(2k + \frac{3}{2}\right)\pi\varepsilon$. Let (t, x) be a point in the region $P \subset \mathbb{R}^2$ defined by $t \geq x > 0$ and let k be such that (t, x) lies between I_k and I'_{k+1}. These two isoclines define a tube in which the trajectory passing through (t, x) is trapped. Since these two isoclines are also infinitely close, in the region P the solutions of the differential equation are infinitely close to the hyperbolas $tx = c$, where c is constant. This argument does not work in the region $S \subset \mathbb{R}^2$ defined by $x > t \geq 0$. A solution with initial condition $x(0) = a > 0$ will cross the isoclines I_k until it reaches the diagonal $t = x$. Let (t_k, x_k) be the point where the solution crosses the isocline I_k. The microscope

$$T = \frac{t - t_k}{\varepsilon} \qquad X = \frac{x - x_k}{\varepsilon}$$

transforms our equation in

$$\frac{dX}{dT} = \sin\left(x_k T + t_k X + \varepsilon T X + \frac{t_k x_k}{\varepsilon}\right) \qquad X(0) = 0$$

Since $t_k x_k = 2k\pi\varepsilon$, by Theorem 3 the solutions of this equation are infinitely close to the solutions of the differential equation

$$\frac{dX}{dT} = \sin\left(x_k T + t_k X\right)$$

as long as T is limited. The change of variable $\theta = x_k T + t_k X$ transforms this equation in

$$\frac{d\theta}{dT} = x_k + t_k \sin\theta$$

We have $t_{k+1} = t_k + \varepsilon T$ for a value $T = p$ for which $x_k p + t_k X(p) + \varepsilon p X(p) = 2\pi$, that is $\theta(p) \simeq 2\pi$. Then we have

$$p \simeq \int_0^{2\pi} \frac{d\theta}{x_k + t_k \sin\theta} = \frac{2\pi}{\sqrt{x_k^2 - t_k^2}} \qquad X(p) \simeq \frac{2\pi - x_k p}{t_k} = \frac{2\pi}{t_k}\left(1 - \frac{x_k}{\sqrt{x_k^2 - t_k^2}}\right)$$

Finally we have

$$\frac{x_{k+1} - x_k}{t_{k+1} - t_k} = \frac{X(p)}{p} \simeq g(t_k, x_k) \qquad g(t, x) = \frac{\sqrt{x^2 - t^2} - x}{t}.$$

By stroboscopy we conclude that the solutions of our differential equation are infinitely close, in the region S, to the solutions of the differential equation $x' = g(t, x)$. This description agreees with the numerical experiment presented in Fig. 1[4].

[4]Figures 1 and 3 of the present paper was obtained by using the software package VV of Jean Louis Callot.

FIGURE 1. The solutions of $x' = \sin(2tx)$ with initial conditions 2.5, 5, 7.5 and 10 (at left) and the corresponding solutions of $x' = g(t, x)$ (at right).

5.2. The Lorentz pendulum. The small oscillations of a pendulum are described by the differential equation

$$\frac{d^2x}{d\tau^2} + \omega^2 x = 0 \qquad \omega^2 = \frac{g}{l}$$

where l is the lenght of the pendulum, g is the gravitation constant, τ is the time and x the angular deviation. The associated vector field in the phase space is

$$\frac{dx}{d\tau} = y \qquad \frac{dy}{d\tau} = -\omega^2 x. \tag{11}$$

Along a trajectory $(x(\tau), y(\tau))$, the energy $E(\tau) = H(x(\tau), y(\tau))$ where

$$H(x, y) = \frac{y^2 + \omega^2 x^2}{2}$$

is constant. Hence the orbits are the ellipses of constant energy.

If ω is a slowly varying parameter, that is ω is a function of $\varepsilon\tau$ where ε is small, the ratio $E(\tau)/\omega(\varepsilon\tau)$ remains nearly constant for times of order $1/\varepsilon$ despite of the fact that E and ω may vary considerably. This model was proposed by Einstein and Lorentz at the Soloway Congress in Bruxelles in 1911 to explain the behaviour of an electron for which the ratio of the energy to the frequence is contant, even if this electron moves in a varying electromagnetic field. Their explanation is based on the fact that the variation of the surrounding electromagnetic field is slow with respect to the high frequency of the electron. Shortly afterwards it appeared that quantum mechanic was more suitable to understand atomic behaviour, but the interest in such phenomenons remains. A nearly constant quantity as above is called an adiabatic invariant of the system.

When $\omega = \omega(\varepsilon\tau)$ is slowly varying, the energy $E(\tau) = H(x(\tau), y(\tau))$ satisfies the differential equation

$$\frac{dE}{d\tau} = \varepsilon\omega(\varepsilon\tau)\omega'(\varepsilon\tau)x^2 \tag{12}$$

where ω' is the derivative of the function ω. The change of time $t = \varepsilon\tau$ transforms the system (11-12) into

$$\frac{dx}{dt} = \frac{y}{\varepsilon} \qquad \frac{dy}{dt} = -\frac{\omega^2(t)x}{\varepsilon} \qquad \frac{dE}{dt} = \omega(t)\omega'(t)x^2. \tag{13}$$

Let $\phi(t) = (x(t), y(t), E(t))$ be a solution of this system. Let t_0 be limited such that $\omega(t_0)$ is not infinitesimal and $\phi(t_0)$ is limited. We may assume that $y_0 = y(t_0) = 0$, so that $E_0 = E(t_0) = \omega(t_0)^2 x_0^2/2$ where $x_0 = x(t_0)$. The successive instant t_1 of observation of the method of stroboscopy is choosen as follows. The microscope

$$T = \frac{t - t_0}{\varepsilon} \qquad F = \frac{E - E_0}{\varepsilon}$$

transforms the system (13) in

$$\frac{dx}{dT} = y \qquad \frac{dy}{dT} = -\omega^2(t_0 + \varepsilon T)x \qquad \frac{dF}{dT} = \omega(t_0 + \varepsilon T)\omega'(t_0 + \varepsilon T)x^2 \tag{14}$$

with initial conditions $x(0) = x_0$, $y(0) = 0$ and $F(0) = 0$. By Theorem 3 the solutions of this system are infinitely close to the solutions of the system

$$\frac{d\overline{x}}{dT} = \overline{y} \qquad \frac{d\overline{y}}{dT} = -\omega^2(t_0)\overline{x} \qquad \frac{d\overline{F}}{dT} = \omega(t_0)\omega'(t_0)\overline{x}^2. \tag{15}$$

This system may be solved explicitely ; we get

$$x(T) \simeq \overline{x}(T) = \frac{\sqrt{2E_0}}{\omega(t_0)}\cos(\omega(t_0)T) \qquad y(T) \simeq \overline{y}(T) = -\sqrt{2E_0}\sin(\omega(t_0)T)$$

$$F(T) \simeq \overline{F}(T) = \int_0^T \frac{2E_0}{\omega^2(t_0)}\omega(t_0)\omega'(t_0)\cos^2(\omega(t_0)r)dr.$$

Define $t_1 = t_0 + \varepsilon p$ where $p = \frac{2\pi}{\omega(t_0)}$ is the period the solution $(\overline{x}(T), \overline{y}(T))$ of the system (15). Then we have

$$F(p) \simeq \overline{F}(p) = 2\pi\frac{E_0\omega'(t_0)}{\omega^2(t_0)}.$$

Finally we have

$$\frac{E_1 - E_0}{t_1 - t_0} = \frac{F(p)}{p} \simeq E_0\frac{\omega'(t_0)}{\omega(t_0)}.$$

By stroboscopy we obtain that $E(t)$ is infinitely close to a solution of the differential equation

$$\frac{dE}{dt} = E\frac{\omega'}{\omega}$$

that is $E(t) \simeq E(0)\omega(t)/\omega(0)$, and the ratio $E(t)/\omega(t)$ is nearly contstant for all limited t such that $\omega(t)$ is not infinitesimal.

This ratio is the adiabatic invariant of the energy divided by the frequency for a pendulum with slowly varying frequency. The same approach works in the more general case of Hamiltonian systems with slowly variable parameters and small perturbations (see [S2,S4]).

5.3. The Van der Pol oscillator and the Einstein equation for the planet Mercury. We consider here another kind of perturbation of a linear oscillator, namely equations of type

$$\frac{d^2x}{d\tau^2}+x=\varepsilon f(x,\frac{dx}{d\tau}) \tag{16}$$

The Van der Pol oscillator

$$\frac{d^2x}{d\tau^2}+x=\varepsilon(1-x^2)\frac{dx}{d\tau}$$

is an example of this type. This also yieldss a famous example of a differential equation in the plane with a unique limit cycle.

The Einstein equation for Mercury is

$$\frac{d^2u}{d\theta^2}+u=a+\varepsilon u^2$$

which is equivalent to

$$\frac{d^2x}{d\tau^2}+x=\varepsilon(x+a)^2$$

with $x=u-a$ and $\tau=\theta$. This equation is of type (16). Here $u=1/r$, where (r,θ) are the polar coordinates of Mercury in the plane, centered at the sun. For $\varepsilon=0$ we have the Newtonian model : the trajectory is an ellipse of the form $u=a+(b-a)\cos\theta$. The usual perturbation theory predicts that, under the gravitational effects of the other planets, this orbit should rotate slowly in space, so that the major axis of the ellipse should advance (precess) by about 530 seconds of arc per century, leading to a complete revolution in 250.000 years. However the precession is actually observed to be of 570 seconds of an per century. The difference is explained by gravitational effects, using the general relativity theory.

The vector field associated to equation (16) is

$$\frac{dx}{d\tau}=y \qquad \frac{dy}{d\tau}=-x+\varepsilon f(x,y)$$

whose solutions are close to those of the system

$$\frac{dx}{d\tau}=y \qquad \frac{dy}{d\tau}=-x$$

for all limited τ. This property suggests to use the so-called the Van der Pol change of variables :

$$x=A\sin(\tau+B) \qquad y=A\cos(\tau+B)$$

Denote $\phi=\tau+B$ then we have

$$A'\sin\phi+AB'\cos\phi=0 \qquad A'\cos\phi-AB'\sin\phi=\varepsilon f.$$

Consequently

$$\frac{dA}{d\tau} = \varepsilon f\big[A\sin(\tau+B), A\cos(\tau+B)\big]\cos(\tau+B)$$
$$\frac{dB}{d\tau} = \frac{\varepsilon}{A} f\big[A\sin(\tau+B), A\cos(\tau+B)\big]\sin(\tau+B)$$

This system is of type

$$\frac{dX}{d\tau} = \varepsilon F(\tau, X)$$

where $X = (A, B)$ and F is 2π-periodic in τ (this is a particular case of the KBM method of averaging). Since we look for the long time behaviour of the solutions, it is more suitable to consider this system at the time scale $t = \varepsilon\tau$. We have

$$\frac{dX}{dt} = F(\frac{t}{\varepsilon}, X)$$

By the Theorem 5 any solution $X(t)$ of this equation is approximated by a solution of the averaged equation

$$\frac{dX}{dt} = F_0(X)$$

where F_0 is the average of the function F over one period :

$$F_0(X) = \frac{1}{2\pi}\int_0^{2\pi} F(\tau, X)d\tau$$

For the Van der Pol oscillator we have $f(x, y) = (1 - x^2)y$ and the averaged system is

$$\frac{dA}{dt} = \frac{A}{8}(4 - A^2) \qquad \frac{dB}{dt} = 0$$

The phase B is nearly constant (there is no precession) but the amplitude of the solution is slowly varying. The value $A = 2$, which is an asymptotically stable equilibrium of the averaged equation, corresponds to the limit cycle of the Van der Pol equation, which is infinitely close to the circle of radius 2 in the plane (x, y).

For the Einstein equation of Mercury we have $f(x, y) = (x + a)^2$. The averaged system is

$$\frac{dA}{dt} = 0 \qquad \frac{dB}{dt} = a$$

Hence the amplitude is nearly constant (there is no secular variation of the amplitude of the orbit of Mercury due to the gravitationnal effects) but the phase is slowly varying (there is a precession). More precisely

$$A(\tau) \simeq A_0 \qquad B(\tau) \simeq -\varepsilon a\tau + B_0.$$

The solution $u(\tau)$ of Einstein's equation of Mercury satisfies

$$u(\tau) \simeq a + A_0\sin\big[\tau(1 - \varepsilon a) + B_0\big]$$

for all times τ such that $\varepsilon\tau$ is limited.

Then the amplitude of Mercury remains nearly constant (during centuries) and the precession due to the gravitational effects of general relativity theory is $2\pi\varepsilon a$ radians per orbit of Mercurry. This explains the difference of 40 seconds of arc per century which was observed, and was not understood in the Newtonian model.

5.4. Numerical instability and ghost solutions. We study the behaviour of the solutions of the mixed difference scheme

$$(1-\mu)\frac{u_{n+1}-u_{n-1}}{2h}+\mu\frac{u_{n+1}-u_n}{h}=g(u_n)\qquad 0\leq\mu\leq 1\quad h>0.$$

This scheme is used to solve numerically the differential equation $u'=g(u)$ where g is a continuous function, with the starting procedure

$$u_0\quad\text{given}\quad\text{and}\quad u_1=u_0+hg(u_0).$$

This scheme is called a multistep method since the values of both u_{n-1} and u_n are necessary to compute u_{n+1}. The problem was studied by Yamaguti and Ushiki [YU] in the case of logistic equation $u'=u(1-u)$ (see also [CJL] and [S3]). The logistic equation has two equilibrium points $u=1$ and $u=0$. The first is asymptotically stable and the second is unstable. The numerical solution approximates the true solution quite well until the true solution enters a small neighborhood of the equilibrium point $u=1$. After this a numerical instability appears. This instability is illustrated on Fig. 2 ; several cases are distinguished :

a) If $k=\mu/h=0$, the numerical solution almost converges to $u=1$, starts to oscillate and then converges to the unstable equilibrium point $u=0$. The numerical solution repeates alternating cycles of smooth and oscillatory behaviour. Finally it goes to minus infinity. Such a solution was called a ghost solution.

b) If $0<k<\frac{1}{2}$, the solution after having almost converged to $u=1$, presents cycles (when k is small these cycles are analogous to those of the case $k=0$), the form of the cycles gradually changes until the numerical solution oscillates between two values α and β such that $\alpha<1<\beta$. The difference $\beta-\alpha$ tends to 0 when k tends to $\frac{1}{2}$.

c) If $k\geq\frac{1}{2}$ there are no ghost solutions.

The mixed difference scheme may be rewritten as follows

$$u_{n+1}=u_{n-1}+\frac{2h}{1+kh}\left[k\left(u_n-u_{n-1}\right)+g(u_n)\right].$$

Let us write this multistep method as a one step method in a higher dimensional space. We have

$$\begin{pmatrix}u_n\\u_{n+1}\end{pmatrix}=T\begin{pmatrix}u_{n-1}\\u_n\end{pmatrix}$$

where the mapping $T:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is defined by

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\x+\dfrac{2h}{1+kh}\left[k(y-x)+g(y)\right]\end{pmatrix}.$$

The orbit of the starting point $(u_0,u_1=u_0+hg(u_0))$ produced by iteration of the mapping T is the same as the sequence of points produced by the mixed difference

Figure 2. The numerical solutions of the logistic equation $u' = u(1-u)$ produced from $u_0 = 0.5$ by the mixed difference scheme for $k = 0$, $k = 0.1$, $k = 0.4$ and $k = 0.5$. The sequences (u_{2n}, u_{2n+1}) are represented on the left of the figure.

scheme, except that every point u_n is produced twice. So it suffices to consider the orbit of (u_0, u_1) under the mapping T^2. This mapping has the following expression :

$$T^2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \dfrac{2h}{1+kh}X \\ y + \dfrac{2h}{1+kh}Y \end{pmatrix}$$

where $X = k(y-x) + g(y)$ and $Y = k(x-y) + g\left(x + \frac{2h}{1+kh}X\right) + \frac{2hk}{1+kh}X$. The sequence (x_n, y_n) produced by the iteration of T^2 is simply the sequence (u_{2n}, u_{2n+1}) produced by the mixed difference scheme. Let $h > 0$ be infinitesimal. Then we have

$$\frac{x_{n+1} - x_n}{2h} \simeq k(y_n - x_n) + g(y_n) \qquad \frac{y_{n+1} - y_n}{2h} \simeq k(x_n - y_n) + g(x_n).$$

By stroboscopy we have the following result :

Theorem 9. *The sequence $(x_n = u_{2n}, y_n = u_{2n+1})$ produced by the mixed difference scheme is infinitely close to a reunion of orbits of the augmented differential system*

$$\begin{aligned} x' &= k(y-x) + g(y) \\ y' &= k(x-y) + g(x) \end{aligned}$$

whenever (x_n, y_n) is limited.

Let us give now some properties of this augmented system. The diagonal plane $x = y$ is invariant. On this plane the sytem reduces simply to the initial single equation $x' = g(x)$. Hence the numerical solution approximates the true solutions whenever these solutions are limited and nh is limited. After this unstabilities may occur. Let us examine this phenomenon more precisely in the particular case of the logistic equation. The associated differential system in $\mathbb{R}^2$ is

$$\begin{aligned} x' &= k(y-x) + y(1-y) \\ y' &= k(x-y) + x(1-x). \end{aligned}$$

For k such that $0 \le k < \frac{1}{2}$ we have four singular points $(0,0)$, $(1,1)$, (α,β) and (β,α) where

$$\alpha = \frac{1 + 2k - \sqrt{1-4k^2}}{2} \qquad \beta = \frac{1 + 2k + \sqrt{1-4k^2}}{2}.$$

For $k \ge \frac{1}{2}$ we have two singular points $(0,0)$ and $(1,1)$. The nature of these singular points is as follows (see Fig. 3) :

a) if $k = 0$ then $(0,0)$ and $(1,1)$ are saddle points and $(\alpha,\beta) = (0,1)$ and $(\beta,\alpha) = (1,0)$ are centers,

FIGURE 3. The augmented system for the logistic equation for $k = 0$, $k = 0.1$, $k = 0.4$ and $k = 0.5$.

b) if $0 < k < \frac{\sqrt{5}}{5}$ then $(0,0)$ and $(1,1)$ are saddle points and (α,β) and (β,α) are stable focuses,

c) if $\frac{\sqrt{5}}{5} \leq k < \frac{1}{2}$ then $(0,0)$ and $(1,1)$ are saddle points and (α,β) and (β,α) are stable nodes,

d) if $k \geq \frac{1}{2}$ then $(0,0)$ is a saddle point and $(1,1)$ is a stable node.

The explanation of the behaviour of the numerical solution follows easily from the properties just mentioned of the augmented system (see Fig. 2 and 3). Consider a numerical solution which approximates the true solution until it reaches the halo of $u = 1$. Then it may diverge from $u = 1$ along the unstable separatrix of the saddle point point $(1,1)$. The phenomenon cannot occur if $k \geq \frac{1}{2}$ since $(1,1)$ is a stable node. When $k = 0$ the unstable separatrix of $(1,1)$ and the stable separatrix of $(0,0)$ coincide, which leads of the succession of smooth behaviour and oscillatory behaviour. When $0 < k < \frac{1}{2}$ the observed behaviour is explained by the fact that the numerical solution is captured by the stable focus or node (α,β) or (β,α). These observation are consequences of the following result (see [S6] for other similar results) :

Lemma 6. *A numerical solution which enters the halo of an asymptotically stable equilibrium point of the augmented system, cannot leave this halo. A numerical solution which enters the halo of a saddle point along the stable separatrix can leave the halo of this saddle point only along the unstable separatrix.*

Proof. We give the proof in the case of a saddle point. Suppose that an iterate (x_p, y_p) is infinitely close to a saddle point σ, and suppose that for some $m > p$ the iterate (x_m, y_m) is at an appreciable distance a from this saddle point. Let $r > 0$ be standard such that $r < a$ and let $B = B(\sigma, r)$ be the ball of center σ and radius r. Let k be the first index such that $k > p$ and $(x_k, y_k) \notin B$. By the stroboscopy method the sequence (x_n, y_n) is infinitely close to the orbit of the augmented system passing through $({}^o x_k, {}^o y_k)$ whenever $n - k$ is of order $1/h$. If the point $({}^o x_k, {}^o y_k)$ is not on the unstable separatrix we get a contradiction since the sequence (x_n, y_n) would have been outside B for some n between p and k. $\square$

This example shows that the behaviour of the mixed difference scheme depends on the nature of the equilibria of the augmented system. Let us give some comments on this behaviour in the more general case of equations of the form $u' = g(u)$ where $g : \mathbb{R} \to \mathbb{R}$. Two cases have to be distinguished

a. *The central difference scheme* $(k = 0)$. The augmented system is

$$x' = g(y) \qquad y' = g(x).$$

The equilibrium points are (α, β) where α and β are roots of equation $g(u) = 0$. Assume $g'(\alpha) \neq 0$ for all α. The equilibrium point (α, β) is a center if $g'(\alpha)g'(\beta) < 0$ and a saddle point if $g'(\alpha)g'(\beta) > 0$. In particular the diagonal equilibria are saddle points whose stable (resp. unstable) manifolds lie in the diagonal $x = y$ if $g'(\alpha) < 0$ (resp. $g'(\alpha) > 0$).

Consider two roots α and β of g with the property that $g'(\alpha) > 0$ and $g'(\beta) < 0$. Then $u = \beta$ (resp. $u = \alpha$) is a stable (resp. unstable) equilibrium point of equation $u' = g(u)$. The other solutions of equation $u' = g(u)$ between $u = \alpha$ and $u = \beta$ are monotically increasing (resp. decreasing) if $\alpha < \beta$ (resp. if $\alpha > \beta$) and tend to α when t tends to $-\infty$ and to β when t tends to $+\infty$.

Let a be between α and β, then the unstability of the numerical solution starting from $(u_0, u_1) \simeq (a, a)$ occurs near the stable solution $u = \beta$, along the unstable separatrix of the saddle point (β, β) of the augmented system. The unstable manifolds of this singular points are the stable manifolds of two other singular points (α_1, β_1) and (β_1, α_1), which are symmetric with respect to the diagonal. These two singular points may coincide with the diagonal singular point (α, α) as in the case of the logistic equation. However, in general they are distinct from (α, α) as in the case of the equation $u' = \cos u$ (see [S3] for the details).

b. *The mixed difference scheme* $(k > 0)$. The diagonal equilibria of the augmented system are (α, α) with α a root of equation $g(u) = 0$. The eigenvalues of the linear part of the system are $\lambda_1 = f'(\alpha)$ and $\lambda_2 = -2k - f'(\alpha)$. If $f'(\alpha) > 0$ the singular point (α, α) is a saddle point whose unstable manifolds are included in the diagonal. If $f'(\alpha) < -2k < 0$ the singular point (α, α) is a saddle point whose stable manifolds are included in the diagonal. If $-2k \leq f'(\alpha) < 0$ the singular point (α, α) is a stable node.

Let α and β be two roots of g such that $g'(\alpha) > 0$ and $g'(\beta) < 0$ and let a be between α and β. If $(u_0, u_1) \simeq (a, a)$ and $2k \geq -f'(\beta)$ there is no instability (the sequence u_n is infinitely close to $u(nh)$ for all n where $u(t)$ is the solution of the initial value problem $u' = g(u)$, $u(0) = a$). If $2k < -f'(\beta)$, the numerical solution diverges from the true solution near the stable equilibrium $u = \beta$ of the equation $u' = g(u)$, along the unstable separatrix of the saddle point (β, β) of the augmented system.

The non diagonal equilibria of the augmented system are solutions of the nonlinear system

$$\begin{aligned} k(y-x) + g(y) &= 0 \\ k(x-y) + g(x) &= 0. \end{aligned}$$

Let α be a root of the nonlinear equation

$$g\left(x + \frac{g(x)}{k}\right) + g(x) = 0, \tag{16}$$

then $(\alpha, \alpha + \frac{g(\alpha)}{k})$ is an equilibrium point of the augmented system. It has to be remarked that if α is a root of equation (16), then $\beta = \alpha + \frac{g(\alpha)}{k}$ is also a root of this equation and $\beta + \frac{g(\beta)}{k} = \alpha$, so that $(\beta, \beta + \frac{g(\beta)}{k}) = (\alpha + \frac{g(\alpha)}{k}, \alpha)$ is the equilibrium point of the augmented system which is symmetric to the equilibrium point $(\alpha, \alpha + \frac{g(\alpha)}{k})$, with respect to the diagonal $x = y$.

5.5. Error propagation in numerical schemes. Let $f_0, f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be standard and continuous functions. Let $h > 0$ be infinitesimal. The sequence of points obtained by the scheme

$$x_{n+1} = x_n + hf_0(x_n, nh) + h^2 f_1(x_n, nh) \tag{17}$$

is infinitely close to a solution $\phi(t)$ of the differential equation

$$x' = f_0(x, t) \qquad x(0) = x_0, \tag{18}$$

whenever nh and $\phi(nh)$ are both limited. This follows by an application of the method of stroboscopy. The error $x_n - \phi(nh)$ is in general of order ε but it growths after iterations, leading to instability. We study this phenomenon making the assumption that f_0 is continuously differentiable in x. We have the following result

Theorem 10. *Let x_n is the sequence produced by the numerical scheme (17). Let $\phi(t)$ be the solution of problem (18) such that $x_n \simeq \phi(nh)$ whenever nh and $\phi(nh)$ are limited. Let $e(t)$ be the solution of the nonhomogeneous linear equation*

$$\frac{de}{dt} = \frac{\partial f_0}{\partial x}(\phi(t), t)e + f_1(\phi(t), t) - \frac{1}{2}\phi''(t) \tag{19}$$

with the initial condition $e(0) = 0$. Then we have

$$x_n = \phi(nh) + he(nh) + h\eta_n \quad \text{where} \quad \eta_n \simeq 0.$$

Proof. We define a sequence (t_n, e_n) by

$$t_n = nh \qquad e_n = \frac{x_n - \phi(t_n)}{h}.$$

Then we have

$$\begin{aligned} e_{n+1} - e_n &= \frac{x_{n+1} - x_n}{h} - \frac{\phi(t_n + h) - \phi(t_n)}{h} \\ &= f_0(\phi(t_n) + he_n, t_n) + hf_1(\phi(t_n) + he_n, t_n) - \phi'(t_n) - \frac{h}{2}\phi''(t_n) + h\alpha_n \end{aligned}$$

where $\alpha_n \simeq 0$. Using $\phi'(t_n) = f_0(\phi(t_n), t_n)$ and applying the Taylor formula we get $\beta_n \simeq 0$ such that

$$e_{n+1} - e_n \simeq h\left[\frac{\partial f_0}{\partial x}(\phi(t_n), t_n)e_n + f_1(\phi(t_n), t_n) - \frac{1}{2}\phi''(t_n)\right] + h\beta_n.$$

Then we have

$$\frac{e_{n+1} - e_n}{t_{n+1} - t_n} \simeq \frac{\partial f_0}{\partial x}(\phi(t_n), t_n)e_n + f_1(\phi(t_n), t_n) - \frac{1}{2}\phi''(t_n).$$

By stroboscopy the sequence of points e_n is infinitely close to the solution $e(t)$ of the differential equation (19) with initial condition $e(0) = 0$, whenever both t_n and $\phi(t_n)$ are limited. □

When $f_1 = 0$, the scheme (17) is simply the Euler scheme and the associated differential equation (19) for the error reduces to

$$\frac{de}{dt} = \frac{\partial f_0}{\partial x}(\phi(t), t)e - \frac{1}{2}\phi''(t)$$

which is the well-known differential equation for the error propagation for the Euler scheme (see [H]).

It is easy to obtain similar results for higher order errors and other numerical schemes. For example if the sequence x_n is defined by

$$x_{n+1} = x_n + hf_0(x_n, nh) + h^2 f_1(x_n + nh) + h^3 f_2(x_n + nh)$$

with f_0 f_1 and f_2 standard and continuous functions, f_0 and f_1 being continuously differentiable. Then the e_n is infinitely close to a solution $e(t)$ of equation (19) as shown above. We wish to evaluate the second-order error. Define a sequence g_n by

$$g_n = \frac{e_n - e(nh)}{h}.$$

Computing the ratio $\frac{g_{n+1} - g_n}{h}$ and using stroboscopy as above we get that g_n is infinitely close to a solution g(t) of the differential equation

$$\begin{aligned} \frac{dg}{dt} = \frac{\partial f_0}{\partial x}(\phi(t), t)g + \frac{1}{2}\frac{\partial^2 f_0}{\partial x^2}(\phi(t), t)e^2(t) \\ + \frac{\partial f_1}{\partial x}(\phi(t), t)e(t) + f_2(\phi(t), t) - \frac{1}{6}\phi'''(t) - \frac{1}{2}e''(t). \end{aligned}$$

For the Euler scheme ($f_1 = f_2 = 0$) we obtain the following differential equation for the propagation of the second order error g_n :

$$\frac{dg}{dt} = \frac{\partial f_0}{\partial x}(\phi(t),t)g + \frac{1}{2}\frac{\partial^2 f_0}{\partial x^2}(\phi(t),t)e^2(t) - \frac{1}{6}\phi'''(t) - \frac{1}{2}e''(t).$$

The differential equations describing the higher order error propagations are all linear nonhomegeneous equations of type

$$\frac{dE}{dt} = \frac{\partial f_0}{\partial x}(\phi(t),t)E + h(t)$$

where $h(t)$ is depending only on $\phi(t)$ and the errors of lower order. Such an approach seems to be usefull in the determination of the asymptotic expansions of the error for general multistep method (see [HL]).

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